Analytic continuation of the Sudakov form factor in QCD

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We exhibit a solution to the evolution equation for the Sudakov form factor in QCD with massless quarks, which exponentiates infrared poles in dimensional continuation as well as logarithms of momentum transfer. We use this solution to construct an expression for the absolute value of the ratio of timelike to spacelike form factors, in which the infrared finiteness of the ratio is manifest. Finally, we compare this result to explicit calculations of the form factor available in the literature. Most of the large two-loop corrections to the absolute value of the ratio come from the exponentiation of one-loop corrections, including the effect of the running coupling.

I. INTRODUCTION

The high-energy behavior of the fermion electromagnetic form factor in gauge theories has been the subject of interest for a long time.[1] The Sudakov form factor is the simplest amplitude to exhibit the double-logarithmic infrared and collinear singularities characteristic of vector-boson radiative corrections.[2,3] It arises naturally in a number of high-energy cross sections.[3–6] Perhaps most directly, it is relevant to the total Drell-Yan cross section normalized to deeply inelastic scattering, which is plagued by large perturbative corrections at both one[7,8] and two[9] loops.

It was observed some time ago[7,8,10] that much of these large corrections can be associated with the ratio of the absolute value squared of the Sudakov form factor at timelike and spacelike momentum transfer. Large corrections in this ratio may be easily identified by treating the form factor in the leading-logarithm approximation (LLA) in the momentum transfer squared q^2, taken to be much larger than the fermion (or vector) mass. Neglecting the running of the coupling, these enhancements in momentum transfer exponentiate in the LLA, in both Abelian and non-Abelian theories:[11]

$$\Gamma_\mu(q^2)|_{\text{LLA}}=\gamma_\mu\Gamma(q^2)|_{\text{LLA}} =\gamma_\mu \exp\left[-\frac{\alpha_s}{4\pi}C_F \ln^2(-q^2)\right] . \tag{1.1}$$

Note that at leading power in $q^2, \Gamma_\mu(q^2)$ has only a vector structure, and that no scale is necessary in the LLA. The exponentiation in Eq. (1.1) is directly related to the exponentiation of infrared divergences in QED. Using Eq. (1.1), the ratio of the timelike to the spacelike form factor is

$$\left.\frac{\Gamma(Q^2)}{\Gamma(-Q^2)}\right|_{\text{LLA}}=\exp\left[-\frac{\alpha_s C_F}{2\pi}(i\pi \ln q^2)+\frac{\alpha_s C_F}{4\pi}\pi^2\right] . \tag{1.2}$$

The π^2 term is responsible for the bulk of the large one-loop corrections in the Drell-Yan cross section mentioned above. Equation (1.1) has been improved by including the effects of the running coupling. The q^2 dependence can be resummed, including all logarithmic behavior,[12–14] in terms of an evolution equation for $\Gamma(q^2)$.[15]

Solutions to the Sudakov evolution equation in the literature have generally been constructed with the purpose of resumming the large logarithms of q^2, and are expressed in terms of an undetermined integration constant, which contains the dependence on the particle masses and the infrared regulator. Such solutions do not immediately supply an expression for the form factor that is directly comparable with the results of diagrammatic calculations.

In this paper, we shall study the Sudakov form factor in QCD in dimensional regularization, the form in which it usually occurs in calculations in perturbative QCD. As we shall see, in this case it is easy to give a simple solution to the evolution equation, by using the electromagnetic charge at $q^2=0$ as a boundary condition. Then, both logarithms in q^2 and poles in $n-4$, with n the number of dimensions, exponentiate explicitly. As such, our results can be readily compared to existing calculations. They will also make it possible to show explicitly that all divergences in the ratio of timelike to spacelike form factors are in an infinite phase, a generalization of the Coulomb phase of QED.

Even aside from the infinite phase, the ratio of form factors has, as expected, a large two-loop correction, which can be easily derived from the explicit calculations of Kramer and Lampe[16] or Matsuura, van der Marck, and Van Neerven.[9] We will show that most of this large two-loop correction may be understood in terms of the exponentiation of the one-loop correction, including the effects of the running coupling. We suggest that this is an encouraging sign for attempts to resume large correction in perturbative QCD.[3–6,10,17–19]

II. EVOLUTION EQUATION AND SOLUTION FOR $\Gamma(q^2)$ IN DIMENSIONAL REGULARIZATION

The central result in Ref. 12–14 is that the form factor obeys an evolution equation of the form

$$\frac{\partial}{\partial \ln q^2}\Gamma\left[\frac{q^2}{\mu^2},\right]=\frac{1}{2}\left[K(\epsilon,\alpha_s)+G\left[\frac{q^2}{\mu^2},\alpha_s,\epsilon\right]\right]\times\Gamma\left[\frac{q^2}{\mu^2},\epsilon\right] . \tag{2.1}$$

Here we work in a massless theory, with μ the renormalization scale and define

$$\epsilon=2-\frac{n}{2} , \tag{2.2}$$

with n the number of dimensions. We also introduce the notation, for any perturbative function $F(\alpha_s)$,

$$F(\alpha_s)=\sum_{n=0}^{\infty}\left[\frac{\alpha}{\pi}\right]^n F^{(n)}(\alpha_s) . \tag{2.3}$$

At this stage, renormalization has already been carried out, so that we can take $\epsilon<0$ to regulate infrared and collinear divergences. $K(\epsilon,\alpha_s)$ and $G(q^2/\mu^2,\alpha_s,\epsilon)$ are perturbatively calculable functions, with Feynman rules that can be found in Ref. 5. $K(\epsilon,\alpha_s)$ is defined as a pure counterterm, and therefore is a series of poles in any minimal renormalization scheme, while $G(q^2/\mu^2,\alpha_2,\epsilon)$ is finite as $\alpha\rightarrow 0$, and contains all the q^2 dependence.

We shall use modified minimal subtraction scheme ($\overline{\text{MS}}$) renormalization, and we accordingly define

$$\mu^2=\mu_0^2\exp[-\epsilon(\gamma_E-\ln 4\pi)] , \tag{2.4}$$

with μ_0 the mass scale appearing in the interaction Lagrangian. One can find explicit expressions for K and G simply by differentiating the form factor and using their defining properties described above. At one loop,[7,9] with μ given by Eq. (2.4),

$$\Gamma\left[\frac{q^2}{\mu^2},\epsilon\right]=1+\frac{\alpha_s}{4\pi}\left[\frac{\mu^2}{-q^2}\right]^{\epsilon}C_F\left[-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}+\zeta(2)-8-2\epsilon[8-\tfrac{3}{4}\zeta(2)-\tfrac{7}{3}\zeta(3)]+O(\epsilon^2)\right] . \tag{2.5}$$

In this notation, we find

$$K^{(1)}(\epsilon,\alpha_s)=C_F\frac{1}{\epsilon} ,$$

$$G^{(1)}\left[\frac{q^2}{\mu^2},\alpha_s,\epsilon\right]=C_F\left[\frac{\mu^2}{-q^2}\right]^{\epsilon}\left[\frac{1}{\epsilon}+\frac{3}{2}-\frac{\epsilon}{2}[\zeta(2)-8]+\epsilon^2[8-\tfrac{3}{4}\zeta(2)-\tfrac{7}{3}\zeta(3)]+O(\epsilon^3)\right]-C_F\frac{1}{\epsilon} . \tag{2.6}$$

In four dimensions G reduces to[15]

$$G\left[\frac{q^2}{\mu^2},\alpha_s,\epsilon=0\right]=-\frac{\alpha_s}{\pi}C_F\left[\ln\frac{-q^2}{\mu^2}-\frac{3}{2}\right] . \tag{2.7}$$

It will be useful to review some of the known properties of K and G.

Defined as above, K and G have a simple behavior under the renormalization group. First, because the electromagnetic form factor is unrenormalized by the strong interactions, so must be the combination K and G:[15]

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}\right](K+G)=0 ,$$

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}\right]K=-\gamma_K(\alpha_s)=-\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}\right]G , \tag{2.8}$$

with $\gamma_K(\alpha_s)$ an anomalous dimension, which depends only on the coupling. Here and in the following, we will need to use the n-dimensional β function defined by

$$\beta(g,\epsilon)=-\epsilon g-b_0\frac{g^3}{(4\pi)^2}+O(g^5) . \tag{2.9}$$

The corresponding n-dimensional expression for the running coupling is

$$\bar{\alpha}\left[\frac{\mu^2}{\mu_0^2},\alpha_s(\mu_0),\epsilon\right]=\left[\frac{\alpha(\mu_0)}{\left[\frac{\mu}{\mu_0}\right]^{2\epsilon}-\frac{1}{\epsilon}\left[1-\left[\frac{\mu}{\mu_0}\right]^{2\epsilon}\right]\frac{b_0}{4\pi}\alpha_s(\mu_0)}\right] . \tag{2.10}$$

The one-loop result (2.6) may be substituted in (2.9) to give

$$\gamma_k{}^{(1)}=2C_F . \tag{2.11}$$

Equations (2.8) and (2.11), by standard methods,[20] determine the leading behavior of K in $1/\epsilon$ and of G in $\ln(q^2/\mu^2)$ at each order of perturbation theory. To see how, we expand K in powers of α_s and inverse powers of ϵ,

$$K=\sum_{h=1}^{\infty}\frac{K_n(\alpha_s)}{\epsilon^n},\quad K_n(\alpha_s)=\sum_{m=n}^{\infty}K_n^{(m)}\left[\frac{\alpha_s}{\pi}\right]^m . \tag{2.12}$$

From the fact that the $K_n(\alpha_s)$ are by construction independent of μ, we can use the renormalization group Eq. (2.8) to show that

$$\gamma_K = \frac{g\,\partial K_1}{\partial g}\ ,$$
$$\beta(\epsilon=0)\frac{\partial K_n}{\partial g} = g\frac{\partial K_{n+1}}{\partial g}\ . \tag{2.13}$$

These recursion relations show, as usual, that the single pole terms in ϵ determine the entire series for K order by order.

An expression for G at $\epsilon \neq 0$ may be found by solving the renormalization group equation (2.8), to give[15]

$$G\left[\frac{q^2}{\mu^2},\alpha_s(\mu),\epsilon\right] = G\left[-1,\bar{\alpha}\left[\frac{-q^2}{\mu^2},\alpha_s(\mu),\epsilon\right]\epsilon\right] + \frac{1}{2}\int_{-q^2}^{\mu^2}\frac{d\mu'^2}{\mu'^2}\gamma_K\left[\bar{\alpha}\left[\frac{\mu'^2}{\mu^2},\alpha_s(\mu),\epsilon\right]\right]\ . \tag{2.14}$$

From the reality of $\Gamma(q^2)$ for $q^2<0$, $G(-1,\bar{\alpha},\epsilon)$ is purely real. Note, however, that, even if q^2 is chosen real, nothing stops us from extending Eq. (2.14) into the complex μ^2 plane. Equation (2.14) shows that the logarithms of G are determined directly by γ_K, and that the lowest-order term in γ_K (2.11) generates, through the running coupling, leading logarithms in all orders of G. We are now ready to construct our solution to Eq. (2.1).

To solve Eq. (2.1), we use the fact that $\Gamma_\mu(q^2)$ obeys the quantum electrodynamics Ward identity

$$q^\mu\Gamma_\mu(q^2)=0\ , \tag{2.15}$$

for arbitrary q^μ, so that according to standard arguments all overall counterterms cancel (and $Z_1=Z_2$ at the QED vertex). As a result, the term of order α_s^n in the perturbative expression for the form factor must be of the form

$$\Gamma^{(n)}\left[\frac{q^2}{\mu^2},\epsilon\right] = \sum_{m=1}^{n}\Gamma_m^{(n)}(\epsilon)\left[\frac{\mu^2}{-q^2}\right]^{m\epsilon}\ , \tag{2.16}$$

where each term is proportional to at least one power of $(\mu^2/(-q^2))^\epsilon$. Then, since $\epsilon<0$,

$$\Gamma^{(n)}(q^2=0)=0\quad(n>0) \tag{2.17}$$

and

$$\Gamma(q^2=0)=1\ . \tag{2.18}$$

Another important consequence, which follows from Eqs. (2.1) and (2.16), is

$$\lim_{q^2\to 0}\left[K(\epsilon,\alpha_s)+G\left[\frac{q^2}{\mu^2}\right]\right]=0 \tag{2.19}$$

as a power of q^2 for $\epsilon<0$, at least in perturbation theory.

We can now use Eq. (2.18) as a boundary condition for the evolution equation (2.1), which then has the explicit solution

$$\Gamma\left[\frac{q^2}{\mu^2},\epsilon\right] = \exp\left\{\frac{1}{2}\int_0^{q^2}\frac{d\eta^2}{\eta^2}\left[K(\epsilon,\alpha_s(\mu))+G\left[\frac{\eta^2}{\mu^2},\alpha_s(\mu),\epsilon\right]\right]\right\}$$
$$= \exp\left\{\frac{1}{2}\int_0^{-q^2}\frac{d\xi^2}{\xi^2}\left[K(\epsilon,\alpha_s(\mu))+G\left[-1,\bar{\alpha}\left[\frac{\xi^2}{\mu^2},\alpha_s(\mu),\epsilon\right],\epsilon\right]+\frac{1}{2}\int_{\xi^2}^{\mu^2}\frac{d\mu'^2}{\mu'^2}\gamma_K\left[\bar{\alpha}\left[\frac{\mu'^2}{\mu^2},\alpha_s(\mu),\epsilon\right]\right]\right]\right\}\ . \tag{2.20}$$

In the second expression we have used Eq. (2.14) for G and have changed variables to $\xi^2=-\eta^2$. Because of Eq. (2.19), the exponent vanishes as $q^2\to 0$, and Eq. (2.18) is satisfied. The apparent logarithmic divergence at $\xi\to 0$ from K, which is independent of ξ, cancels against the ξ-independent pole parts from the integration of γ_K over μ'. The ξ^2 integral is defined order by order in perturbation theory, and the effective couplings should be thought of as expanded in powers of $\alpha_s(\mu)$. The series generated in this way is at best asymptotic, because the resummed effective couplings will diverge at $\xi^2=\Lambda_{\rm QCD}^2$.[17]

III. THE RATIO OF THE TIMELIKE TO SPACELIKE FORM FACTOR

Equation (2.20) clearly displays the q^2 dependence of the form factor, so that it is easy to write an expression for the ratio $\Gamma(q^2)/\Gamma(-q^2)$, with $q^2\equiv Q^2>0$. It is given by a contour integral extending from $+Q^2$ to $-Q^2$ in the ξ^2 plane. The contour runs above the branch cut from $\xi^2=0$ to $-\infty$, but is fixed to go through the singularity at $\xi^2=0$, which is integrable in dimensional regularization.

If we close the contour with a semicircle of radius Q^2 in the upper ξ^2 plane, we can shrink the resulting close curve to a small circle at its only fixed point, the origin, where the integral vanishes because the singularity there is integrable. Therefore the ratio of form factors can be written as

$$\ln\frac{\Gamma(Q^2)}{\Gamma(-Q^2)} = \frac{1}{2}\int_C\frac{d\xi^2}{\xi^2}\left[K(\epsilon,\alpha_s(\mu))+G\left[-1,\bar{\alpha}\left[\frac{\xi^2}{\mu^2},\alpha_s(\mu),\epsilon\right],\epsilon\right]+\frac{1}{2}\int_{\xi^2}^{\mu^2}\frac{d\mu'^2}{\mu'^2}\gamma_K\left[\bar{\alpha}\left[\frac{\mu'^2}{\mu^2},\alpha_s(\mu),\epsilon\right]\right]\right]\ , \tag{3.1}$$

where C is the semicircle $\xi^2=Q^2e^{i\theta}$, with $0<\theta<\pi$. We can then change variables to θ, and set $\mu=Q$ in $G+K$. (Recall that $G+K$ is independent of μ, Eq. (2.8). Noting that $K(\epsilon,\alpha(Q))$ is independent of θ, we find our fundamental result

$$\ln\frac{\Gamma(Q^2)}{\Gamma(-Q^2)}=i\frac{\pi}{2}K(\epsilon,\alpha_s(Q))+\frac{i}{2}\int_0^\pi d\theta\left[G\left[-1,\bar{\alpha}(e^{i\theta},\alpha_s(Q),\epsilon),\epsilon\right]-\frac{i}{2}\int_0^\theta d\phi\,\gamma_K\left[\bar{\alpha}(e^{i\phi},\alpha_s(Q),\epsilon)\right]\right] . \tag{3.2}$$

This expression gives the ratio of the timelike to the spacelike form factor as an explicit sum of a divergent phase, given entirely by $K(\epsilon,\alpha(Q))$, and a term which is manifestly finite (although complex) as $\epsilon\to 0$. As in Eq. (2.13), higher orders in $\alpha_s(Q)$ are influenced by lower orders through expansions of the effective coupling. For example, the recursion relation (2.13) can be solved using the one-loop β function (2.10), to give

$$K_n^{(m)}=\frac{1}{2m}\left[-\frac{b_0}{4}\right]^{n-1}\gamma_K^{(m-n+1)} . \tag{3.3}$$

Thus at this level $\gamma_K^{(1)}$ determines all $K_p^{(q)}$ with $p=q$, i.e., the most singular terms, $\gamma_K^{(2)}$ those with $p=q-1$, and so on. In particular, the resummation of the leading $1/\epsilon$ singularities [all the terms proportional to $(\alpha_s/\epsilon)^n$] in the infinite phase is given by

$$\exp\left[i\frac{\pi}{2}\sum_{m=1}^{\infty}\left[\frac{\alpha_s}{\pi}\right]^m\left[-\frac{b_0}{4}\right]^{m-1}\frac{\gamma_K^{(1)}}{2m}\frac{1}{\epsilon^m}\right] . \tag{3.4}$$

Similarly, the finite contributions to both real and imaginary parts may be derived from G and γ_K, expanded to the available order in the effective coupling. We use the expansion

$$G(-1,\bar{\alpha},\epsilon=0)=\frac{\bar{\alpha}}{\pi}G^{(1)}+\left[\frac{\bar{\alpha}}{\pi}\right]^2G^{(2)}+O(\bar{\alpha}^3) , \tag{3.5}$$

with the effective coupling

$$\bar{\alpha}(e^{i\theta},\alpha_s(Q),\epsilon=0)=\frac{\alpha_s(Q)}{1+i\theta b_0\alpha(Q)/4\pi} , \tag{3.6}$$

and define for the sake of simplicity

$$\nu\equiv 1+\frac{i}{4}b_0\alpha(Q) . \tag{3.7}$$

Then we get the following simple expression for the full two-loop exponent in the ratio:

$$\begin{aligned}\ln\frac{\Gamma(Q^2)}{\Gamma(-Q^2)}=\frac{i\pi}{2}\Bigg[\frac{\alpha_s(Q)}{\pi}K^{(1)}+\left[\frac{\alpha_s(Q)}{\pi}\right]^2K^{(2)}\Bigg[&\\ +\frac{2G^{(1)}}{b_0}\ln\nu+\frac{2G^{(2)}}{b_0}\frac{\alpha_s(Q)}{\pi}\left[1-\frac{1}{\nu}\right]&\\ -\frac{4\pi}{b_0^2}\frac{\gamma_K^{(1)}}{\alpha_s(Q)}(\nu\ln\nu-\nu+1)&\\ +\frac{4\gamma_K^{(2)}}{b_0^2}(\ln\nu-\nu+1) .&\end{aligned} \tag{3.8}$$

Note that for small ν, $\ln\nu\sim\nu-1$ is purely imaginary.

IV. COMPARISON WITH TWO-LOOP CALCULATIONS

The renormalized two-loop Sudakov form factor of QCD has been calculated in Refs. 9 and 16. The result[9] is of the form of Eq. (2.16), with $n=2$ and coefficients

$$\begin{aligned}\Gamma_2^{(2)}(\epsilon)=\frac{C_F^2}{16}&\left[\frac{2}{\epsilon^4}+\frac{6}{\epsilon^3}+\frac{1}{\epsilon^2}[\tfrac{41}{2}-2\zeta(2)]+\frac{1}{\epsilon}[\tfrac{221}{4}-\tfrac{64}{3}\zeta(2)]+[\tfrac{1151}{8}+\tfrac{17}{2}\zeta(2)-58\zeta(3)-13\zeta^2(2)]\right]\\ &+C_AC_F\left[-\frac{11}{6}\frac{1}{\epsilon^3}+\frac{1}{\epsilon^2}[\zeta(2)-\tfrac{83}{9}]+\frac{1}{\epsilon}[13\zeta(3)-\tfrac{11}{6}\zeta(2)-\tfrac{4129}{108}]+[\tfrac{44}{5}\zeta^2(2)+\tfrac{467}{9}\zeta(3)-\tfrac{119}{9}\zeta(2)-\tfrac{89173}{648}]\right]\\ &+n_fC_F\left[\frac{1}{3}\frac{1}{\epsilon^3}+\frac{14}{9}\frac{1}{\epsilon^2}+\frac{1}{\epsilon}[\tfrac{1}{3}\zeta(2)+\tfrac{353}{54}]+[-\tfrac{26}{9}\zeta(3)+\tfrac{14}{9}\zeta(2)+\tfrac{7541}{324}]\right]\end{aligned} \tag{4.1}$$

and

$$\Gamma_1^{(2)}(\epsilon)=\frac{C_Fb_0}{16}\left[\frac{2}{\epsilon^3}+\frac{3}{\epsilon^2}+\frac{1}{\epsilon}[8-\zeta(2)]+16-\tfrac{3}{2}\zeta(2)-\tfrac{14}{3}\zeta(3)\right] . \tag{4.2}$$

The former is the sum of all two-loop diagrams directly, the latter the sum of one-loop diagrams dressed by one-loop $\overline{\text{MS}}$ counterterms.

These expressions allow us to derive the two-loop contributions to K and G in Eq. (2.1), from the relation

$$\begin{aligned}K^{(2)}+G^{(2)}=-&\left[\frac{\mu^2}{-q^2}\right]^{2\epsilon}\tfrac{1}{4}\epsilon\Gamma_2^{(2)}-\left[\frac{\mu^2}{q^2}\right]^{\epsilon}\tfrac{1}{8}\epsilon\Gamma_1^{(2)}\\ &-(K^{(1)}+G^{(1)})\tfrac{1}{4}\epsilon\Gamma^{(1)} ,\end{aligned} \tag{4.3}$$

which follows by equating the expansion of the right-hand side of Eq. (2.1) to the derivative with respect to $\ln q^2$ of Γ, using Eq. (2.16). $K^{(2)}$ and $G^{(2)}$ can be read off from the right-hand side of Eq. (4.3), which may be evaluated from Eqs. (2.5), (2.6), (4.1), and (4.2). Note that to determine $G^{(2)}$ to order ϵ^n, it is necessary to expand $G^{(1)}$ to order ϵ^{n+1}, since $\epsilon\Gamma_1^{(1)}=O(1/\epsilon)$. Of interest to us here, however, is only the explicit form of $K^{(2)}$, which is given by

$$K^{(2)}=\frac{K_2{}^{(2)}}{\epsilon^2}+\frac{K_1{}^{(2)}}{\epsilon} \quad (4.4)$$

with

$$K_2{}^{(2)}=-\tfrac{1}{8}C_F b_0=-\tfrac{1}{16}b_0\gamma_K{}^{(1)}\ , \quad (4.5)$$

as expected from Eq. (3.3). Similarly, we have

$$K_1{}^{(2)}=\frac{\gamma_k{}^{(2)}}{4} \quad (4.6)$$

with $\gamma_K^{(2)}$ the two-loop anomalous dimension computed first by Kodaira and Trentedue:[21,14]

$$\gamma_K^{(2)}=C_A C_F[\tfrac{67}{18}-\zeta(2)]+n_f C_F(-\tfrac{5}{9})\ . \quad (4.7)$$

It is straightforward to verify that all of the two-loop pole terms in $\Gamma(Q^2)/\Gamma(-Q^2)$ may be derived from $K^{(2)}$ and cross terms between $K^{(1)}$ and $\gamma_K{}^{(1)}$ or $G^{(1)}$, according to the expansion of Eq. (3.8).

Of more physical interest is the ratio of the absolute values $|\Gamma(Q^2)/\Gamma(-Q^2)|$, which, by Eq. (3.2) is (as claimed above) free of infrared poles to all orders. To compute this quantity, it is only necessary to know that $\epsilon\to 0$ limits of the $G^{(n)}$, while for $\Gamma(Q^2)$ itself, all powers of ϵ in the $G^{(n)}$ contribute, through cross terms with $1/\epsilon$ poles in K.

The squared ratio of absolute values may be calculated directly, using Eqs. (2.5), (4.1), and (4.2), which gives

$$\left|\frac{\Gamma(Q^2)}{\Gamma(-Q^2)}\right|^2=1+\frac{\alpha_s(Q)}{2\pi}\pi^2 C_F+\left[\frac{\alpha_s(Q)}{4\pi}\right]^2[2\pi^4 C_F^2+\pi^2(\tfrac{233}{9}-\tfrac{2}{3}\pi^2)C_A C_F-\tfrac{38}{9}\pi^2 n_f C_F]\ . \quad (4.8)$$

Alternately, we may expand the absolute value squared of Eq. (3.8):

$$\left|\frac{\Gamma(Q^2)}{\Gamma(-Q^2)}\right|^2=1+\frac{\alpha_s(Q)}{2\pi}3\zeta(2)\gamma_K^{(1)}+\left[\frac{\alpha_s(Q)}{4\pi}\right]^2[18\zeta^2(2)\gamma_K^{(1)^2}+12\zeta(2)b_0 G^{(1)}+24\zeta(2)\gamma_K^{(2)}]\ . \quad (4.9)$$

By substituting Eqs. (2.7), (2.11), and (4.7) in (4.9), we easily check that Eqs. (4.8) and (4.9) agree.

The ratio $|\Gamma(Q^2)/\Gamma(-Q^2)|^2$, which enters directly into the Drell-Yan cross section, gives a typical example of large corrections in an infrared stable, and hence nominally perturbatively calculable, quantity. Suppose we take for the purpose of illustration $n_f=3$, for which $b_0=9$ in color SU(3). Then the numerical values of the coefficients in Eq. (4.8) are

$$\left|\frac{\Gamma(Q^2)}{\Gamma(-Q^2)}\right|^2=1+\left[\frac{\alpha}{\pi}\right](6.6)+\left[\frac{\alpha}{\pi}\right]^2(58.9)\ , \quad (4.10)$$

which hardly has a convergent look. On the other hand, referring to Eq. (4.9), we see that the coefficient of $(\alpha/\pi)^2$ is made up of three terms, two of which follow from using one-loop results only in the exponential form of Eq. (3.8). The only "intrinsically two-loop" contribution is the one proportional to $\gamma_K^{(2)}$. Numerically, it makes up 15.0 of the 58.9 in Eq. (4.10), a value not too dissimilar from the one-loop coefficient. The size of truly two-loop effects in the ratio squared is thus moderate for small (α/π). We note the importance of the $b_0 G^{(1)}$ term in Eq. (4.9), which is due to the running of the coupling. This is consistent with the results of Ref. 18 for large corrections in Drell-Yan cross sections. The combination of the conclusions of this paper with those of Ref. 18 is quite straightforward.[19]

We conclude by reemphasizing that the choice of $\overline{\rm MS}$ subtraction to define the function K in Eq. (2.1) is for convenience only. Other choices will lead to different expressions for $\gamma_K^{(2)}$, and hence to better or worse numerical reduction of two-loop corrections by exponentiated one-loop results. We believe, however, that the example of $\overline{\rm MS}$ is adequate to illustrate the value of resummation techniques in understanding large higher-order corrections.

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[1]V. V. Sudakov, Zh. Eksp. Teor. Fiz., **30**, 87 (1956) [Sov. Phys. JETP **3**, 65 (1956)].

[2]G. Sterman, Phys. Rev. D **17**, 2773 (1978); **17**, 2789 (1978).

[3]Yu. L. Dokshitzer, D. I. Dyakanov, and S. I. Troyan, Phys. Rep. **58**, 271 (1980).

[4]G. Parisi and R. Petronzio, Nucl. Phys. **B154**, 427 (1979); G. Curci, M. Greco, and R. Srivastava, *ibid.* **B159**, 451 (1979).

[5]J. C. Collins and D. E. Soper, Nucl. Phys. **B193**, 381 (1981); **B213**, 545(E) (1983).

[6]P. V. Landshoff and D. J. Pritchard, Z. Phys. C **6**, 69 (1980); G. P. Lepage and S. J. Brodsky, Phys. Rev. D **22**, 2157 (1980); J. Botts and G. Sterman, Nucl. Phys. **B325**, 62 (1989).

[7]G. Altarelli, R. K. Ellis, and G. Martinelli, Nucl. Phys. **B157**, 461 (1979); J. Kubar-Andre and F. E. Paige, Phys. Rev. D **19**, 221 (1979); K. Harada, T. Kaneko, and N. Sakai, Nucl. Phys. **B155**, 169 (1979).

[8]G. Parisi, Phys. Lett. **90B**, 295 (1980); G. Curci and M. Greco, *ibid.* **92B**, 175 (1980).

[9]T. Matsuura, S. C. van der Marck, and W. L. van Neerven, Nucl. Phys. **B319**, 570 (1989).

[10]G. Sterman, Nucl. Phys. **B281**, 310 (1987).

[11]P. M. Fishbane and J. D. Sullivan, Phys. Rev. D **4**, 458 (1971);

J. M. Cornwall and G. Tiktopoulos, *ibid.* **13**, 3370 (1976); J. Frenkel and J. C. Taylor, Nucl. Phys. **B116**, 185 (1976); E. C. Poggio and G. Pollack, Phys. Lett. **71B**, 135 (1977).

[12]A. H. Mueller, Phys. Rev. D **20**, 2037 (1979); J. C. Collins, *ibid.* **22**, 1478 (1980).

[13]A. Sen, Phys. Rev. D **24**, 3281 (1981).

[14]G. P. Korchemsky and A. V. Radyushkin, Nucl. Phys. **B283**, 342 (1987); G. P. Korchemsky, Phys. Lett. B **220**, 629 (1989).

[15]J. C. Collins in *Perturbative QCD,* edited by A. H. Mueller, Advanced Series on Directions in High Energy Physics, Vol. 5 (World Scientific, Singapore, 1989).

[16]G. Kramer and B. Lampe, Z. Phys. C **34**, 497 (1987); R. J. Gonsalves, Phys. Rev. D **28**, 1542 (1983).

[17]D. Amati, A. Bassetto, M. Ciafaloni, G. Marchesini, and G. Veneziano, Nucl. Phys. **B173**, 429 (1980); M. Ciafaloni, Phys. Lett. **95B**, 113 (1980); M. Ciafaloni and G. Curci, *ibid.* **102B**, 352 (1981); L. V. Gribov, E. M. Levin, and M. G. Ryskin, Phys. Rep. **100**, 1 (1983); P. Chiapetta, T. Grandou, M. Le Bellac, and J. L. Meunier, Nucl. Phys. **B207**, 251 (1982); B. R. Webber, Annu. Rev. Nucl. Part. Sci. **36**, 253 (1986); Yu. L. Dokshitzer, V. A. Khoze, A. H. Mueller, and S. I. Troyan, Rev. Mod. Phys. **60**, 373 (1988); M. Ciafaloni, Nucl. Phys. **B296**, 49 (1988); D. Appel, P. Mackenzie, and G. Sterman, *ibid.* **B309**, 259 (1988); S. Catani and L. Trentedue, *ibid.* **B327**, 323 (1989); P. Aurenche, R. Baier, and M. Fontanez, Fermilab Report No. FERMILAB-PUB-90/27-T (1990).

[18]L. Magnea, Stony Brook Report No. ITP-SB-90-28 (1990).

[19]L. Magnea and G. Sterman, presented at the Workshop on Hadron Structure Functions and Parton Distributions, Fermilab, Batavia, Illinois, 1990 (unpublished).

[20]G.'t Hooft, Nucl. Phys. **B61**, 455 (1973); J. C. Collins, *Renormalization* (Cambridge University Press, Cambridge, England, 1984).

[21]T. Kodaira and L. Trentedue, Phys. Lett. **112B**, 66 (1982); C. T. H. Davies and W. J. Stirling, Nucl. Phys. **B244**, 337 (1984).