Baryon-number dissipation at finite temperature in the standard model

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We analyze the phenomenon of baryon-number violation at finite temperature in the standard model, and derive the relaxation rate for the baryon density in the high-temperature electroweak plasma. The relaxation rate γ is given in terms of real-time correlation functions of the operator $\mathbf{E}\cdot\mathbf{B}$, and is directly proportional to the sphaleron transition rate Γ: $\gamma \propto n_f \Gamma/T^3$. Hence it is *not* instanton suppressed, as claimed by Cohen, Dugan, and Manohar (CDM). We show explicitly how this result is consistent with the methods of CDM, once it is recognized that a new anomalous commutator is required in their approach.

I. INTRODUCTION

Baryon (and lepton) number is not conserved in the standard $SU(2)_L \times U(1)$ electroweak theory. This derives from the fact that the pure SU(2) vacuum is a periodic structure labeled by an integer Chern-Simons winding number:

$$N_{CS} = \frac{g^2}{16\pi^2} \int d^3\mathbf{x}\, \epsilon_{ijk} \left[A_i^a \partial_j A_k^a - \frac{g}{3} \epsilon_{abc} A_i^a A_j^b A_k^c \right] . \tag{1}$$

In order to change from a vacuum configuration with one integer value of N_{CS} to that with another integer value, it is necessary to pass through nonvacuum, i.e., *finite-energy* field configurations: Fig. 1. The height of the potential barrier between adjacent vacua is given by the energy of a certain static solution of the coupled Yang-Mills-Higgs classical field equations, called a sphaleron. In the Weinberg-Salam theory this energy barrier is of order M_W/α_W, or 7 to 10 TeV.[1]

Necessarily associated with the twisting of the gauge field from one vacuum state to another is the violation of chiral fermion number through the chiral anomaly. Because of (maximal) parity violation, the chiral anomaly becomes an anomaly in the lepton- and baryon-number currents as well:

$$\partial_\mu b^\mu = \partial_\mu l^\mu = \frac{n_f}{32\pi^2}(-2g^2 F_{\mu\nu}^a \tilde{F}^{a\mu\nu} + g'^2 F'_{\mu\nu} \tilde{F}'^{\mu\nu}) . \tag{2}$$

Here $F_{\mu\nu}^a$ and $F'_{\mu\nu}$ are the field strength tensors for the $SU(2)_L$ and U(1) hypercharge gauge fields of the Weinberg-Salam theory, g and g' are the corresponding coupling constants, and n_f is the number of sequential generations (families) of quarks and leptons. Since $F_{\mu\nu}\tilde{F}^{\mu\nu}$ may be expressed as the total divergence of a four-vector whose time component is just that appearing in the definition of N_{CS}, the nonconservation of $B+L$ is related to the change in N_{CS} of the $SU(2)_L$ gauge vacuum. Because the anomalies in the baryon and lepton currents are identical, the difference $B-L$ is exactly conserved in the standard model.

At temperatures and energies low compared to 10 TeV, such transitions and concomitant $B+L$ violation are very severely suppressed. 't Hooft showed that instanton-induced baryon-number-violating processes involving 12 fermions (for $n_f=3$) are suppressed by a factor of

$$\exp(-4\pi \sin^2\theta_W/\alpha) \sim 10^{-170} , \tag{3}$$

and hence are entirely negligible at zero temperature.[2]

At high temperatures, the situation is quite different. Because the energy barrier represented by the classical sphaleron solution is finite, the rate of classical real-time thermal transitions changing N_{CS} and, therefore, $B+L$ has no such exponential suppression in electroweak theory.[3–7] The rate of such B- and L-violating processes has been computed in the Weinberg-Salam theory by semiclassical methods for the temperature range $M_W(T) \ll T \ll M_W(T)/\alpha_W$.[4,5] At temperatures greater than $M_W(T)/\alpha_W$, the semiclassical analysis fails because perturbation theory around the zero-temperature ground state is unreliable. The failure of the semiclassical approximation for the rate does not mean that the rate is

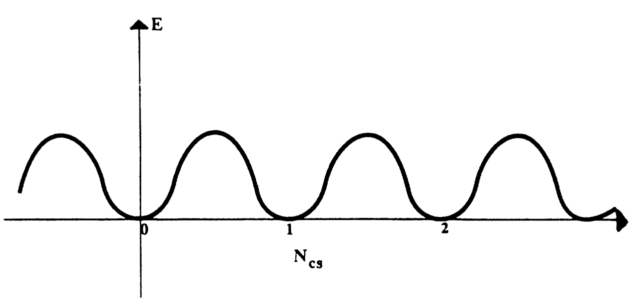

FIG. 1. The periodic vacuum structure of non-Abelian gauge theory in the absence of fermions.

small. Though this might seem paradoxical from the point of view of instanton methods,[8] it is borne out by calculations (both analytic and numerical) in two-dimensional models.[5,9,10] It is also possible to argue from general properties of scaling in the high-temperature phase that the rate of such transitions per unit volume is of order $\alpha_W^4 T^4$.[4,11]

Another line of objection to unsuppressed fermion number violation in the electroweak theory has been raised by Cohen, Dugan, and Manohar[12] (CDM). These authors have argued that the rate of *dissipation* for any $B+L$ asymmetry remains exponentially small, even though the rate of *fluctuations* of N_{CS} is large at high temperatures. Although such a situation would be contrary to quite general statistical considerations which relate fluctuation rates to relaxation processes,[13] the critical role of a quantum anomaly in this case has generated some degree of confusion and controversy.

The resolution of this controversy is important for cosmology. Since the seminal work of Sakharov[14] it has been recognized that the observed baryon number of the Universe could be produced by out-of-equilibrium reactions which simultaneously violate baryon number, charge conjugation, and *CP*. Moreover, the baryon-number-violating reactions must turn off (i.e., become insignificant) before the system returns to thermal equilibrium; otherwise any baryon density produced will relax to its equilibrium value, namely, zero. A high rate of electroweak $B+L$ nonconservation at $T>M_W$ therefore carries with it the implication that any preexisting $B+L$ asymmetry would be eliminated by the time of the electroweak phase transition.[3,4,15] Thus, in order to obtain the observed baryon number either $B-L\neq 0$ at temperatures much greater than M_W, or baryogenesis must occur at the time of the electroweak phase transition.[16] This is a strong constraint on any theory of baryogenesis, and excludes some grand unified models [such as the minimal SU(5) model] for generating the observed baryon excess in the Universe, quite apart from the bounds provided by recent proton decay searches.

Khlebnikov and Shaposhnikov[11] (KS) used a well-defined formalism to evaluate the nonequilibrium dynamics of relaxation, and found a large relaxation rate at high temperatures. However, they did not explicitly evaluate *fermionic* quantities, which is at the heart of the CDM objection. In this paper we redo the calculation of KS with fermions, and in the process obtain a closed form relation between the baryon-number relaxation rate and the (sphaleron) transition rate. This relation is quite general, and *independent* of any sphaleron approximation, in accordance with general fluctuation-dissipation considerations. The expression (23) for the rate in terms of a certain spectral density function may provide for techniques of evaluation quite different from sphaleron methods.

Finally, we reexamine the analysis of CDM, and show how the methods of those authors may be used to achieve the *same* result. The new ingredient in our reanalysis of CDM is an anomalous commutator between baryon number and $\mathbf{E}\cdot\mathbf{B}$, neglected in CDM, but required for consistency with the usual anomaly. Since these several different viewpoints all lead to the same conclusion, there ought to be no further controversy about unsuppressed electroweak B and L violation at high temperature and its implication(s) for early Universe cosmology.

II. THE BARYON- AND LEPTON-NUMBER RELAXATION RATE

Consider the standard electroweak theory at temperatures above M_W. In our discussion we neglect the contribution of the weak hypercharge to the baryon-number anomaly. This is done for simplicity of notation. Inclusion of the hypercharge contribution would not change any of our conclusions. Let us assume that all of the dynamical variables of the system are in thermal equilibrium, except two: the baryon and lepton numbers N_B and N_L, which have been driven out of equilibrium by a small amount due to some unspecified process. The initial condition for our problem then is $\langle N_B(t=0)\rangle\neq 0, \langle N_L(t=0)\rangle\neq 0$, and we wish to calculate the relaxation rate γ for B and L to return to their equilibrium value. In statistical mechanics, the time development of the dynamical variable $\dot N_B \equiv dN_B/dt$ is given in terms of the statistical average $\langle \dot N_B\rangle \equiv \mathrm{Tr}(\dot N_B \rho)/Z$ where $\rho(t)$ is the nonequilibrium statistical operator satisfying the quantum Liouville equation

$$\dot\rho + i[H,\rho]=0 , \tag{4}$$

and $Z=\mathrm{Tr}\rho$. Zubarev has shown that the operator

$$\rho=\exp\left[-\beta H+\epsilon\beta\int_{-\infty}^{t} e^{\epsilon(t'-t)}[\mu_B(t')N_B(t')+\mu_L(t')N_L(t')]dt'\right], \quad \epsilon\to 0^+$$
$$\equiv \exp\{-\beta[H+h(t)]\} , \tag{5}$$

satisfies the Liouville equation in the limit $\epsilon\to 0^+$ and should be a good approximation in the case that only a few dynamical variables are out of equilibrium.[17]

Now, the number operators satisfy the anomalous equations of motion,

$$\dot N_B=\dot N_L=-n_f\int d^3\mathbf{x}\, q(t,\mathbf{x})\equiv -n_f\frac{\alpha_W}{2\pi}\int d^3\mathbf{x}\,\mathbf{E}^a\cdot\mathbf{B}^a , \tag{6}$$

where $\mathbf{E}^a$ and $\mathbf{B}^a$ are the $SU(2)_L$ electroweak electric and magnetic field strengths. In terms of the Chern-Simons charge N_{CS}, we have

$$\dot N_B=\dot N_L=+n_f\dot N_{CS} . \tag{7}$$

Following KS we evaluate now ρ/Z to first order in h:

$$\frac{\rho}{Z}=\left[1+\beta\left\langle\int_0^1 d\lambda\, e^{-\beta H\lambda}he^{\beta H\lambda}\right\rangle_0 -\beta\int_0^1 d\lambda\, e^{-\beta H\lambda}he^{\beta H\lambda}\right]\frac{\rho_0}{Z_0}\ , \tag{8}$$

where

$$h=-\mu_B(t)N_B(t)+\int_{-\infty}^{t}e^{\epsilon(t'-t)}[\dot{\mu}_B(t')N_B(t')+\mu_B(t')\dot{N}_B(t')]dt' +(B\leftrightarrow L)\ , \tag{9}$$

and the zero subscript denotes the equilibrium statistical operator with $h=0$.

Let us calculate first the average baryon number to this order. We find

$$\langle N_B(t)\rangle=-\beta\int_0^1 d\lambda\langle N_B(t)e^{-\beta H\lambda}h(t)e^{\beta H\lambda}\rangle_0\ , \tag{10}$$

where we have used $\langle N_B\rangle_0$. Substituting the previous expression for h, we find that the term involving $\dot{N}_B$ vanishes by the time-reversal invariance of ρ in the limit $\epsilon\to 0^+$. Ignoring the term involving $\dot{\mu}_B$ for the moment, we obtain

$$\langle N_B(t)\rangle=\beta\mu_B(t)\int_0^1 d\lambda\langle N_B(0)e^{-\beta H\lambda}N_B(0)e^{\beta H\lambda}\rangle_0 \to\beta\mu_B(t)\langle N_B^2(0)\rangle_0\ , \tag{11}$$

where the last expression is valid in the high-temperature or weak-coupling (classical) limit. An exactly analogous expression holds for $\langle N_L(t)\rangle$. The term involving $\dot{\mu}_B$ is negligible, provided that the relaxation rate γ is slow compared to typical correlation times in the system. This approximation is justified *a posteriori* by Eq. (32) below.

In a similar manner we may compute

$$\frac{d}{dt}\langle N_B(t)\rangle=-\beta\int_0^1 d\lambda\int_{-\infty}^{t}dt'e^{\epsilon(t'-t)}[\mu_B(t')+\mu_L(t')] \times\langle\dot{N}_B(t)e^{-\beta H\lambda}\dot{N}_B(t')e^{\beta H\lambda}\rangle_0\ , \tag{12}$$

using (7),

$$\frac{d}{dt}\langle N_B\rangle_0=0\ ,$$

and

$$\langle\dot{N}_B(t)\textstyle\int_0^1 d\lambda\, e^{-\beta H\lambda}N_B(t)e^{\beta H\lambda}\rangle_0=0$$

by the time-reversal invariance of the equilibrium state. Since $\dot{\mu}_{B,L}$ is negligible, we may replace $\mu_{B,L}(t')$ by $\mu_{B,L}(t)$ in the above expression and remove them from the integral. Then using the previous results for $\langle N_B(t)\rangle$ and $\langle N_L(t)\rangle$, we may eliminate the chemical potentials from (12) entirely, to arrive at

$$\frac{d}{dt}\langle N_B\rangle=\frac{d}{dt}\langle N_L\rangle=-K\left[\frac{\langle N_B\rangle}{\langle N_B^2(0)\rangle_0}+\frac{\langle N_L\rangle}{\langle N_L^2(0)\rangle_0}\right], \tag{13}$$

where

$$K\equiv\int_{-\infty}^{t}dt'e^{\epsilon(t'-t)}\int_0^1 d\lambda\langle\dot{N}_B(t)e^{-\beta H\lambda}\dot{N}_B(t')e^{\beta H\lambda}\rangle_0\ . \tag{14}$$

This derivation exactly parallels that of KS,[11] who derive the equivalent result for N_{CS} instead of the fermionic operator N_B. We now depart from those authors by expressing K and the high-temperature (sphaleron) transition rate Γ in terms of the *same* spectral function, thereby allowing us to find a direct relation between the two, independently of any specific approximation scheme.

To this end let us introduce the retarded response function

$$G_R(t-t',\mathbf{x}-\mathbf{x}')\equiv-i\theta(t-t')\langle[q(t,\mathbf{x}),q(t',\mathbf{x}')]\rangle_0 =\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int\frac{d^3\mathbf{k}}{(2\pi)^3}e^{-i\omega(t-t')} \times e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\tilde{G}_R(\omega,\mathbf{k})\ , \tag{15}$$

whose Fourier transform $\tilde{G}_R$ is analytic in the upper half complex ω plane:

$$\tilde{G}_R(\omega,\mathbf{k})=\int_{-\infty}^{+\infty}d\omega'\frac{\rho(\omega',\mathbf{k})}{\omega-\omega'+i\epsilon}\ . \tag{16}$$

The spectral density ρ (not to be confused with the density matrix of which we have no further use) is determined by the matrix elements of the topological charge density:

$$\rho(\omega,\mathbf{k})=\frac{(2\pi)^3}{Z_0}\sum_{n,m}|\langle n|q(0)|m\rangle_0|^2 \times e^{-E_n/T}(1-e^{-(E_m-E_n)/T}) \times\delta(\omega-E_m+E_n)\delta^3(\mathbf{k}-\mathbf{p}_m+\mathbf{p}_n)\ , \tag{17}$$

where the states $|n\rangle$ are a complete set of eigenstates of the full Hamiltonian with energy eigenvalues E_n.

By using the anomaly *operator* equation (7) and substituting the same complete set of intermediate eigenstates, it may be verified in a direct computation that the quantity K of Eq. (14) is given by

$$\begin{aligned}K&=iVn_f^2T\frac{d}{d\omega}\tilde{G}_R(\omega,\mathbf{k})\Big|_{\omega=\mathbf{k}=0}\\ &=-iVn_f^2T\int_{-\infty}^{\infty}\frac{d\omega}{\omega-i\epsilon}\left[\frac{\rho(\omega,\mathbf{0})}{\omega}\right]\\ &=Vn_f^2T\pi\left[\frac{\rho(\omega,\mathbf{0})}{\omega}\right]\Big|_{\omega=0}\\ &=Vn_f^2T\pi\frac{d\rho}{d\omega}\Big|_{\omega=\mathbf{k}=0}\ ,\end{aligned} \tag{18}$$

where we have made use of the fact that

$$\rho(\omega)=\rho_+(\omega)-\rho_-(\omega)=\rho_+(\omega)-\rho_+(-\omega) \tag{19}$$

is explicitly an odd function of ω when $\mathbf{k}=0$.

The quantity $(d\rho/d\omega)|_{\omega=\mathbf{k}=0}$ occurs in a quite different context, as the rate for the (Brownian) diffusion of the topological charge,

$$Q(t)\equiv\int_0^t dt'\int d^3\mathbf{x}\, q(t',\mathbf{x}) \tag{20}$$

in the periodic potential of Fig. 1. For we may calculate

$$\langle Q^2(t)\rangle_0 = 2V\int_0^\infty d\omega\,\rho_+(\omega,\mathbf{0})\frac{\sin^2(\omega t)}{\omega^2}$$
$$\rightarrow 2\pi V t\rho_+(0,\mathbf{0}),\quad t\rightarrow\infty$$
$$= 2\pi V t T\frac{d\rho}{d\omega}\bigg|_{\omega=\mathbf{k}=0} . \tag{21}$$

Since (in the absence of fermions) we define the diffusion coefficient of the random walk in Chern-Simons number by

$$\lim_{t\rightarrow\infty}\langle Q^2(t)\rangle = \lim_{t\rightarrow\infty}\langle[N_{\rm CS}(t)-N_{\rm CS}(0)]^2\rangle$$
$$= 2Vt\Gamma , \tag{22}$$

we have proven that

$$\Gamma = \pi T\frac{d\rho}{d\omega}\bigg|_{\omega=\mathbf{k}=0}$$
$$= \frac{8\pi^3}{Z_0}\sum_{n,m}|\langle n|q(0)|m\rangle|^2$$
$$\times e^{-E_n/T}\delta(E_n-E_m)\delta^3(\mathbf{p}_n-\mathbf{p}_m) , \tag{23}$$

and, therefore,

$$K = Vn_f^2\Gamma . \tag{24}$$

which relates the baryon relaxation rate to the finite-temperature diffusion rate in the absence of fermions. The last two expressions remain valid in the presence of fermions as well, provided only that the baryon-number density is small compared to T^3, which is the same assumption necessary to derive the linear relations of Eqs. (13).

In the previous literature[4,11] Γ is evaluated in the semiclassical method of Langer,[18] which relates it to the sphaleron energy in a semiclassical approximation. Expression (23) furnishes an *a priori* definition of Γ, which may (in principle) be evaluated from knowledge of the spectral density function near $\omega=0$. In practice, this is quite difficult since it involves the long-time behavior of the response function, which cannot be calculated in perturbation theory. Euclidean methods are also of little use since the long time limit is sensitive to any approximation(s) made in Euclidean time, and hence the continuation is generally unreliable. Nevertheless, we believe it is worthwhile to have a definition of the rate that is independent of any approximate method of evaluating it.

To complete the evaluation of the relaxation rate we must calculate the denominators of Eq. (13). If we were dealing with a single species of left handed fermion this would be straightforward in the regime where the temperature is much higher than fermion masses and chemical potential μ. In that case we would simply compute the partition function of a free fermion gas with a single helicity state:

$$\ln Z(\mu) = \frac{2VT^3}{\pi^2}\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n^4}\left[1+\frac{(\mu\beta n)^2}{2}+O(\mu^4)\right] , \tag{25}$$

where μ is the chemical potential for this particle number, whose average is given by

$$\langle N\rangle = T\frac{\partial}{\partial\mu}\ln Z(\mu) = \frac{\mu VT^2}{6} \tag{26}$$

to linear order in μ. The mean square fluctuation of this number is

$$\langle N^2\rangle = \frac{T^2}{Z}\frac{\partial^2}{\partial\mu^2}Z\bigg|_{\mu=0} = T^2\frac{\partial^2}{\partial\mu^2}\ln Z\bigg|_{\mu=0} = \frac{VT^3}{6} , \tag{27}$$

which is consistent with Eqs. (11) and (26).

In the standard model the accounting is a bit different. We must consider both baryon and lepton numbers together, since both are violated by the transition. Since

$$N_B = \tfrac{1}{3}\sum_f\sum_{c=1}^{3}(N^c_{u_f}+N^c_{d_f}) ,$$
$$N_L = \sum_f(N_{l_f}+N_{\nu_f}) , \tag{28}$$

where f labels the family or sequential generation, we have

$$\langle N_B\rangle = n_f\times3\times2\times\frac{1}{3}\times\frac{\mu_B}{3}\times\frac{2VT^2}{6} = \frac{2}{9}n_f\mu_B VT^2 ,$$
$$\langle N_L\rangle = n_f\mu_L(\tfrac{2}{6}+\tfrac{1}{6})VT^2 = \tfrac{1}{2}n_f\mu_L VT^2 , \tag{29}$$

The fluctuations in these quantities are likewise modified to become

$$\langle N_B^2(0)\rangle_0 = \tfrac{2}{9}n_f VT^3,\quad \langle N_L^2(0)\rangle_0 = \tfrac{1}{2}n_f VT^3 , \tag{30}$$

in the high-temperature or weak-coupling limit. Substituting these last relations into the denominators of (13), and using the earlier result for K, Eq. (24) yields the desired expression for the fermion number relaxation rate:

$$\frac{d}{dt}\langle N_B\rangle = \frac{d}{dt}\langle N_L\rangle = -\frac{n_f\Gamma}{T^3}(\tfrac{9}{2}\langle N_B\rangle + 2\langle N_L\rangle) . \tag{31}$$

If we consider initial conditions with $\langle N_B\rangle = \langle N_L\rangle$, or simply consider the relaxation of the linear combination, $\frac{9}{2}\langle N_B\rangle + 2\langle N_L\rangle$, the fermion number relaxation rate becomes

$$\gamma = \tfrac{13}{2}n_f\frac{\Gamma}{T^3} . \tag{32}$$

Equation (32) is the main result of this paper. It shows that the fermion number relaxation rate is directly proportional to the diffusion rate that is calculated by the usual semiclassical sphaleron method. If the latter is unsuppressed, then so is the former.

Actually, there is a simpler, more intuitive way to derive this same result, based on detailed balance.[4] Suppose for $t<0$ constant chemical potentials μ_B and μ_L are added to the Hamiltonian,

$$H \rightarrow H - \mu_B N_B - \mu_L N_L , \tag{33}$$

so that it becomes energetically favorable to create a net baryon and lepton number in the plasma. From the anomaly equation, this means that the periodic potential of Fig. 1 is replaced by a skewed potential near $N_B=0$: Fig. 2. Notice that the minima of Fig. 1 are forced to be degenerate, since all integer N_{CS} are equivalent to each other by a (topologically nontrivial) gauge transformation. Unlike the Chern-Simons number, N_B is *gauge invariant,* so that states of different baryon number may have (and do have) different energies.

For large enough positive N_B the potential of Fig. 2 turns upward once more. This is because of Fermi-Dirac statistics: even if the fermions are treated as massless, it costs energy to create a fermion-antifermion pair with net chirality, since the pair must be created in an unoccupied momentum state. Since the spacing between states (and hence this energy cost) goes to zero in the infinite-volume limit, the value of N_B at which the potential of Fig. 2 begins to turn upward is of order V. Indeed, to linear order in μ_B explicit evaluation of the thermal average in the Fermi-Dirac distribution just yields the results, (29) to linear order in μ_B and μ_L. The mean N_B is shifted to this positive value, so that the larger population of states with $\langle N_B\rangle>0$ diffusing to lower N_B can compensate for the energy bias to the right. Hence, there is detailed balance and

$$\frac{d}{dt}\langle N_B\rangle=\frac{d}{dt}\langle N_L\rangle=-n_f V\langle q(t,\mathbf{0})\rangle=0,\quad t<0\,. \tag{34}$$

Suppose that the external chemical potentials are removed suddenly at $t=0$. Now the large rate of diffusion to the left from the initial overpopulation with positive N_B is no longer balanced by an energy bias to the right. Hence there will be a net decrease of $\langle N_B\rangle$ with time; i.e., the net baryon number will relax to zero. We may calculate the rate of relaxation if we assume that Eqs. (29) continue to hold for $t>0$ as well, effectively defining a *slowly varying* $\mu_B(t)$ and $\mu_L(t)$ in terms of the decreasing baryon and lepton numbers. That is, we assume that the relaxation is slow enough so that the system may be treated as approximately in equilibrium at all times during the relaxation, with an effective time-dependent chemical potential. This adiabaticity assumption permits us to use detailed balance and equate $(d/dt)\langle N_B\rangle$ for $t>0$ to the *negative* of the transition rate to the right with the original skewed Hamiltonian that set up the distribution for $t<0$ in the first place. Therefore,

$$\langle q(t)\rangle=-\frac{d}{dt}\langle N_{CS}\rangle=-(\Gamma_+-\Gamma_-)$$
$$=+n_f(\mu_B+\mu_L)\frac{\Gamma}{T},\quad t>0 \tag{35}$$

since

$$\Gamma_\pm=\Gamma e^{\mp n_f(\mu_B+\mu_L)\beta/2}$$
$$=\Gamma\left[1\mp\frac{n_f(\mu_B+\mu_L)}{2T}+O(\mu^2)\right] \tag{36}$$

to linear order in μ_B and μ_L in the skewed potential. Then we may eliminate $\mu_B+\mu_L$ from Eq. (35) by using Eqs. (6) and (29) to secure

$$\frac{d}{dt}\langle N_B\rangle=\frac{d}{dt}\langle N_L\rangle=-\frac{n_f\Gamma}{T^3}\left(\tfrac{9}{2}\langle N_B\rangle+2\langle N_L\rangle\right), \tag{37}$$

which is the same result for the fermion relaxation rate obtained by the more formal Zubarev approach.

III. CDM ANALYSIS REEXAMINED

CDM also calculate $(d/dt)\langle N_B(t)\rangle$. However, they use a trick to obtain the thermal average in terms of the derivative of a generating function $F(\theta)$, defined in terms of the generalized electroweak Hamiltonian,

$$H(\theta)=\frac{1}{2}\int d^3\mathbf{x}\left[\left[\mathbf{\Pi}+\alpha_W\frac{\theta}{2\pi}\mathbf{B}\right]^2+(\mathbf{B})^2\right]+H_{\text{fermion}}\,, \tag{38}$$

where

$$\mathbf{\Pi}=-\mathbf{E}-\alpha_W\frac{\theta}{2\pi}\mathbf{B} \tag{39}$$

is the momentum conjugate to the gauge field. Define

$$e^{-\beta F(\theta)}\equiv\mathrm{Tr}(e^{-\beta H(\theta)}) \tag{40}$$

such that

$$\left.\frac{\partial F}{\partial\theta}\right|_{\theta=0}=-\left\langle\int d^3\mathbf{x}\,q\right\rangle_0=0\,. \tag{41}$$

In fact, *all* derivatives of $F(\theta)$ vanish because F is independent of θ, as we now demonstrate. In order to do so it is sufficient to show that

$$e^{-iN_B\phi}H(\theta)e^{iN_B\phi}=\exp\left[n_f\phi\frac{\partial}{\partial\theta}\right]H(\theta)=H(\theta+n_f\phi)\,, \tag{42}$$

i.e., that a baryon number phase rotation can be used to rotate the angle θ to zero in the electroweak theory. Expanding (42) in a power series in ϕ gives

$$H(n_f\phi)=H(0)-i\phi[N_B,H(0)]$$
$$-\frac{\phi^2}{2}[N_B,[N_B,H(0)]]+\cdots\,. \tag{43}$$

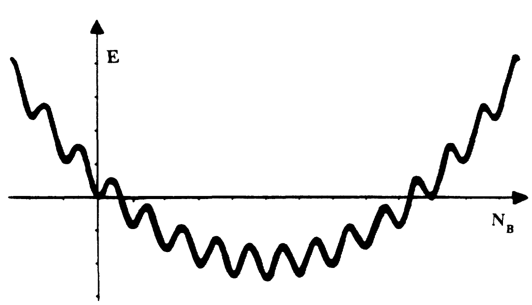

FIG. 2. The potential energy of gauge field plus massless fermion system as a function of fermion number. The potential is concave for large N_B in a finite volume, due to Fermi-Dirac statistics, as explained in the text.

The second term on the right-hand side (RHS) is given by the anomaly Eq. (6):

$$-i\phi[N_B,H(0)]\equiv\phi\dot{N}_B=-n_f\phi\int d^3\mathbf{x}\,q=+n_f\phi\left.\frac{\partial H}{\partial\theta}\right|_{\theta=0}. \tag{44}$$

This verifies the first derivative term of the expansion. Integrating the anomaly relation $\dot{N}_B=n_f\dot{N}_{\rm CS}$ and fixing the gauge by the condition that $N_{\rm CS}=0$ when $N_B=0$ permits us to write the commutator in the third term on the RHS of (43) as

$$\begin{aligned}-[N_B,[N_B,H(0)]]&=-n_f[N_{\rm CS},[N_B,H(0)]]\\&=-n_f\left[N_{\rm CS},-in_f\int d^3\mathbf{x}\,q\right]\\&=+n_f^2\left[\frac{\alpha_W}{2\pi}\right]^2\int d^3\mathbf{x}\,\mathbf{B}^a\cdot\mathbf{B}^a\\&=n_f^2\left.\frac{\partial^2 H}{\partial\theta^2}\right|_{\theta=0}.\end{aligned} \tag{45}$$

This verifies that terms quadratic in ϕ in Eq. (43) are correct. Since $\mathbf{B}^2$ no longer involves the electric field operator, its commutator with $N_{\rm CS}$ and N_B vanishes, as do all the higher-order commutators in the ellipsis, since

$$\left.\frac{\partial^n H}{\partial\theta^n}\right|_{\theta=0}=0,\quad n>2\;. \tag{46}$$

Thus, consistency requires a new anomalous commutator, viz.,

$$\left[N_B,\int d^3\mathbf{x}\,q\right]=-in_f\left[\frac{\alpha_W}{2\pi}\right]^2\int d^3\mathbf{x}\,\mathbf{B}^2\;, \tag{47}$$

in addition to the original anomaly, Eq. (6). If desired, one may verify this new anomalous commutator directly in terms of the canonical commutation relations of the theory, by defining the operator N_B composed of fermion bilinears in terms of a gauge-invariant point-splitting technique. Insertion of the path-ordered exponential of $\int dx^i A_i$ between the fermion operators yields the anomalous commutator (47), which remains after the point splitting has been removed.

Hence Eq. (42) is proven, and indeed we may rotate away the angle θ in Eq. (40), proving that $F(\theta)=F(0)$ is independent of θ. This conclusion is verifiable directly in a Lagrangian path-integral approach. In the Hamiltonian approach consistency requires the anomalous commutator (47). By taking the second derivative of F with respect to θ and using the fact that F is independent of θ, we find[12]

$$\left\langle\int d^3\mathbf{x}\,\mathbf{B}^2\right\rangle_0=\beta\left\langle\int d^3\mathbf{x}\,\mathbf{E}\cdot\mathbf{B}\int d\lambda\,e^{-\beta H\lambda}\times\int d^3\mathbf{x}'\mathbf{E}'\cdot\mathbf{B}'e^{\beta H\lambda}\right\rangle_0. \tag{48}$$

Let us now consider nonequilibrium dynamics. As a trial nonequilibrium statistical operator, CDM consider the local operator

$$\rho_\theta=N\exp\left[-\beta\left[H(\theta)+\sum_k c_kO_k(\theta)\right]\right], \tag{49}$$

where the O_k are arbitrary operators and the c_k are arbitrary small coefficients. Define $F(\theta)$ as before with this new statistical operator $e^{-\beta F(\theta)}\equiv{\rm Tr}(\rho_\theta)$ with $O_k(\theta)=e^{-iN_B(t)\phi}O_ke^{iN_B(t)\phi}$ and $\phi=\theta/n_f$. Then $F(\theta)=F(0)$ as before. Differentiating F with respect to θ once we obtain

$$n_f\left\langle\int d^3x\,q\right\rangle=-i\sum_k c_k\langle[N_B,O_k]\rangle_0\;, \tag{50}$$

to first order in the small parameters c_k.

CDM consider operators satisfying $[N_B,O_k]=n_kO_k$, but they do not consider operators such as $O=\dot{N}_B=-n_f\int d^3\mathbf{x}\,q$. Using the anomalous commutator, Eq. (47), for this single operator, we obtain

$$\frac{d}{dt}\langle N_B\rangle\approx-c\left[\frac{n_f\alpha_W}{2\pi}\right]^2\left\langle\int d^3\mathbf{x}\,\mathbf{B}^2\right\rangle_0. \tag{51}$$

Notice that this estimate for $(d/dt)\langle N_B\rangle$ is *not* small or instanton suppressed. If we replace the nonequilibrium statistical operator of KS and the perturbing Hamiltonian of Zubarev with the *local* term

$$h(t)\approx\frac{\mu_B+\mu_L}{T}\dot{N}_B\equiv cO\;, \tag{52}$$

which is valid in the limit that the autocorrelation function for $\dot{N}_B$ has support only when the time interval is of order T^{-1}, then Eq. (51), obtained by the CDM local operator method, is identical to Eq. (12) of the previous section, since

$$\begin{aligned}K&=\int_{-\infty}^{t}dt'e^{\epsilon(t'-t)}\left\langle\dot{N}_B(t)\int_0^1 d\lambda\,e^{-\beta H\lambda}\dot{N}_B(t')e^{\beta H\lambda}\right\rangle_0\\&\approx\left[\frac{n_f\alpha_W}{2\pi}\right]^2\beta\left\langle\int d^3\mathbf{x}\,\mathbf{E}\cdot\mathbf{B}\int d\lambda\,e^{-\beta H\lambda}\int d^3\mathbf{x}'\,\mathbf{E}'\cdot\mathbf{B}'e^{\beta H\lambda}\right\rangle_0\\&=\left[\frac{n_f\alpha_W}{2\pi}\right]^2\left\langle\int d^3x\,\mathbf{B}^2\right\rangle_0\end{aligned} \tag{53}$$

by Eq. (48).

Thus, the main results of this paper, Eqs. (23) and (24), and (31) and (32) relating the dissipation of fermion number at high temperature to the fluctuation or diffusion rate over the potential barrier are consistent with the methods of CDM, provided account is taken of the anomalous commutator (47). The relation (32) is a reflection of general fluctuation-dissipation theorems, and is a kind of analog to the relation found by Einstein for Brownian motion in a medium.[13] The local approximation of CDM leads to the estimate

$$\Gamma\approx\left[\frac{\alpha_W}{2\pi}\right]^2\langle\mathbf{B}^2\rangle_0 \tag{54}$$

by combining Eqs. (24) and (53). Actually, we might expect the time scale for the correlation function (14) to decay to be of order $(\alpha_W T)^{-1}$ rather than T^{-1}, since the former is the inverse dimensional coupling of the three-dimensional gauge theory appropriate at high temperature. Then the above estimate for Γ should be enhanced by a factor of α_W^{-1} relative to (54). The same dimensional coupling enters the magnetic screening length,[9] so that we should expect

$$\langle \mathbf{B}^2 \rangle_0 \approx \alpha_W^3 T^4 \tag{55}$$

and

$$\Gamma \approx \frac{\alpha_W^4 T^4}{4\pi^2} \tag{56}$$

at high temperature. If the scaling relation $\Gamma \propto \alpha_W^4 T^4$ is correct, then the dissipation rate of baryon number in the hot electroweak plasma is of order $n_f \alpha_W^4 T$, which is much larger than the expansion rate of the Universe at these temperatures, and certainly relevant for cosmology.

In related work, Cline and Raby[19] have derived relations between the high-energy behavior of B-violating inclusive cross sections that imply results for Γ different from this naive scaling behavior.

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