Exact finite-lattice method for two-dimensional gauge-fermion models: Z_2 gauge fermions

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We present a new method for making exact calculations of physical quantities of two-dimensional gauge-fermion models on a finite lattice. The method is applied to the Z_2 gauge-fermion model with a Kogut-Susskind fermion action. The behavior of the zeros of the partition function, Wilson loops, and other quantities is explicitly examined on the basis of the exact calculation on a 6×4 lattice. A novel symmetry of the partition function and the Wilson loop is also discussed briefly.

I. INTRODUCTION

In recent years, the study of dynamical fermions has become an important theme of lattice field theories. Further to enrich lattice field theories as a nonperturbative method of field theories, analytic approaches must be developed as well as numerical ones.

The purpose of this paper is to present a new method which enables us to integrate exactly two-dimensional gauge-fermion models on a finite lattice. To illustrate the method we shall discuss explicitly the two-dimensional Z_2 gauge-fermion model with the Kogut-Susskind fermion action on a 6×4 lattice.

Both numerical and analytical approaches to lattice gauge theory have some analogy to statistical mechanics and successful results were obtained.[1-3] The analogy, however, does not apply to the models which contain Grassmann fermion fields: (Euclidean) classical fermions. For example, it is not clear whether e^{-S} in such models plays the role of a Boltzmann weight. Furthermore, it is not obvious if the partition function keeps its positivity in general.[4] Nevertheless, it is desirable that the general idea of phase transition in statistical mechanics be applied to the models containing the Grassmann fermion fields to help us understand more deeply their phase transitions.

In this paper, we consider the distribution of zeros of the partition function, first studied by Lee and Yang,[5] in the case of a two-dimensional Z_2 gauge-fermion model. In the course of our argument it will become obvious that our expression of the partition function is positive definite for arbitrary values of real parameters. In Z_N spin and gauge models there are some empirically observed regularities; i.e., the zeros seem to fall naturally on arcs although no simple rule exists for the locus of zeros in the complex temperature plane.[6] To our knowledge there has been no such example including fermion fields, and it will be interesting to see whether or not the distribution of the zeros has different features as compared to the spin of the pure gauge systems previously calculated.

In addition to the above, the two-dimensional pure Z_2 gauge theory[7] has interesting properties such as confinement (the area law of the Wilson loop[8]). Further, the model is one of the simplest examples of two-dimensional gauge-fermion models. A detailed study of this basic model will be useful in finding basic ideas and common methods for solving problems of further interesting models such as the lattice Schwinger model and two-dimensional QCD (QCD_2) on a finite lattice.

We will introduce the model and briefly exhibit its novel symmetry in Sec. II. Our method of the fermionic integration of the partition function is presented in Sec. III. Methods of integration of the gauge fields in the partition function and Wilson loop are described in Sec. IV. The results are discussed in Sec. V.

II. THE MODEL

Let us consider a lattice model of the two-dimensional (2D) Z_2 gauge fermions in which the fermion action is of Kogut-Susskind (KS) type.[9] The fermionic part of the action is given by[10-13]

$$S_f=-\sum_{x,\mu}\eta_\mu(x)\frac{\bar{\psi}(x+\hat{\mu})\sigma_\mu(x)\psi(x)-\bar{\psi}(x)\sigma_\mu(x)\psi(x+\hat{\mu})}{2}, \tag{2.1}$$

where the lattice points are denoted as $x=(x_1,x_2)$ or $n=(n_1,n_2)$; $n_\mu=1,2,\ldots,N_\mu$, and $\hat{\mu}$ is a unit lattice vector. $\eta_1(x)=1$ and $\eta_2(x)=(-1)^{n_1}$ are the standard representation of KS fermion phases arising from the Dirac matrices.

The pure gauge part of the action is given by[7,8]

$$S_g=-\beta\sum_{x,\mu\nu}[\sigma_\mu(x)\sigma_\nu(x+\hat{\mu})\sigma_{-\mu}(x+\hat{\mu}+\hat{\nu})\sigma_{-\nu}(x+\hat{\nu})], \tag{2.2}$$

where $\sigma_\mu(x)$ is a Z_2 gauge variable on a link between sites x and $x+\hat{\mu}$. The partition function of the model is defined by

$$Z=\int\prod_{x,\mu}^{N_1N_2}d\sigma_\mu(x)\int\prod_{x}^{N_1N_2}d\psi(x)d\bar{\psi}(x)\exp(-S_f-S_g). \tag{2.3}$$

Here, the notation $\int\prod_{x,\mu}^{N_1N_2}d\sigma_\mu(x)$ means the configuration sum

$$\sum_{\sigma(N_1,N_2)=\pm1} \cdots \sum_{\sigma(n_1,n_2)=\pm1} \cdots \sum_{\sigma(1,1)=\pm1} ,$$

and N_1 and N_2 are the total number of the lattice points along the $\mu=1$ and 2 directions, respectively. We call this configuration sum "integration," because the present method can be generalized into continuous gauge groups.

We also calculate the Wilson loop,[8] which is defined by

$$\begin{aligned}W(C)&=\left\langle \prod_{x\in C}\sigma_\mu(x)\right\rangle\\&=Z^{-1}\int\prod_{x,\mu}^{N_1N_2}d\sigma_\mu(x)\\&\quad\times\int\prod_{x}^{N_1N_2}d\psi(x)d\bar\psi(x)\\&\quad\times\prod_{x\in C}\sigma_\mu(x)\exp(-S_f-S_g)\ ,\end{aligned} \tag{2.4}$$

where C is a closed contour of the links.

It is well known that the fermionic integrals in (2.3) can be integrated out formally so that we obtain a usual determinant form and an effective action of the gauge fields. One may doubt, however, if this step represents real progress as it was emphasized in numerical simulations of lattice gauge theory with dynamical fermions.[4] There are at least two difficult problems in making the exact analytic calculations for the partition function and other quantities on the finite lattice.

First, naively, the determinant contains $(6\times4\times2)!\approx10^{61}$ terms by definition in the lattice size in our consideration. This large number make it difficult to combine like terms even by using the latest and fastest computer. Second, even if we could rearrange the terms in the determinant and form an effective gauge field action by exponentiating the determinant, we must integrate it out; this integration is difficult because it is highly nonlocal and nonlinear.

In the fermionic integration, we therefore abandon the above formal method and employ the method developed for nonlinear fermions on a finite lattice,[14] which is somewhat extended in order to deal with gauge field variables. The complexity of the calculation is drastically reduced by this method in the fermionic integration. In the integration of the gauge fields we choose the $\sigma_2(x)=1$ gauge in order to reduce the complexity of calculations further.

For this purpose, we impose free boundary conditions along the $\mu=2$ direction. Along the $\mu=1$ direction we impose both periodic and antiperiodic (periodic) boundary conditions for the fermion (gauge) fields in order to examine the boundary effects. The latter case of the antiperiodic boundary conditions can be considered as a system in a finite temperature $T=(N_1)^{-1}$ if we identify the $\mu=1$ axis as a temporal direction instead of a spatial one.

Finally, we briefly discuss the existence of a novel symmetry which is to be satisfied by the partition function and the Wilson loop; i.e., the partition function of (2.3) is invariant under the transformation

$$\eta_\mu(x)\to1 \ \text{ and } \beta\to-\beta\ . \tag{2.5}$$

On the one hand, the Wilson loops do not change their values by the above transformation up to the sign $W(C)\to\pm W(C)$. Here the plus (minus) sign is the case when the contour C encloses an even (odd) number of plaquettes.

The proof is as follows. We change the gauge field variables as

$$\sigma_\mu(x)\to\tilde\sigma_\mu(x)=\eta_\mu(x)\sigma_\mu(x)\ . \tag{2.6}$$

This enables us to absorb $\eta_\mu(x)$ into the gauge field variables in (2.1). Then, each plaquette in the gauge action (2.2) becomes

$$\beta\prod_{\partial\Box}\sigma_\mu(x)\to\tilde\beta\prod_{\partial\Box}\tilde\sigma_\mu(x)\ , \tag{2.7}$$

where $\tilde\beta=\beta[\prod_{\partial\Box}\eta_\mu(x)]$, and the shorthand notation $\prod_{\partial\Box}$ means the product around each plaquette. On the other hand, there is a condition that the KS fermion phases satisfy

$$\prod_{\partial\Box}\eta_\mu(x)=-1\ , \tag{2.8}$$

which has been already pointed out.[12,13] The measure is obviously invariant under this transformation:

$$\prod_{x,\mu}d\sigma_\mu(x)=\prod_{x,\mu}d\tilde\sigma_\mu(x)\ . \tag{2.9}$$

Thus, it is shown that the partition function is invariant under the transformation of (2.5). Similarly, in the Wilson loop, it can be proved that the Wilson loop operator in (2.4) is transformed so that

$$\prod_{C}\sigma_\mu(x)=\left[\prod_{\Box\in D}\left[\prod_{\partial\Box}\eta_\mu(x)\right]\right]\prod_{C}\tilde\sigma_\mu(x) \tag{2.10}$$

by (2.6), where $\partial D=C$. Q.E.D.

Thus the calculation of the partition function or the Wilson loop is simplified in such a way that, starting from an action, without KS fermion phases $\eta_\mu(x)$ in (2.1), (2.3), and (2.4), we obtain the correct solution by the transformation $\beta\to-\beta$.

Generally, all physical quantities that have a simple transformation rule under (2.6) are easily calculable owing to the invariance of the partition function. This symmetry is valid depending on the gauge groups. It can be extended to the gauge-fermion models with KS fermions on the lattice whose dimension is less than or equal to four and the gauge group contains -1 as an element. The details of the extension and applications will be discussed in a separate paper.[15]

III. FERMIONIC INTEGRATION

We first explain the method for calculating the fermionic integration in the partition function (2.3). We consider the Grassmann functional integration

$$Z_f(\sigma)=\int \prod_x^{N_1N_2} d\psi(x)d\bar{\psi}(x)\exp(-S_f)\ . \tag{3.1}$$

The method of the fermionic integration is almost parallel to our previous work for nonlinear fermion models[14] except for the inclusion of the gauge field variables. Therefore, details can be found in the references.

In order to avoid the complication of the fractional coefficients in (2.1), we rescale the fields as follows: $\psi\to(1/\sqrt{2})\psi$ and $\bar{\psi}\to(1/\sqrt{2})\bar{\psi}$. Then S_f is converted into the form

$$S_f=-\sum_{x,\mu}\eta_\mu(x)[\bar{\psi}(x+\hat{\mu})\sigma_\mu(x)\psi(x) -\bar{\psi}(x)\sigma_\mu(x)\psi(x+\hat{\mu})]\ . \tag{3.2}$$

We can calculate the partition function and the Wilson loops using S_f in (3.2) instead of (2.1), and the final expression of Z will be recovered by multiplying the overall factors $(\frac{1}{2})^{N_1N_2}$.

We can rewrite $Z_f(\sigma)$ of (3.1), by using the algebraic properties of Grassmann numbers,[14,16] in the following form factorized into (N_1+N_2) traces along each axis:

$$Z_f(\sigma)=\sum_{\{p\}=1}^{4}\prod_{n_2=1}^{N_2}\mathrm{Sp}\left[\prod_{n_1=1}^{N_1}(-1)^{\tilde{F}_p(n)}\hat{R}_p(n)\right]\prod_{n_1=1}^{N_1}\mathrm{Tr}\left[\prod_{n_2=1}^{N_2}(-1)^{\delta_{3p}}\mathcal{S}^{n_1}\hat{R}_{5-p}(n)\right]\ , \tag{3.3}$$

where

$$\hat{R}_{p_{(\mu)}}(n)=K\hat{\sigma}_\mu(n)R_p(n)\ . \tag{3.4}$$

Here, $R_p(n)=R_p$ except at the boundaries, and R_p are the matrices

$$\begin{aligned}&R_1=r_1\otimes r_1,\quad R_2=r_2\otimes r_1\ ,\\ &R_3=r_1\otimes r_2,\quad \text{and } R_4=\mathcal{S}'(r_2\otimes r_2)\ .\end{aligned} \tag{3.5}$$

Here $r_1=\frac{1}{2}(1+\tau_3)$, $r_2=\tau_1$, and τ_i are the Pauli matrices, and $\mathcal{S}'$ is the diagonal matrix: $\mathcal{S}'=\mathrm{diag}(1,1,-1,1)$. In (3.4), $\hat{\sigma}_\mu(n)$ is defined by the following diagonal matrix which includes the gauge field variables:

$$\begin{aligned}\hat{\sigma}_\mu(n)&=\mathrm{diag}(1,\sigma_\mu(n),\sigma_\mu(n),1)\\ &=\mathrm{diag}(1,\sigma_\mu(n))\otimes\mathrm{diag}(1,\sigma_\mu(n))\ .\end{aligned} \tag{3.6}$$

$\hat{\sigma}_\mu(n)$ plays the role to couple the fermion and gauge field in this formulation. $\mathcal{S}$ and K are, respectively,

$$\mathcal{S}=\tau_3\otimes\tau_3 \tag{3.7}$$

and

$$K=\frac{1}{2}\sum_{i=1}^{4}\tau_i\otimes\tau_i\quad(\tau_4=1)\ . \tag{3.8}$$

Here Sp and Tr mean traces along the $\mu=1$ and 2 axes, respectively. $\mathcal{S}^{n_1}$ is a rewriting of $\eta_\mu(x)$, and can be removed by assuming the transformation $\beta\to-\beta$ in the present model as stated in Sec. II. $\tilde{F}_p(n)$ is given by

$$\tilde{F}_p(n_1,n_2)=F_p(n_1,n_2)\times\left[\sum_{m_1=n_1}^{N_1}\sum_{m_2=n_2}^{N_2}F_p(m_1,m_2)\right]_{\mathrm{mod}\ 2}\ , \tag{3.9}$$

where F_p $(p=1,\ldots,4)$ are fermion numbers of θ^p which are elements of the basis of the Grassmann algebra $\{\theta^p\}=(1,\psi,\bar{\psi},\psi\bar{\psi})$ $(p=1,\ldots,4)$. We can further simplify $\tilde{F}_p(n)$, if we use $\sum_{n_\mu=1}^{N_\mu}F_p(n_\mu)=0$ $(\mu=1,2)$.

At the boundary along the $\mu=1$ direction, they become $R_p(N_1,n_2)=R_p\mathcal{S}$ or $R_p(N_1,n_2)=R_p$ corresponding to the choice of periodic or antiperiodic boundary conditions, respectively. On the other hand, at the boundary along the $\mu=2$ direction, $R_p(n_1,N_2)=R_pR_1$ corresponding to the choice of the free boundary conditions. This expression with the free boundary conditions is derived from the fact that the system is strictly confined to the region $1\le n_2\le N_2$ [i.e., $\psi(n)=\bar{\psi}(n)=0$ for $n_2<1$ and $n_2>N_2$], and an identity $\mathrm{tr}(AR_1)=(A_{11})$ for arbitrary 4×4 matrix $A=(A_{ij})$.

We should have accessibility to a computer to calculate the partition function and the Wilson loop with lattices as large as possible. To evaluate (2.3) and (2.4), two processes of the integrations (fermionic and gauge field ones) are required. Here we describe the algorithm with only fermionic integration in order to treat (3.3). The one of the gauge field integration will be described in the next section. The fermionic integration by a FORTRAN program is carried out on the basis of the following algorithm in order to obtain the analytic result. Each term of (3.3) can be expressed by an integer matrix $(p_{n_1n_2})$ formed from the subscripts of $\hat{R}_p(n_1,n_2)$ $(n_\mu=1,\ldots,N_\mu)$. For example, the column $(p_{n_1n_2})$ (n_2 fixed) is uniquely associated with the operation

$$\mathrm{Sp}\left[\prod_{n_1=1}^{N_1}\hat{R}_p(n_1,n_2)\right]\qquad(n_2\ \text{fixed})\ ,$$

which can be easily evaluated. It gives the form

$$\sum_{j(n_2)}\left[\bar{C}(\bar{b}(n_2))\prod_{n_1=1}^{N_1}\sigma_1(n_1,n_2)^{\bar{b}_{n_1n_2}}\right]_{j(n_2)}\qquad(n_2\ \text{fixed})$$

(there are 6^2 nontraceless ones for $N_1=4$). Here the exponents $\bar{b}_{n_1n_2}$ take values 1 or 0. The subscript $j(n_2)$ (n_2

fixed) specifies the kind of terms in this calculation, which are less than or equal to 2, and the coefficient $[\bar{C}(\bar{b}(n_2))]_{j(n_2)}$ (n_2 fixed) of each term takes values ± 1 or exceptionally 2. Similarly, each row $(p_{n_1 n_2})$ (n_1 fixed) is associated with Tr[ ] in (3.3) (there are 13^2 nontraceless ones for $N_2=6$). All of the nontraceless ones, however, are simply reduced to sign factor ± 1, if we take $\sigma_2(x)=1$ gauge.

Although (3.3) has $4^{N_1 N_2}$ terms, almost all of them will vanish. Therefore, it is desirable to generate only a set of nonvanishing (nontraceless) terms. This is actually possible by using the method described in Ref. 14. As a result, only 517^2 nonvanishing terms in (3.3) are picked up and evaluated in 6×4 lattice size, before the translational symmetries are considered. Thus, if a matrix $(\bar{p}_{n_1 n_2})$ expressing the nonvanishing term of (3.3) is given by this efficient method, we can associate it with a polynomial in $\sigma_1(n_1,n_2)$. This polynomial is a product of N_1 traces (Tr), N_2 traces (Sp), and other factors such as $(-1)^{\bar{F}_p(n)}$ and $(-1)^{\delta_{3p}}$ (which depend on the indices p). The polynomial can be easily expressed by a FORTRAN program, because each term of the polynomial is specified by numerical information such as the exponents and coefficient. Finally, we note the following point to avoid misunderstanding. Since the fermionic integration is not the final goal, it is not necessary to obtain the most compact form of $Z_f(\sigma)$, (3.3). Once a nonvanishing term of $Z_f(\sigma)$ is obtained, it can be transformed to the expression after executing the gauge field integration discussed in the next section. It will be more efficient and convenient to combine like terms with it. Therefore, the remaining part of the algorithm will be described along this line in the next section.

IV. GAUGE FIELD INTEGRATION

If we carry out the traces Tr and Sp in (3.3), the following polynomial will be obtained in the case of $\sigma_2(x)=1$ gauge:[17]

$$Z_f(\sigma)=\sum_{\{b_{n_1 n_2}\}} \tilde{C}(b_{n_1 n_2}) \prod_{n_1=1}^{N_1} \left[\prod_{n_2=1}^{N_2} \sigma_1(n_1,n_2)^{b_{n_1 n_2}} \right], \tag{4.1}$$

where exponents $b_{n_1 n_2}$ take values 1 or 0. We notice that the partition function is essentially none other than expectation value of $Z_f(\sigma)$ over the gauge field

$$Z=\int \prod_x^{N_1 N_2} d\sigma_1(x) Z_f(\sigma)\exp(-S_g)=\langle Z_f(\sigma)\rangle_g Z_g . \tag{4.2}$$

Here, Z_g is the partition function of the pure gauge model, and can be evaluated as $Z_g=2^{N_1 N_2}(\cosh\beta)^{N_p}$ with the free boundary conditions.[1] Here N_p is the total number of plaquettes, and $N_p=N_1(N_2-1)$. At the same time we can factorize each term in (4.2) along the $\mu=1$ direction, and the equivalence between the 2D Z_2 pure gauge model and the 1D Ising model enables us to rewrite (4.2) by the language of the 1D Ising model

$$Z=Z_{\mathrm{I}}^{N_1}\left[\sum_{\{b_{n_1 n_2}\}} \tilde{C}(b_{n_1 n_2}) \prod_{n_1=1}^{N_1} \left\langle \prod_{n_2=1}^{N_2} \sigma_1(n_1,n_2)^{b_{n_1 n_2}} \right\rangle_{\mathrm{I}} \right], \tag{4.3}$$

where the notations Z_{I} and $\langle\ \rangle_{\mathrm{I}}$ mean the partition function $[=2^{N_2}(\cosh\beta)^{N_2-1}]$ and the expectation value in the 1D Ising model of the chain of N_2 spins, respectively.

We can evaluate the arbitrary n-point correlation functions in (4.3). The $2n$-point correlation functions are obtained as follows [we note that $(2n+1)$-point correlation functions vanish]. Assuming the coordinates to be in the order $x_1<x_2<\cdots<x_{2n}$, the correlation function is given by

$$\langle \sigma_{x_1}\sigma_{x_2}\cdots\sigma_{x_{2n}}\rangle_{\mathrm{I}}=(\tanh\beta)^{\sum_{m=1}^{n} d_m} , \tag{4.4}$$

where $d_m=x_{2m}-x_{2m-1}$. This formula can be obtained by a straightforward extension of the two-point correlation function.[18]

Consequently, the partition function (2.3) is given by

$$Z=(\cosh\beta)^{N_p}\left[\sum_{m=0}^{N_p} C(m)\tanh^m\beta \right] . \tag{4.5}$$

Thus the evaluation of the partition function means the determination of $C(m)$. We can do it, because the terms

$$\prod_{n_1=1}^{N_1} \left\langle \prod_{n_2=1}^{N_2} \sigma_1(n_1,n_2)^{b_{n_1 n_2}} \right\rangle_{\mathrm{I}}$$

in (4.3) can be evaluated by (4.4) and the calculation, equivalent to the evaluation of the coefficient $\tilde{C}(b_{n_1 n_2})$, can be done by the algorithm discussed in the latter half of Secs. III and IV (see below). Hence the partition function on the finite lattice is calculated exactly.

Next we will discuss a method of calculation of the Wilson loop (2.4). This is similar to one for the partition function. From (2.4) and (4.3), in $\sigma_2(x)=1$ gauge, the Wilson loop $W(C)$ can be written as

$$W(C)=Z_g\left[\sum_{\{\tilde{b}_{n_1 n_2}\}} \tilde{C}(\tilde{b}_{n_1 n_2}) \prod_{n_1=1}^{N_1} \left\langle \prod_{n_2=1}^{N_2} \sigma_1(n_1,n_2)^{\tilde{b}_{n_1 n_2}} \right\rangle_{\mathrm{I}} \right] \Big/ Z , \tag{4.6a}$$

where

$$\tilde{b}_{n_1 n_2} = \begin{cases} b_{n_1 n_2} + 1 & \text{if } (n_1, n_2) \in C, \\ b_{n_1 n_2} & \text{if } (n_1, n_2) \notin C. \end{cases} \tag{4.6b}$$

This turns to the following expression by similar derivation of (4.5):

$$W(C) = \left[\sum_{m=0}^{N_p} C(m,C) \tanh^m \beta \right] \Bigg/ \left[\sum_{m=0}^{N_p} C(m) \tanh^m \beta \right] . \tag{4.7}$$

We can evaluate the Wilson loop (4.7), because we can determine the coefficient $C(m,C)$ by a similar method as in the calculation of the partition function; the difference is only an addition of unity to the exponents $b_{n_1 n_2}$ belonging to links along the $\mu = 1$ direction in a contour of the Wilson loop.

Now, we briefly describe the FORTRAN algorithm for the gauge field integration. In the preceding section it was discussed that if the integer matrix $(\tilde{p}_{n_1 n_2})$ expressing each nonvanishing term of (3.3) is given we can correlate it with the polynomial in $\sigma_1(n_1, n_2)$ for the treatment of (3.3). On the FORTRAN program each term of the polynomial can be numerically expressed by a set of the exponents and the coefficient. The exponents can be correlated with the one of $\tanh\beta$ on the basis of (4.4) by a searching algorithm (there are several quick searching methods, e.g., the binary search). Also, combining like terms of the polynomial can be easily accomplished in the following manner: Numbers of addresses of the memories are assigned by the exponent of $\tanh\beta$, and each coefficient of like terms is accumulated there. For the Wilson loop, it can easily be understood that the algorithm is similar to the one for the partition function mentioned above.

Finally, we comment on the following equality which is satisfied between either the coefficients $C(m)$ in the partition function (4.5) or the coefficients $C(m,C)$ in the Wilson loops (4.7):

$$\sum_{m=0}^{N_p} C(m,C) = \sum_{m=0}^{N_p} C(m) , \tag{4.8}$$

for arbitrary contours C of the Wilson loops (i.e., this means that the sum of the coefficients is independent of the loop size and the location).

The reason can be easily understood by (4.3), (4.4), and (4.6); i.e., it is due to the identical coefficients $\tilde{C}({}^{(\sim)}\!b_{n_1 n_2})$ in (4.3) and (4.6). Therefore, $W(C) \to 1$ with $\beta \to \infty$ is satisfied.

TABLE I. Coefficients $C(m)$ of the partition function in $\tanh\beta$ [see (4.5)].

m	$C(m)$
0	1 681
1	9 512
2	27 808
3	56 168
4	87 174
5	109 392
6	115 824
7	108 992
8	95 994
9	82 072
10	69 824
11	59 640
12	50 208
13	40 872
14	32 160
15	24 120
16	16 588
17	10 000
18	4 856
19	1 584
20	242

V. RESULTS AND DISCUSSIONS

In Table I, we show the coefficients $C(m)$ as being a result of the calculation of the partition function (4.5) on the 6×4 lattice. The gauge invariance is explicit, because $\cosh\beta$ and $\tanh\beta$ (or $\sinh\beta$) are expressed by the integration(s) of the plaquette variables.[1,19] Since all $C(m)$ have positive signs, the positivity of the partition function is obvious for the positive β. The positivity of the partition function is also satisfied for the negative β. However, the positivity of all $C(m)$ is nontrivial. For example, the signs of the coefficients change alternatively when $\eta_\mu(x) \to 1$; the positivity of the coefficients will be recovered by the simultaneous transformation $\beta \to -\beta$ as stated in Sec. II.

In Fig. 1, we show the distribution of zeros of the partition function in the complex $t = \tanh\beta$ plane. The plot in the t plane is natural from the expression of the partition function (4.5) as it is the standard one for the Z_2

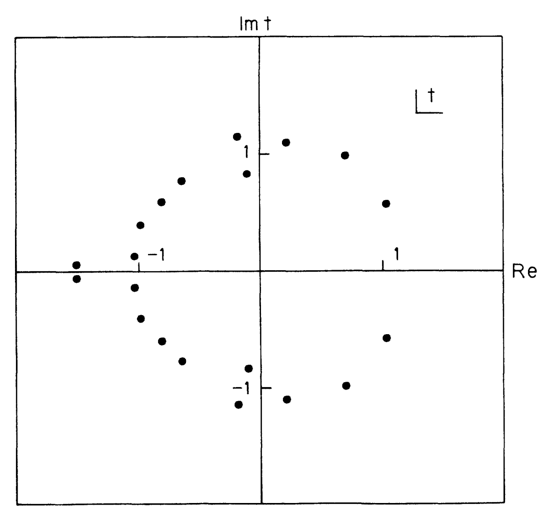

FIG. 1. The zeros of the partition function on the 6×4 lattice in the t ($= \tanh\beta$) plane.

variables.[6,20] It is interesting to see how the distribution of the zeros occurs as compared to the ones of the pure spin or gauge models. The result shows a tendency for the zeros to approximately fall on an arc which looks like a circle. In the physical region $0 \le |\mathrm{Re}t| \le 1$, the closest zero is sufficiently separated from the positive Ret axis. This is consistent with the fact that the specific heat does not indicate a phase transition (see below). On the other hand, in the negative Ret region, the zeros have a richer structure than the positive Ret, particularly the existence of zeros near the Ret axis.

For more details about the negative Ret region, let us comment on the model of negative β. The model describes a coupled system of the two-dimensional fermions and the antiferromagnetic Ising model with the anisotropic limit (quasi-one-dimensional one) in an analogy with the positive β model. As stated in Sec. II, the partition function with negative β is identical to the one of the positive β without $\eta_\mu(x)$. This means that loci of the zeros of the partition function of the original model and the one without $\eta_\mu(x)$ are related by mirror reflection along the Imt axis.

In the region of negative t, pairs of the zeros exist close to Ret axis. These are at Re$t \simeq -1.5$ and -1. We consider that the former pair does not indicate the existence of phase transition in the thermodynamic limit, because it is outside the physical region $0 \le |\mathrm{Re}t| \le 1$. In the 2D ferro- and antiferromagnetic Ising model, circles of the zeros of the partition functions in the t plane intersect the Ret axis at such nonphysical points and the physical ones at $t_c = \pm(\sqrt{2}-1)$.[20] On the other hand, it is not obvious if the latter is nonphysical on the basis of similar reasoning of the former. The point $t=-1$ (i.e., $\beta=-\infty$) is a trivial phase transition point of the pure gauge model (or 1D antiferromagnetic Ising model), although this is not shown by root. The present zeros are controlled by the contribution of the dynamical fermions as understandable from (4.5). Therefore, if this is an indication of some physics, it would be with regard to some symmetry of fermions.

In Fig. 2 we show (normalized) internal energy and (normalized) specific heat of this model by solid lines; they are defined by

$$U = \frac{1}{N_p} \frac{\partial}{\partial \beta} \ln Z \tag{5.1}$$

and

$$C = \frac{1}{N_p} \frac{\partial^2}{\partial \beta^2} \ln Z \ , \tag{5.2}$$

respectively. Also in Fig. 2, the internal energy and specific heat of the pure gauge model are plotted by dashed lines.

The differences are effects of the dynamical fermions. A conspicuous difference is that U holds positive value even in the limit $\beta \to 0$, in contrast with the pure gauge model. The reason is as follows. In the limit $\beta \to 0$, only the linear term of $\tanh\beta$ contributes to U. The expression $\tanh\beta$ is the same as the expectation value of one plaquette in the pure gauge model. On the other hand, the

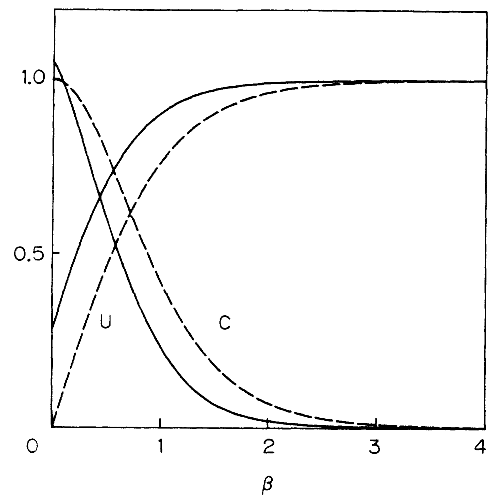

FIG. 2. The solid lines show the specific heat C and internal energy U for the 6×4 lattice for this model, and dashed lines show those for the pure gauge model.

determinant represents closed fermion loops, and depends on the gauge link variables in (2.1) forming closed loops. From these, the positive value of U means that one plaquette having positive coefficient is created by the fermion loop. More explicitly, this positivity itself requires the existence of $\eta_\mu(x)$ as stated before, and it can be proved that the property $\prod_{\partial\Box} \eta_\mu(x) = -1$ acts to cancel the statistical minus sign factor accompanied to the fermion loop. (To avoid a lengthy discourse, proof is omitted.)

As for the specific heat of this model, it decreases monotonically so that there is no indication of a phase transition. This is consistent with the fact that the closest zero is sufficiently far away from positive Ret axis.

In order to examine the boundary effects, we change the periodic boundary conditions along the $\mu=1$ axis into the antiperiodic boundary conditions for the fermion fields. We find, however, that there is no change at all in the partition function and other quantities. Therefore, we infer that the boundary effect is relatively small. The antiperiodic case means that the system can be interpreted as it is in the finite temperature $T^{-1}=N_1=4$; since the present calculations are in the Euclidean, the $\mu=1$ axis can be considered to be the temporal axis instead of the spatial one and the $\mu=2$ axis to be the spatial axis instead of the temporal one.

On the other hand, we examine the boundary effect of the $\mu=2$ direction, along which the free boundary conditions are imposed, in the following manner: We gradually shift the 1×1 Wilson loop on the lattice from the center to the edge through intervals of the unit lattice spacing along the $\mu=2$ axis.

In Fig. 3, we show these 1×1 Wilson loops, which are placed at $n_2=3$ (placed at the center), $n_2=4$, and $n_2=5$ (placed at the edge). In this figure, the solid line denotes $n_2=3$, the dashed-dotted line is $n_2=4$, and the dashed-double-dotted line is $n_2=5$, respectively. Similarly the

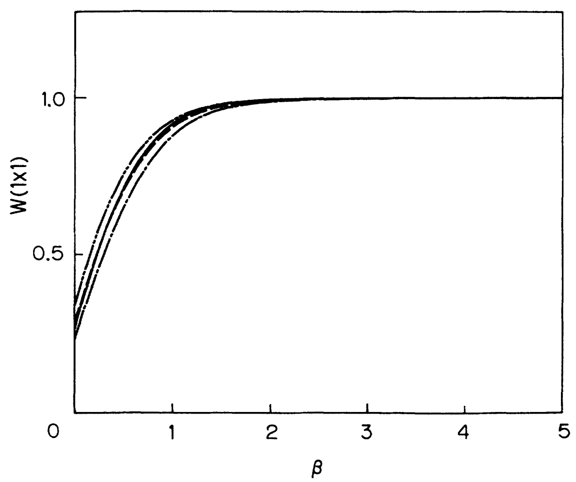

FIG. 3. The solid line denotes $n_2=3$ (center placed), the dashed-dotted line is $n_2=4$, and the dashed-double-dotted line is $n_2=5$ (ending placed) with the $\mu=2$ coordinates, respectively. The dashed line shows the internal energy U.

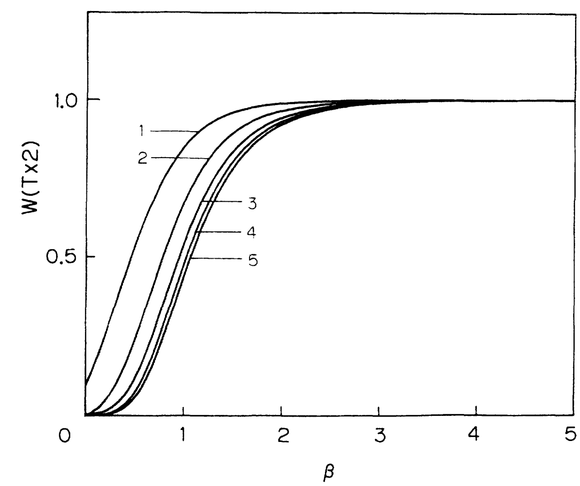

FIG. 4. The behavior of β dependence of $T\times 2$ Wilson loops in this model. The numbers of lines denote T along the $\mu=2$ direction.

dashed line shows the internal energy U, (5.1) (this is the 1×1 Wilson loop averaged over all lattice points) in order to compare with the 1×1 Wilson loops placed at different locations. We observed that the dependence on the location does not vary uniformly. Consequently, U agrees well with the center placed 1×1 Wilson loop due to the cancellation of the deviation as shown in Fig. 3. As a result, from Fig. 3 we would conclude that as a whole the discrepancy of the values due to location of the loops can practically be neglected. In its pure gauge model, the Wilson loop has a peculiar property, such as translation invariance, even if the free boundary conditions are imposed on a finite lattice. Therefore, in the present calculation, the small violation of the translation invariance along the $\mu=2$ direction must be caused by the fermionic part.

In Fig. 4, the behavior of β dependence of $T\times 2$ $(=R)$ Wilson loops of this model is shown. The numbers of the lines denote the size T of loops along the $\mu=2$ direction.

Figure 5 shows the size dependence of $T\times 2$ Wilson loops of this model compared with the one of the pure gauge model for various β in logarithmic scale. In Fig. 5, the effects of the dynamical fermions to the Wilson loops are clearly seen at each loop size in the small β region. On the other hand, in the large β region, it is difficult to distinguish the Wilson loops between this model and the pure gauge model in the present loop sizes. The values of the Wilson loops incorporating dynamical fermions deviate upward from the ones of the pure gauge model as the size increases, as seen in Fig. 5.

This behavior will be expected when the dynamical fermions are incorporated. In the pure gauge model, the Wilson loop having a size of $T\times R$ is given by $W[T\times R]=\exp\{-[\ln(\coth\beta)]TR\}$. Therefore the area law is exact in any loop size and any finite value of β, as seen in Fig. 5. On the other hand, the expected main effect of the fermions to the Wilson loop can be considered such that many dynamical loops of the fermions rotate to cancel out the gauge field strength and to make holes in the Wilson loop. As a result, the area enclosed by the Wilson loop will be effectively reduced compared to the one for the pure gauge.

Finally, we comment on the extensions of our method. The present method tells us how to make extensions for calculating further physically interesting two-dimensional gauge-fermion models such as the lattice Schwinger model or $SU(2)_c$ QCD_2. The essential difference between the lattice Schwinger model and the present model is as follows: after the gauge fixing, such as $A_2(x)=0$, the exact calculation of arbitrary n-point correlation functions of the 1D planar spin model instead of the 1D Ising model becomes a matter of great concern. Fortunately, all n-point correlation functions are exactly calculable on a finite lattice. Similarly, as to the $SU(2)_c$ QCD_2, except a

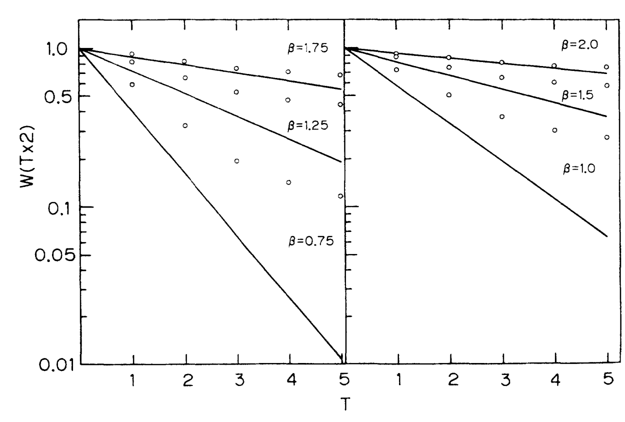

FIG. 5. The size dependence of $T\times 2$ Wilson loops of this model (open circles) are compared with those of the pure gauge model (solid lines) for various β in logarithmic scale.

soluble extension to fermions having SU(2)-color indices, a critical problem becomes whether or not the arbitrary n-point correlation functions of the 1D O(4) [$\simeq$SU(2)$\otimes$SU(2)] classical Heisenberg model are exactly calculable. They can be evaluated on finite lattices. Therefore, $SU(2)_c$ QCD_2 is a solvable model on a finite lattice. Such models will be fully discussed in a separate paper.[21]

A reduced version of KS fermions is considered to be advantageous for the calculation on larger lattices.[11,12,22,23] Therefore, examinations by the method described in this paper would be an interesting subject for a future investigation.

In conclusion, we presented a new method for calculating exact physical quantities of the two-dimensional gauge-fermion models, such as the partition function on the finite lattices. We explained the method with a concrete example of the Z_2 gauge-fermion model, which is the simplest model having the essential framework. At the same time we showed the behavior of the quantities, such as zeros of the partition function, the internal energy, the specific heat, the Wilson loop, and the boundary effects, on the basis of the exact calculations on the 6×4 lattice. We also showed that the model has the novel symmetry in the partition function and the Wilson loop with respect to the KS fermion phases and the coupling.

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