PHYSICAL REVIEW D VOLUME 42, NUMBER 12 15 DECEMBER 1990

Calculations on infinite lattices applied to lattice gauge theory

R. Sinclair*

Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-1000 Berlin 33, Federal Republic of Germany

(Received 22 August 1990)

A procedure is introduced allowing the infinite lattice limit for particular lattice quantities to be obtained by integrating over boundary conditions on small, finite lattices. Numerical results are given for a simple lattice model. Relevance to the measurement of lattice finite-temperature effects is discussed.

GENERAL APPROACH TO INFINITE-LATTICE CALCULATIONS

Matrix expansions defined upon a torus include paths which would not belong on an infinite, simply connected lattice. We discuss their removal with the aim of approximating infinite-lattice calculations using finite lattices.

An infinite regular lattice in d-dimensional flat space can be fully described through a set of d basis vectors $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$. We can label the sites in the lattice using vectors i $(i_1, \ldots, i_d)$ with integer components. Consider the typical case of a matrix W with nonzero elements linking nearest neighbors and perhaps diagonal terms. The quantities of interest for the present discussion are those which can be expressed as functions of the elements of the matrix expansion

$$E_{ij} \equiv \sum_{l=0}^{\infty} a_l W_{ij}^l \; , \tag{1}$$

where the indices (i,j) are to be understood as those of the exponentiated matrix.

Examples are

$$-\ln \det W = \mathrm{Tr} \sum_{l=1}^{\infty} \frac{1}{l}(1-W)^l \tag{2}$$

and

$$W_{ij}^{-1} = \sum_{l=0}^{\infty} (1-W)_{ij}^l \; , \tag{3}$$

which converge when the modulus of the largest eigenvalue of $1-W$ is less than unity. Both of the above are of interest in lattice gauge theory (where the matrix W is taken to be the fermion matrix), as an effective action term for calculating expectation values in the full theory (2), and as the lattice fermion propagator (3) (see, for example, Ref. 1). These converge given a large enough bare mass (this is discussed in Ref. 2 and references therein). Meson propagators cannot however be written in this form (rather as the product of two expansions, of W and W^H, respectively).

We can interpret E_{ij} as a weighted sum of products of matrix elements over all paths beginning with site j and ending at site i. On an infinite regular lattice, the space through which these paths wander is simply connected.

On a finite lattice, boundary conditions must be chosen. We restrict the lattice to a parallelogram of side lengths L $(L_1, \ldots, L_d)$, having $M = \prod L_r$ lattice sites. We can define boundary conditions with reference to a vector (Ψ), upon which the matrix W acts. In a lattice gauge model, this would be a vector of Grassmann variables representing fermion fields. We consider here only torodial topologies, and write

$$\Psi_{i+L_r e_r} = e^{i\theta_r}\Psi_i \; . \tag{4}$$

Periodic or antiperiodic boundary conditions in the direction r would be achieved by $\theta_r = 0$ or $\theta_r = \pi$, respectively. We will however treat the general case where the boundary phase θ_r is a real number (or in the range $(-\pi, \pi]$, say).

We now calculate, with an $M \times M$ matrix,

$$\overline{W}_{pq} \equiv W_{pq} + \sum_{r=1}^{d} \left(e^{i\theta_r} W_{L_r, L_r - 1} \delta_{p_r,0} \delta_{q_r, L_r - 1} + e^{-i\theta_r} W_{L_r - 1, L_r} \delta_{q_r,0} \delta_{p_r, L_r - 1} \right) , \tag{5}$$

which is defined for $0 \le p_r, q_r < L_r$, but must be imagined as periodically continued (this can be seen when one notices that traveling once around the toroidal lattice in any direction brings one back to exactly the same link from which one started—this is independent of the boundary conditions chosen for Ψ). It will in fact be of use to define an infinite matrix V as the periodic continuation of W outside our finite lattice volume.

$$V_{ij} \equiv W_{i \bmod L,\, j \bmod L} \delta_{i,j} + \sum_{r=1}^{d} \left(W_{(j \bmod L)+e_r,\, j \bmod L} \delta_{i,j+e_r} + W_{i \bmod L,\, (j \bmod L)+e_r} \delta_{j,i+e_r} \right) . \tag{6}$$

The matrix expression (1) may now be written for this finite lattice.

$$\overline{E}_{pq} \equiv \sum_{l=0}^{\infty} a_l \overline{W}_{pq}^l = \sum_{l=0}^{\infty} a_l \sum_{w_1=-\infty}^{\infty} \cdots \sum_{w_d=-\infty}^{\infty} e^{i\theta \cdot w} [\overline{W}_{pq}^l]_w \; , \tag{7}$$

where the terms

$$[\overline{W}^l_{pq}]_w \equiv V^l_{p+\sum w_r L_r e_r, q} = (2\pi)^{-d} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{-i\theta\cdot w} \overline{W}^l_{pq} d\theta_1 \cdots d\theta_d \tag{8}$$

correspond to the paths in the sum with winding numbers $w(w_1, \ldots, w_d)$. See Fig. 1. One may interpret this as a Fourier transformation between boundary-phase and winding-number spaces.

In order to make calculations for the infinite lattice using the finite matrix $\overline{W}$, we must select out only those paths in $\overline{E}$ which appear on the infinite lattice. These are paths of winding number zero along every axis. They can be separated out by integrating over boundary conditions:

$$\begin{aligned}\langle \overline{E}_{pq} \rangle &\equiv (2\pi)^{-d} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \overline{E}_{pq} d\theta_1 \cdots d\theta_d \\ &= \sum_{l=0}^{\infty} a_l [\overline{W}^l_{pq}]_0 \\ &= \sum_{l=0}^{\infty} a_l V^l_{pq} \\ &\simeq E_{pq} .\end{aligned} \tag{9}$$

Equality with E_{pq} is only achieved if the elements of the infinite matrix W_{ij} already have a periodicity of L (i.e., $V=W$). This includes the case where they only depend upon the difference $i-j$, which occurs for the free-field situation in lattice gauge models. It is also interesting to note that, if the integral over θ in (9) is replaced by a sum over $Z(N)$ [i.e., $\int d\theta \rightarrow \sum \exp(i\theta), \exp(i\theta) \in Z(N)$], particular topologies other than that of the infinite lattice may be selected out.

The above considerations show how the topological properties of an infinite lattice can be realized in calculations upon a finite, torodial, lattice. It is, moreover, also possible to calculate quantities for an infinite lattice exactly in a finite calculation, but only if the problem already involves a periodicity of the finite lattice size used. The seriousness of the errors arising from the absence of such a periodicity in the infinite system will of course depend upon the properties of the quantity being measured. The method proposed is restricted to linear functions of the elements of the matrix series expansion (1), necessarily requiring that the series converges.

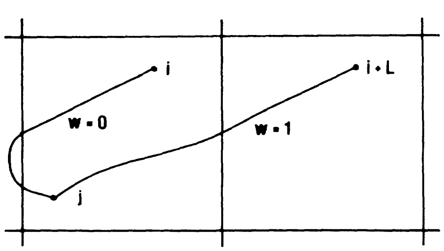

FIG. 1. Two paths with different winding numbers from site j to site i on a toroidal lattice represented on an infinite lattice. The path with nonzero winding number is unwanted on an infinite lattice, since it leads to a "ghost" copy of i.

RESULTS AND DISCUSSION RELEVANT TO LATTICE GAUGE THEORY

As a simple example, a two-dimensional $Z(2)$ lattice gauge model was chosen. Gauge fields $U \in Z(2)$ are defined upon links. The lattices studied were restricted to sizes $1 \times L$. The model can be imagined as a closed chain of L sites, with a single loop tied to every one. Links between adjacent sites will be written U_{ij}. The loops require only one site index, and will be referred to as U_i. We choose the "quenched" action[1]

$$S = \beta \sum_i (1 - U_i U_{i+1}) . \tag{10}$$

The absence of the U_{ij} in the action allows them to be regarded as random variables. One can define a type of fermion propagator as the inverse of the matrix

$$W_{ij} = \delta_{ij} - \kappa[2U_i\delta_{ij} + (1+\sigma)U_{ij}\delta_{i,j+1} + (1-\sigma)U_{ij}\delta_{i,j-1}] , \tag{11}$$

where indices on the Pauli matrices have been suppressed. The inverse of this matrix is of the form (1), and therefore a candidate for being calculated on an infinite lattice.

If we calculate in a "free-field" situation ($U=1$ everywhere on the lattice), then we can expect that integration over boundary conditions will lead to the infinite lattice results exactly. The inverse of the matrix W for the free-field case on an infinite lattice may be found analytically. Calculations on finite lattices were performed numerically by explicit summing over the finite number of states involved with $\kappa=0.1$. Summation over elements of $Z(N)$ was used to simulate the integration in (9) over all boundary conditions. A translationally invariant form for the propagator (of which the trace with respect to Pauli matrix indices will be referred to as G) was used. The results are shown in Table I. One sees that infinite lattice results are already obtained with the lattice of length $L=4$ by a summation over $Z(4)$. The same accuracy for fixed, antiperiodic boundary conditions was only reached by $L=16$ and above. Assuming that the calculation of the full inverse of the matrix W requires of the order of L^3 operations, the advantage of summing over $Z(4)$ to obtain the infinite lattice limit is seen as a reduction in the number of operations required by a factor $16^3/(4^3 \times 4) = 16$.

The results obtained are also interesting in that the propagator values in Table I for the $L=4$ lattice with summing over $iZ(2) \equiv \{i, -i\}$ are identical to those found on the $L=8$ lattice with fixed antiperiodic boundary conditions. This is to be expected from Eq. (7). Summing over $iZ(2)$ eliminates those paths with an odd winding number. The remaining paths have lengths which are multiples of 8, and acquire a minus sign if the multiple is odd. These are, however, exactly those products one would expect on a closed chain of length $L=8$ with fixed antiperiodic boundary conditions, explaining the numerical result.

TABLE I. Fermion propagators calculated for the free-field case on a $1 \times L$ lattice.

L	$\exp(i\theta)$	$G(0)$	$G(1)$	$G(2)$	$G(3)$
4	-1	1.245 136 1	0.145 914 4	0.000 000 0	-0.145 914 4
8	-1	1.249 980 9	0.156 209 5	0.038 909 3	0.009 155 1
16	-1	1.250 000 0	0.156 250 0	0.039 062 5	0.009 765 6
∞		1.250 000 0	0.156 250 0	0.039 062 5	0.009 765 6
4	$Z(4)$	1.250 000 0	0.156 250 0	0.039 062 5	0.009 765 6
4	$iZ(2)$	1.249 980 9	0.156 209 5	0.038 909 3	0.009 155 1
4	$Z(2)$	1.250 019 1	0.156 290 5	0.039 215 7	0.010 376 1

Calculations were also carried out for $\beta=0$, which is the "strong coupling" limit of such a model where all field configurations are equally likely. In order to see a nonzero propagator after summing over all states, some fixing of the gauge fields[3] is required. We choose a "temporal gauge" here, setting all links between adjacent sites to 1 except the one at the end (U_{cc}), which closes the chain. U_{cc} may still be considered a random variable. The corresponding infinite lattice propagator G can also be found analytically. The results are given in Table II. κ was again set to 0.1. This low value suppresses the errors due to the fact that the gauge fields in the strong-coupling limit do not have a periodicity of L ($V \neq W$), allowing us to concentrate on topological aspects. The infinite lattice results are reached much earlier (already by $L=8$) than in the free-field case, and the antisymmetric nature $[G(\tau)=-G(L-\tau)]$ of the free-field propagators with fixed antiperiodic boundary conditions (see the first row in Table I) is gone. This is to be understood as a direct result of the randomness in U_{cc}, which is unavoidable since we are using the "quenched" action (10) and given that we only fix the field using gauge transformations (one cannot fix every link in a lattice[3]). Making the substitution $U_{cc} \to bU_{cc}$, where $b \in Z(2)$, and summing over b with a fixed U_{cc} is therefore equivalent to U_{cc} itself being random, which it is. This implies an effective summing over periodic ($b=1$) and antiperiodic ($b=-1$) boundary conditions. The propagator G measured on a lattice of length L will be topologically equivalent to a free propagator measured with periodic boundary conditions on a lattice of length $2L$ [consider Eq. (7)]. This explains the rapid approach to infinite lattice results in Table II, and the disappearance of the symmetry $G(\tau)=-G(L-\tau)$, since the effective length of the lattice is no longer L.

The above topological considerations would be of importance in the measurement of finite temperature properties of fermion propagators in quenched SU(N) [or U(1)] lattice gauge models. In lattice gauge theory, the physical temperature is proportional to the inverse of the length of the lattice in the time direction.[4] If temporal gauge fixing[3] is used, transformations at the boundaries of the form $U_{cc} \to bU_{cc}$, where $b \in Z(N)$ [or $\in U(1)$, respectively] do not disturb the gauge fixing, since one does not fix links at the boundaries, and are a symmetry of the quenched action (see Ref. 5). In summing over all states one therefore automatically and unavoidably sums over boundary conditions, effectively lengthening the lattice in the time direction by a factor of N for SU(N), or to infinity for U(1), for the fermion propagator. Related effects for meson propagators were reported in Ref. 5. One would nevertheless continue to use the original lattice periodicity to define the temperature, since it is the gauge fields which determine the physics in the quenched approximation.

One should also be wary of this effective change in length if masses are to be found by fitting to free-field propagators,[6] since the effective length of the lattices, and certain symmetries of the propagators, differ between the two cases.

The fact that antiperiodic boundary conditions in the time direction for fermion fields are required when calculating finite-temperature properties[4] must also be considered. The effective boundary conditions are not always those of the original lattice (as was seen above). In particular, effective antiperiodic boundary conditions are achieved by choosing antiperiodic boundary conditions for the finite matrix [Eqs. (4) and (5)] if N is odd, but complex boundary conditions ($\theta_t=\pi/2$) if N is even.

A consistent approach to approximating the infinite lattice limit in a nonquenched calculation must use a corresponding fermion determinant in the lattice action. An appropriate form is Eq. (2) averaged over boundary conditions. To approach the infinite lattice limit, one should integrate over all θ_r, irrespective of what group is chosen for the gauge fields.

TABLE II. Fermion propagators on a $1 \times L$ lattice at strong coupling.

L	$\exp(i\theta)$	$G(0)$	$G(1)$	$G(2)$	$G(3)$
4	-1	1.041 671 3	0.108 517 8	0.022 653 6	0.004 930 6
8	-1	1.041 666 7	0.108 506 9	0.022 605 6	0.004 709 5
∞		1.041 666 7	0.108 506 9	0.022 605 6	0.004 709 5
4	$Z(3)$	1.041 666 7	0.108 506 9	0.022 605 6	0.004 709 5

ACKNOWLEDGMENTS

I would like to thank the Deutscher Akademischer Austauschdienst for financial support in Berlin, and A. Nakamura, V. Linke, M. Plewnia, P. Rakow, and W. Theis for many useful discussions and critical reading of this manuscript.

*Present address: IPS, ETH-Zentrum, CH-8092 Zurich, Switzerland.

[1]C. Rebbi, *Lattice Gauge Theories and Monte Carlo Simulations* (World Scientific, Singapore, 1983).

[2]I. O. Stamatescu, Phys. Rev. D **25**, 1130 (1982).

[3]M. Creutz, Phys. Rev. D **15**, 1128 (1977).

[4]D. Gross, R. Pisarski, and L. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).

[5]G. Martinelli, G. Parisi, R. Petronzio, and F. Rapuano, Phys. Lett. **122B**, 283 (1983).

[6]R. Roskies and J. C. Wu, Phys. Rev. D **33**, 2469 (1986).