Truncating the Schwinger-Dyson equations: How multiplicative renormalizability and the Ward identity restrict the three-point vertex in QED

D. C. Curtis and M. R. Pennington
Centre for Particle Theory, University of Durham, Durham DH1 3LE, United Kingdom
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Nonperturbative studies of field theory require the Schwinger-Dyson equations to be truncated to make them tractable. Thus, when investigating the behavior of the fermion propagator, for example, an *Ansatz* for the three-point vertex has to be made. While the well-known Ward identity determines the longitudinal part of this vertex in terms of the fermion propagator as shown by Ball and Chiu, it leaves the transverse part unconstrained. However, Brown and Dorey have recently emphasized that the requirement of multiplicative renormalizability is not satisfied by arbitrary *Ansätze* for the vertex. We show how this requirement restricts the form of the transverse part. By considering the example of QED in the quenched approximation, we present a form for the vertex that not only satisfies the Ward identity but is multiplicatively renormalizable to all orders in leading and next-to-leading logarithms in perturbation theory and so provides a suitable *Ansatz* for the full three-point vertex.

I. INTRODUCTION

The Schwinger-Dyson equations embody the full structure of any field theory[1] and consequently are the natural way to study their dynamics. Unfortunately, being an infinite set of coupled equations they are intractable without some simplifying assumptions. The best known of these is, of course, perturbation theory. However, to study the nonperturbative behavior of any Green's function requires some other, nonperturbative, approximation. The structure of these equations is such that they relate the n-point Green's function to the $(n+1)$-point function; at its simplest, propagators are related to three-point vertices, three-point vertices to four-point couplings and so on, *ad infinitum*. Since practicality dictates that we can only investigate the behavior of a few Green's functions simultaneously,[2] we must find some way to truncate this set of equations. A well-known way to do this is illustrated by considering the photon and fermion propagators $\Delta_{\mu\nu}(p)$ and $S_F(p)$, respectively, in QED. We denote their inverses by $\Pi_{\mu\nu}(p)$ and $S_F^{-1}(p)$ in an obvious manner. If we consider a world with N_f identical fermions, then the Schwinger-Dyson equations are

$$i\Pi_{\mu\nu}(p)=i\Pi^0_{\mu\nu}(p)+\frac{e^2N_f}{(2\pi)^4}\int d^4k\,\gamma^\mu S_F(k)\Gamma^\nu(k,p)S_F(q)\ , \qquad (1)$$

$$iS_F^{-1}(p)=iS_F^{0^{-1}}(p)-\frac{e^2}{(2\pi)^4}\int d^4k\,\gamma^\mu S_F(k)\Gamma^\nu(k,p)\Delta_{\mu\nu}(q)\ , \qquad (2)$$

where $q=k-p$ and Γ^ν is the fermion-boson three-point function and the superscript 0 denotes the bare quantities. Figure 1 illustrates the fermion equation, Eq. (2).

These equations will form a closed system for the two-point functions if we can make an *Ansatz* for the three-point vertex in terms of these. The most trivial of these is the so-called rainbow approximation[3] that treats the vertex Γ^μ as bare, i.e., as just γ^μ. However, this form violates one of the most fundamental of consequences of gauge invariance, namely, the Ward-Takahashi identities. Relevant here is the relation between the full fermion-boson vertex, $\Gamma^\mu(k,p)$, and the fermion propagator expressed in the well-known identity

$$q^\mu\Gamma_\mu(k,p)=S_F^{-1}(k)-S_F^{-1}(p)\ , \qquad (3)$$

where again $q=k-p$. The fact that this relates the full three-point function to the full two-point function means that with its aid we can imagine solving Eqs. (1) and (2) in terms of just two-point functions. How to do this in the case of a massless theory will be discussed in Sec. II. We shall see that the solution to Eq. (3) for the vertex is, of course, not unique and show in Sec. III how multiplicative renormalizability acts as a powerful constraint on the unknown transverse part. In Sec. IV we treat the massive fermion case and summarize our results.

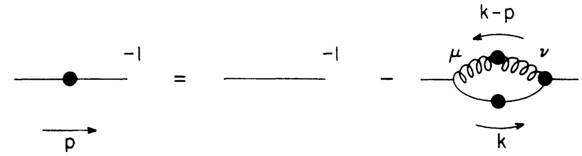

FIG. 1. Schwinger-Dyson equation for the fermion propagator. The straight lines represent fermions and the wavy line the photon. The solid dots indicate full, as opposed to bare, quantities.

II. SOLVING THE WARD IDENTITY—MASSLESS FERMION CASE

From the work of Ball and Chiu,[4] how to solve Eq. (3) for Γ_μ should be well known. However, erroneous solutions continue to appear in the literature. Consequently, we discuss these first since they illustrate how to find the right solution. Clearly, we can write the vertex

$$\Gamma^\mu(k,p)=\left[g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right]\Gamma_\nu+\frac{q^\mu q^\nu}{q^2}\Gamma_\nu$$

as a trivial identity. Then using Eq. (3) this can be reexpressed as

$$\Gamma^\mu(k,p)=\left[g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right]\Gamma_\nu+\frac{q^\mu}{q^2}[S_F^{-1}(k)-S_F^{-1}(p)]\ .$$

This clearly satisfies the Ward-Takahashi identity. Indeed, it continues to appear to do so, even if we replace the first term, i.e., the transverse projection of Γ^μ, by *any other* transverse vector Γ_T^μ, for which $q_\mu\Gamma_T^\mu\equiv 0$, so that[5]

$$\Gamma^\mu(k,p)=\Gamma_T^\mu(k,p)+\frac{q^\mu}{q^2}[S_F^{-1}(k)-S_F^{-1}(p)]\ . \tag{4}$$

Such forms have often been used. However, such *Ansätze* take no account of the fact that the Ward-Takahashi identity has a well-known nonsingular limit when $k\to p$, viz., the original Ward identity:

$$\frac{\partial S_F^{-1}(p)}{\partial p_\mu}=\Gamma^\mu(p,p)\ . \tag{5}$$

It is easy to see that this will not be satisfied by an arbitrary choice of transverse vector Γ_T^μ in Eq. (4), but only by the unique form that ensures the kinematic singularity induced by the $1/q^2$ cancels; i.e., it requires

$$\Gamma_T^\mu(k,p)=\left[g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right]\frac{\partial S_F^{-1}(p)}{\partial p^\nu}$$

for $k\to p$.

The importance of Eq. (5) as the limit of Eq. (3) suggests a natural way to construct a form for the full vertex is to start with its limit Eq. (5). To see the idea let us first consider massless QED for simplicity. Then we can write

$$S_F(p)=\frac{\mathcal{F}(p^2)}{\not p} \tag{6}$$

so that Eq. (5) requires

$$\Gamma^\mu(p,p)=\frac{\partial}{\partial p_\mu}\left[\frac{\not p}{\mathcal{F}(p^2)}\right]$$
$$=\frac{\gamma^\mu}{\mathcal{F}(p^2)}+\frac{\partial}{\partial p^2}\left[\frac{1}{\mathcal{F}(p^2)}\right]2p^\mu\not p\ . \tag{7}$$

It is then natural to write this limit in a k,p symmetric way and so represent the longitudinal part of the vertex by

$$\Gamma_L^\mu(k,p)=\frac{1}{2}\left[\frac{1}{\mathcal{F}(k^2)}+\frac{1}{\mathcal{F}(p^2)}\right]\gamma^\mu+\frac{1}{2}\left[\frac{1}{\mathcal{F}(k^2)}-\frac{1}{\mathcal{F}(p^2)}\right]\frac{(k+p)^\mu(\not k+\not p)}{k^2-p^2}\ . \tag{8}$$

This is the *Ansatz* proposed by Ball and Chiu,[4] which is, of course, free of kinematic singularities. To this we can add any transverse part Γ_T^μ that is also free of kinematic singularities and satisfies both

$$\text{(i)}\ \ q_\mu\Gamma_T^\mu(k,p)=0,\quad \text{(ii)}\ \ \Gamma_T^\mu(p,p)=0\ ,$$

to ensure the full vertex

$$\Gamma^\mu(k,p)=\Gamma_L^\mu(k,p)+\Gamma_T^\mu(k,p) \tag{9}$$

satisfies Eqs. (3) and (5).

For the transverse part, Ball and Chiu enumerate a suitable basis of eight independent tensors. Of these only four can have nonzero coefficients if the fermion in Eq. (2) is to remain massless, which is the case we consider first for simplicity. Then we can write

$$\Gamma_T^\mu(k,p)=\sum_{i=2,3,6,8}\tau_i(k^2,p^2,q^2)T_i^\mu(k,p)\ , \tag{10}$$

where the T_i^μ are given in the Appendix and the dynamical coefficients τ_i ensure that the full vertex is k, p symmetric, but are otherwise arbitrary.

The truncation of the Schwinger-Dyson equations at the level of the two-point functions of Eqs. (1) and (2) means that any *Ansatz* for Γ_T^μ, i.e., for the τ_i in Eq. (10), can only involve the fermion and photon renormalization functions as the only unknown functions. An obvious way to achieve this is to set $\Gamma_T^\mu=0$, i.e., $\tau_i=0$, but it is clear, we have no real criterion for such a simple choice. We shall see in the next section that the requirement of multiplicative renormalizability provides a powerful constraint.

III. MULTIPLICATIVE RENORMALIZABILITY

It has long been known that the renormalization of the Schwinger-Dyson equations is highly nontrivial,[1] except in perturbation theory. In general, a nonperturbative *Ansatz* for some n-point function does not respect the property of multiplicative renormalizability (MR). This problem has been studied in the context of a spectral representation for the fermion propagator in QED by King,[6] for instance, and has recently been highlighted in a way more suited to practical calculations by Brown and Dorey.[7] Here, we will use the criterion of MR to restrict the form of the three-point function. This, of course, means that since the longitudinal part is fixed by the Ward identity, it is the coefficients of the transverse tensors, τ_i, of Eq. (10) that will be constrained.

In general, MR for massless QED is accomplished by introducing the renormalized fermion field $\psi_R=Z_2^{-1/2}\psi$, the photon field $A_R^\mu=Z_3^{-1/2}A^\mu$, and the coupling $e_R=Z_2Z_3^{1/2}e/Z_1$. The renormalized fermion function then satisfies

$$\mathcal{F}_R(p,\mu)=Z_2^{-1}(\Lambda,\mu)\mathcal{F}(p,\Lambda)\ , \tag{11}$$

where Λ is the ultraviolet cutoff introduced to make the loop integrals in Eqs. (1) and (2) finite and μ is the arbitrary renormalization scale.

To see how MR imposes constraints at its simplest, let us follow Brown and Dorey[7] and investigate Eqs. (1) and (2) in the quenched approximation, when we let the number of fermions N_f tend to zero, so that the photon propagator becomes bare. Then $Z_3 \equiv 1$ and the photon propagator is simply

$$\Delta^{\mu\nu}(q)=\frac{1}{q^2}\left[g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right]+\xi\frac{q^\mu q^\nu}{q^4}\ , \tag{12}$$

where ξ is the covariant gauge parameter. We then just have to consider the fermion equation, Eq. (2). In the ultraviolet leading-logarithm approximation in perturbation theory, when

$$\mathcal{F}(p,\Lambda)=1+\alpha A_1 \ln p^2/\Lambda^2+\alpha^2 A_2 \ln^2 p^2/\Lambda^2+\cdots\ ,$$

where $\alpha=e^2/4\pi$ as usual, MR requires $A_2=A_1^2/2$. Indeed, in general it requires that if

$$\mathcal{F}(p,\Lambda)=1+\sum_{n=1}^{\infty}\alpha^n A_n \ln^n p^2/\Lambda^2 \tag{13}$$

then

$$A_n=A_1^n/n! \tag{14}$$

so that

$$\mathcal{F}(p,\Lambda)=\exp[\alpha A_1 \ln(p^2/\Lambda^2)]=\left[\frac{p^2}{\Lambda^2}\right]^{\alpha A_1}\ ,$$

$$Z_2^{-1}(\mu,\Lambda)=\exp[\alpha A_1 \ln(\Lambda^2/\mu^2)]=\left[\frac{\Lambda^2}{\mu^2}\right]^{\alpha A_1}\ ,$$

then

$$\mathcal{F}_R(p,\mu)=\exp[\alpha A_1 \ln(p^2/\mu^2)]=\left[\frac{p^2}{\mu^2}\right]^{\alpha A_1}$$

independent of Λ as MR requires.

As noted by Brown and Dorey, the vertex with $\Gamma_T^\mu=0$ does not give $A_2=A_1^2/2$. Of course, this relation must be satisfied in perturbation theory, so guided by that we compute the $O(\alpha)$ contribution to Γ^μ from Fig. 2. Then again in the leading-logarithm approximation, when $k^2 \gg p^2$, we obtain

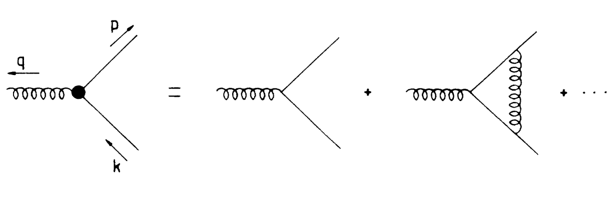

FIG. 2. Lowest-order graphs in the perturbative expansion of the fermion-boson vertex. The lines are as in Fig. 1.

$$\Gamma^\mu_{\rm pert}(k,p)=\gamma^\mu\left[1-\frac{\alpha\xi}{4\pi}\ln\left[\frac{k^2}{\Lambda^2}\right]\right] -\frac{\alpha}{4\pi}[\not p\gamma^\mu \not k+(\xi-1)k^\mu \not p]\frac{1}{k^2}\ln\left[\frac{k^2}{p^2}\right]\ . \tag{15}$$

When this is substituted into Eq. (2), this, of course, gives $O(\alpha^2)$ corrections to $\mathcal{F}(p,\Lambda)$ with $A_2=A_1^2/2$. To see the form of the effective transverse part of the vertex implied by Eq. (15), let us subtract from $\Gamma^\mu_{\rm pert}$ the longitudinal part of Eq. (8) and dropping the ξ-independent terms that vanish under the integral of Eq. (2), we have

$$\Gamma^\mu_{T,\rm pert}=\frac{\alpha\xi}{8\pi}\left[-\gamma^\mu-\frac{2k^\mu \not p}{k^2}+\frac{(k+p)^\mu(\not k+\not p)}{k^2-p^2}\right]\ln\left[\frac{k^2}{p^2}\right] \tag{16}$$

which, in the leading-logarithm approximation of $k^2 \gg p^2$, can be written as

$$\Gamma^\mu_{T,\rm pert}\simeq-\frac{\alpha\xi}{8\pi}T_6^\mu\frac{\ln(k^2/p^2)}{k^2}\ . \tag{17}$$

To this order in perturbation theory we note that since

$$\mathcal{F}(k^2)=1+\frac{\alpha\xi}{4\pi}\ln(k^2/\Lambda^2)+\cdots \tag{18}$$

the forms of Eq. (17) can be written as

$$\Gamma^\mu_T(k,p)=\frac{1}{2}\left[\frac{1}{\mathcal{F}(k^2)}-\frac{1}{\mathcal{F}(p^2)}\right]\frac{T_6^\mu}{k^2} \tag{19}$$

as a possible form for the full transverse part for $k^2 \gg p^2$. Of course, Γ^μ has to be symmetric in k, p, while T_6^μ is antisymmetric. Consequently, the factor of k^2 in the denominator must be the large-k^2 limit of an appropriate symmetric, kinematic-singularity free function of k, p, we denote by $d(k,p)$.

Now the first thing to note is that forming a full vertex from the present Eq. (19) and Eqs. (8) and (9) and substituting into Eq. (2) gives a leading-logarithm result for $\mathcal{F}$ that satisfies MR not just to $O(\alpha)$, as we have already guaranteed, but to *all* orders; i.e., Eq. (14) is satisfied. Furthermore, if the corrections to the k^2 in the denominator of Eq. (19) are $O(p^2/k^2)$, then straightforward, but tedious, algebra shows Eq. (19) gives a full vertex that satisfies MR to all orders in next-to-leading logarithms, too: that is, if

$$\mathcal{F}(p,\Lambda)=1+\sum_{n=1}^{\infty}\alpha^n A_n \ln^n(p^2/\Lambda^2) +\sum_{n=1}^{\infty}\alpha^n B_n \ln^{n-1}(p^2/\Lambda^2)+\cdots \tag{20}$$

then not only does $A_n=A_1^n/n!$ [Eq. (14)], but

$$B_n=A_{n-2}B_2-A_{n-1}B_1\ . \tag{21}$$

To show that we have indeed found a satisfactory form for the transverse part of the full vertex, Eq. (19) we consider the massive fermion case.

IV. MASSIVE FERMIONS

With the fermion propagator now given by

$$S_F(p)=\frac{\mathcal{F}(p^2)}{\not p-\Sigma(p^2)} \tag{22}$$

rather than Eq. (6), the longitudinal vertex that satisfies the Ward identity, Eqs. (3) and (5), is obtained by adding to Eq. (8) the piece[4]

$$\Gamma^\mu_{\text{mass},L}(k,p)=-\left[\frac{\Sigma(k^2)}{\mathcal{F}(k^2)}-\frac{\Sigma(p^2)}{\mathcal{F}(p^2)}\right]\frac{(k+p)^\mu}{k^2-p^2} . \tag{23}$$

Now Eq. (2) becomes two equations, one for $1/\mathcal{F}(p^2)$ and the other for $\Sigma(p^2)/\mathcal{F}(p^2)$, readily projected by tracing Eq. (2) with $\not p$ and with the unit matrix, respectively. To see how we may have to alter the transverse part to attain MR with a mass term, let us again be guided by perturbation theory to $O(\alpha)$ in leading logarithms. We find from the graphs of Fig. 2 that we must add to Eq. (15), for $k^2 \gg p^2 \gg m^2$:

$$\Gamma^\mu_{\text{mass,pert}}(k,p)=\frac{\alpha}{4\pi}(3+\xi)\frac{k^\mu}{k^2}m\ln\left[\frac{k^2}{p^2}\right] , \tag{24}$$

where m is the zeroth-order value for $\Sigma(p^2)$. This is exactly the form that comes from Eq. (23), since, to $O(\alpha)$,

$$\Sigma(p^2)=m\left[1-\frac{3\alpha}{4\pi}\ln\left[\frac{p^2}{\Lambda^2}\right]\right] , \tag{25}$$

and so we have to add nothing to our *Ansatz* for Γ^μ_T in the massive case to lowest order. However, we now have to ensure the appropriate mass renormalization, so that

$$\Sigma_R(p,\mu)=Z_m^{-1}(\mu,\Lambda)\Sigma(p,\Lambda) . \tag{26}$$

Again, if the leading and next-to-leading logarithms in the expansion of $\Sigma(p,\Lambda)$ are C_n and D_n, respectively, analogously to A_n,B_n in Eq. (20), then these in turn must satisfy analogues of Eqs. (14) and (21) for MR. Indeed, our vertex constructed using Eq. (9) for the longitudinal parts of Eqs. (8) and (23) and the transverse part of Eq. (19) does, when substituted into Eq. (2) for the fermion propagator, make this multiplicatively renormalizable to all orders in both leading and next-to-leading logarithms for both $\mathcal{F}(p,\Lambda)$ and $\Sigma(p,\Lambda)$. This is provided that the corrections to the k^2 in the denominator of Eq. (19) are again of $O(p^2/k^2)$. Since this factor must be analytic and symmetric in k, p, this suggests a form such as

$$d(k,p)=\frac{(k^2-p^2)^2+[\Sigma(k^2)^2+\Sigma(p^2)^2]^2}{k^2+p^2} \tag{27}$$

which gives a transverse component free of kinematic singularities for real k^2, p^2. Thus our *Ansatz* for the full vertex that depends only on the fermion renormalization functions is

$$\Gamma^\mu(k,p)=\frac{1}{2}\left[\frac{1}{\mathcal{F}(k^2)}+\frac{1}{\mathcal{F}(p^2)}\right]\gamma^\mu+\frac{1}{2}\left[\frac{1}{\mathcal{F}(k^2)}-\frac{1}{\mathcal{F}(p^2)}\right]\frac{(k+p)^\mu(\not k+\not p)}{k^2-p^2}-\left[\frac{\Sigma(k^2)}{\mathcal{F}(k^2)}-\frac{\Sigma(p^2)}{\mathcal{F}(p^2)}\right]\frac{(k+p)^\mu}{k^2-p^2}$$
$$+\frac{1}{2}\left[\frac{1}{\mathcal{F}(k^2)}-\frac{1}{\mathcal{F}(p^2)}\right]\frac{\gamma^\mu(k^2-p^2)-(k+p)^\mu(\not k-\not p)}{d(k,p)} . \tag{28}$$

The use of this form is presently being investigated in an extensive nonperturbative study of three-dimensional QED.[8]

If we had not chosen to be guided by the perturbative calculation of Γ^μ_T explicitly from Fig. 2, but had just sought an effective transverse vertex given by Eq. (10), then in the leading-logarithm approximation when the functions τ_i are taken to be constants t_i times powers of k^2 to ensure each term in Γ^μ_T is dimensionless and a factor of $-(\alpha\xi/4\pi)\ln(k^2/p^2)$ for $i=2$, 3, 6, 8 and $(\alpha m/4\pi)(3+\xi)\ln(k^2/p^2)$ for $i=1$, 4, 5, 7, we would find that the $O(\alpha)$ component of Γ^μ_T gives the contributions to $\mathcal{F}$ and to Σ to $O(\alpha^2)$ listed in Table I. These contributions must sum together with the known component from Eqs. (8) and (23) to ensure MR. Of course, these transverse parts must thereby sum to the same result as the total transverse component of perturbation theory, which is also given in Table I. Thus MR requires

TABLE I. The leading logarithm (LL) contributions to the fermion renormalization functions to $O(\alpha^2)$ from the $O(\alpha)$ components of the transverse part of the vertex given by the tensors T_i^μ ($i=1,\ldots,8$) of the Appendix, together with the total perturbative answer for these transverse parts.

Tensor i	LL contributions to $1/\mathcal{F}(p^2)$ $\times-\frac{3\alpha^2}{128\pi^2}\ln^2\frac{p^2}{\Lambda^2}$	LL contributions to $\Sigma(p^2)/\mathcal{F}(p^2)$ $\times-\frac{3\alpha^2 m}{64\pi^2}\ln^2\frac{p^2}{\Lambda^2}$
1		
2	$-\xi$	
3	-2ξ	$+2\xi$
4		
5		$+2(3+\xi)$
6	$+2\xi$	$+2\xi$
7		$-(3+\xi)$
8	$+2\xi$	
Pert. answer	$+\xi$	$+\xi$

$$(-t_2-2t_3+2t_6+2t_8)\xi=\xi ,$$

$$2(t_3+t_6)\xi+(2t_5-t_7)(3+\xi)=\xi .$$

These imply

$$t_5=t_7/2,\quad t_3+t_6=\tfrac{1}{2},\quad -\frac{t_2}{4}+t_6+\frac{t_8}{2}=\frac{1}{2} .$$

The only one tensor that can satisfy these is T_6 with $t_6=\frac{1}{2}$ with the other $t_i=0$, as given by Eq. (19), making Eq. (28) a most suitable, straightforward form for the full vertex.

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APPENDIX

We collect here the formulas for the eight basis vectors T_i^μ for the transverse vertex as defined by Ball and Chiu.[4] As in Fig. 2, the momenta of the fermion legs are k, p and that of the boson line is $q=k-p$:

$$T_1^\mu=p^\mu(k\cdot q)-k^\mu(p\cdot q),\quad T_2^\mu=T_1^\mu(\not k+\not p) ,$$

$$T_3^\mu=q^2\gamma^\mu-q^\mu\not q,\quad T_4^\mu=T_1^\mu p^\nu k^\rho\sigma_{\nu\rho} ,$$

$$T_5^\mu=\sigma^{\mu\nu}q_\nu ,$$

$$T_6^\mu=\gamma^\mu(k^2-p^2)-(k+p)^\mu(\not k-\not p) ,$$

$$T_7^\mu=\tfrac{1}{2}(k^2-p^2)[\gamma^\mu(\not k+\not p)-p^\mu-k^\mu]+(k+p)^\mu p^\nu k^\rho\sigma_{\nu\rho} ,$$

$$T_8^\mu=-\gamma^\mu p^\nu k^\rho\sigma_{\nu\rho}+p^\mu\not k-k^\mu\not p ,$$

where $\sigma_{\mu\nu}=\frac{1}{2}[\gamma_\mu,\gamma_\nu]$.

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