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Conformal invariance beyond the leading order in the supersymmetric nonlinear σ model with dilaton

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We calculate the $O(\alpha'^3)$ contributions to the renormalization-group β functions in the $N=1$ supersymmetric σ model with a dilaton. At this order both metric and dilaton β functions are found to depend nontrivially on the dilaton field and vanish if the dilaton satisfies $\nabla_\mu\nabla_\nu\phi=0$. By employing the Curci-Paffuti relation it is shown that such dilaton backgrounds in Ricci-flat spaces $R_{\mu\nu}=0$ satisfy the conformal invariance conditions up to this order. The particular class of Ricci-flat, compact, and orientable manifolds naturally emerge as appropriate internal-space configurations consistent with local scale invariance. We further explore the cosmological consequences of these dilaton configurations. In a Robertson-Walker four-dimensional background we find all dilatons satisfying $\nabla_\mu\nabla_\nu\phi=0$. Except for the constant and the time-dependent dilaton $\phi(t)=-2\ln t+\lambda$ whose cosmological implications have been already discussed in the literature, additional solutions are found. These may be of relevance beyond leading order and for nonvanishing background values for the antisymmetric tensor $B_{\mu\nu}$. For these solutions, also the cosmic scale factor is at most linear in time therefore giving rise to either a static or a linearly expanding (contracting) universe.

I. INTRODUCTION

Two-dimensional σ models still continue to attract the interest of particle physicists because of their close connection with string theories. Their action represents the motion of the string in nontrivial backgrounds for the target-space metric $G_{\mu\nu}$, antisymmetric tensor $B_{\mu\nu}$, and dilaton ϕ which are known to be the massless modes of the string excitations. The low-energy effective action of these modes, derived from the string S-matrix dynamics, yields equations of motion that can be derived from conformal invariance of the two-dimensional σ model.[1-5] Conformal invariance is equivalent to the vanishing of the reparametrization-invariant β functions $\bar\beta_{\mu\nu}$, $\bar\beta_B$, $\bar\beta_\phi$ which up to diffeomorphisms are the renormalization β functions $\beta_{\mu\nu}$, β_B, β_ϕ, respectively.[4,5] The nonvanishing component of the energy-momentum tensor generates the Virasoro algebra whose central charge $\tilde\beta_\phi$ turns out to depend on $\bar\beta_\phi$, $\bar\beta_{\mu\nu}$. The constancy of $\tilde\beta_\phi$ is guaranteed by the Curci-Paffuti[5,6] equation that relates the derivative of $\tilde\beta_\phi$ to $\bar\beta_{\mu\nu}$, $\bar\beta_B$. At the fixed points of $\bar\beta_{\mu\nu}$, $\bar\beta_B$ therefore, $\tilde\beta_\phi$ consistently can be taken equal to zero. Conformal invariance is intimately connected with the consistent quantization of string theory[7] and to this effect calculations of the β functions have been carried out in numerous works in the case of the ordinary nonlinear σ model.[7,8] In the supersymmetric σ model $\beta_{\mu\nu}$ has been calculated up to four-loop order[9-11] when the dilaton field is absent. At the fourth-loop order $\beta_{\mu\nu}$ was found not to vanish even on Ricci-flat spaces.[11] The introduction of the dilaton field in the supersymmetric case is rather nontrivial making matter interacting with the two-dimensional geometry (zweibein). Explicit calculations of the β functions in the presence of the dilaton field in the $N=1$ supersymmetric σ model do not exist, to our knowledge, beyond the two-loop order. Because of the significance of the β functions we undertake this problem and in this work we present a three-loop calculation of the renormalization-group β functions in the $N=1$ supersymmetric nonlinear σ model in the presence of metric and dilaton fields. We will use component formalism since the mixing of the zweibein and the matter multiplet makes it rather difficult to use the otherwise more economical language of the superfields.

This paper is organized as follows. In Sec. II we briefly discuss the role of the auxiliary fields of the supersymmetric nonlinear σ model and we calculate the $O(\alpha'^3)$ corrections to both metric $\beta_{\mu\nu}$ and dilaton β_ϕ renormalization-group β functions. In Sec. III we discuss the conformal invariance conditions and show that the special backgrounds $R_{\mu\nu}=0$, $\nabla_\mu\nabla_\nu\phi=0$ which are fixed points of the renormalization-group β functions satisfy the conformal invariance conditions at this order. We argue that these are the simplest possible backgrounds having this property. In Sec. IV we carry on to discuss the consequences of these conditions for Robertson-Walker (RW) cosmologies. We find all possible dilaton solutions satisfying $\nabla_\mu\nabla_\nu\phi=0$ and discuss their implications for cosmology. Finally we end up with the conclusions.

II. THREE-LOOP β FUNCTIONS

In order to treat properly the supersymmetric σ model in the presence of a dilaton field $\phi(x)$ one has to take into account the auxiliary fields of both the string multiplet and the zweibein multiplet Λ^μ and H, respectively. These mix with each other and the auxiliary field part of the action is given by[12]

$$4\pi\alpha' S'_{\rm aux}=\int d^2x[G_{\mu\nu}\Lambda^\mu\Lambda^\nu+2\alpha'\phi_{,\lambda}\Lambda^\lambda H+\bar\lambda^\mu\lambda^\nu(\Gamma^\lambda_{\mu\nu}\Lambda_\lambda+\alpha'\phi_{,\mu\nu}H)]\ . \quad (1)$$

H is the auxiliary field associated with the multiplet accommodating the conformal factor σ in the gauge $g_{\alpha\beta}=n_{\alpha\beta}e^{2\sigma}$. Elimination of Λ^μ, H brings down a four-fermion dilaton-dependent coupling in addition to the well-known Riemann tensor coupling. This is given by

$$T_{\mu\nu\kappa\lambda}=-\tfrac{1}{12}R_{\mu\nu\kappa\lambda}+A_{\mu\nu\kappa\lambda}(\phi)\ , \quad (2)$$

where

$$A_{\mu\nu\kappa\lambda}(\phi)=\frac{1}{12(\nabla\phi)^2}[(\nabla_\mu\nabla_\kappa\phi)(\nabla_\nu\nabla_\lambda\phi)-(\kappa\leftrightarrow\lambda)]\ .$$

Both tensors $R_{\mu\nu\kappa\lambda}$ and $A_{\mu\nu\kappa\lambda}$ have the same weight under rigid conformal transformations $G_{\mu\nu}\to\Omega G_{\mu\nu}$. $R_{\mu\nu\kappa\lambda}$ and $A_{\mu\nu\kappa\lambda}$ have the same antisymmetry/symmetry properties as far as their indices are concerned. Also $A_{\mu\nu\kappa\lambda}$ possesses the cyclicity property $A_{\mu\nu\kappa\lambda}+A_{\mu\kappa\lambda\nu}+A_{\mu\lambda\nu\kappa}=0$, the same as the Riemann tensor, but it does not satisfy the Bianchi identity. The appearance of this coupling is not hard to understand. From Eq. (1) one can find that the two-point functions of the auxiliary fields are

$$\langle\Lambda_\mu\Lambda_\nu\rangle=G_{\mu\nu}-\frac{\phi_{,\mu}\phi_{,\nu}}{(\nabla\phi)^2}\ ,$$
$$\alpha'\langle\Lambda_\mu H\rangle=\frac{\phi_{,\mu}}{(\nabla\phi)^2}\ , \quad (3)$$
$$\alpha'^2\langle HH\rangle=\frac{1}{(\nabla\phi)^2}\ ,$$

as can be found by inverting their mixing matrix. Then the graph shown in Fig. 1 yields a result for the four-fermion coupling of the form

$$(\bar\lambda\lambda)^2\left[\left[G-\frac{\phi^2_{,}}{(\nabla\phi)^2}\right]\Gamma^2+\frac{(\Gamma\phi,)\phi_{,,}}{(\nabla\phi)^2}+\frac{\phi^2_{,,}}{(\nabla\phi)^2}\right]\ , \quad (4)$$

where we have suppressed for simplicity all indices. The Γ^2G term in Eq. (4) with the $\partial\Gamma$ already present in the supersymmetric σ-model action yields the Riemann tensor, and the rest which depends on ϕ gives the $A_{\mu\nu\kappa\lambda}$ coupling. This is of course equivalent to eliminating

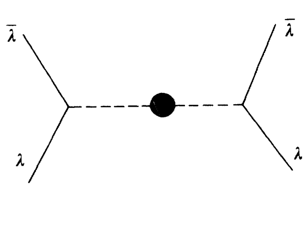

FIG. 1. Graph giving rise to a four-fermion coupling. The dashed line denotes auxiliary fields.

straightforwardly the auxiliary fields via their equations of motions in the action (1). The $A_{\mu\nu\kappa\lambda}$ term has been shown[12] to contribute to the β function of the metric at the three-loop order and hence it has to be taken into account in the conformal-invariance conditions. Its presence may impose stringent restrictions on the dilaton background if conformal-invariance conditions are demanded but for this to be known a complete three-loop calculation of the β functions should be performed. Such a calculation can be carried out using covariant techniques developed in Ref. 13 which have been also applied for the calculation of the β functions of the ordinary σ model.[14] These use the propagation of strings in curved backgrounds and are suitable for extracting covariant results directly in configuration space. In this work we found it convenient to work in momentum space, and in the conformal gauge $g_{\alpha\beta}=\eta_{\alpha\beta}e^{2\sigma}$, but our results can be converted to configuration space and in a covariant form as we shall see.

Our calculation, especially for the dilaton β function, is greatly facilitated if we rescale the bosonic quantum fields ξ^a as $\xi^a=e^{\epsilon\sigma/2}\tilde\xi^a$ while for fermions we rescale as $\lambda^\mu=e^{(\epsilon-1)\sigma/2}\tilde\lambda^\mu$. Such a redefinition makes the bilinear in the quantum field terms be brought into the form

$$-\frac{1}{4\pi\alpha'}\int d^{2-\epsilon}x\, n_{ab}\left[\tilde\xi^a\partial^2\tilde\xi^b+\frac{\epsilon}{4(1-\epsilon)}e^{2\sigma}R^{(2)}\tilde\xi^a\tilde\xi^b-\frac{i}{2}\bar{\tilde\lambda}^a\partial\!\!\!/\tilde\lambda^b\right]\ . \quad (5)$$

The signature of the d-dimensional world-sheet flat metric is $(+,-,\ldots,-)$ and $R^{(2)}$ is given by $R^{(2)}=(1-\epsilon)e^{-2\sigma}[2\partial^2\sigma-\epsilon(2\sigma)^2]$ where $\epsilon\equiv 2-d$. With such a rescaling all interaction terms in the normal coordinate expansion have a σ dependence of the form $e^{\kappa\epsilon\sigma}$ (κ=integer or half-integer) without involving derivatives of σ. Then a contribution to the dilaton β function may arise from these factors through quadratically divergent graphs which when converted to configuration space yield among other things $\epsilon\partial^2\sigma$ which is essentially $\epsilon R^{(2)}$. Thus these σ dependences may in principle give rise to contributions to the dilaton β-function β_ϕ provided that the corresponding graph is quadratically divergent and carries at the same time a $1/\epsilon^2$ pole. Another contribution to β_ϕ may arise from the $\epsilon R^{(2)}\tilde\xi\tilde\xi$ term appearing in Eq. (5) which is of order ϵ so that when inserted in a bosonic line of a given graph may give a simple pole multiplying $R^{(2)}$ provided that the graph itself carries $1/\epsilon^2$ pole. These are the sources from which a contribution to the dilaton β function may arise as well as graphs involving vertices of the type $\nabla_{\mu_1}\cdots\nabla_{\mu_n}\phi\tilde\xi^{\mu_1}\cdots\tilde\xi^{\mu_n}R^{(2)}$ arising from the normal coordinate expansion[15] of the dilaton term. Before embarking on the details of our calculation we should remark that the redefinitions performed on the fields are ϵ dependent. Therefore a counterterm which in the initial language in terms of the ξ^a,λ^a fields has only poles, when expressed in terms of the $\tilde\xi^a,\tilde\lambda^a$ fields may carry some finite parts from the expansion of the prefactors $e^{\kappa\epsilon\sigma}$ occurring at the vertices which should be taken into account. These finite contributions cause no additional calculational problem since they are determined in a unique way using the fact that the counterterms in the general gauge are covariant expressions[13] of the fields involved. Regarding the calculation itself we should remark that we need calculate only graphs involving at least one fermion loop since the corresponding pure bosonic contributions have been already known up to three-loop order.[8,14]

During the process of the calculation we will consider first graphs having two (∂x) fields as external lines which are logarithmically divergent. These certainly contribute to the $\beta_{\mu\nu}$ function if they have $1/\epsilon$ pole but are not the only ones. Also linearly divergent graphs with only one (∂x) as an external leg or quadratically divergent ones without (∂x)'s at all may in principle contribute to $\beta_{\mu\nu}$. For instance, a contribution $(1/\epsilon)\widetilde{T_i\partial_\alpha x^i}(Q)Q^\alpha\tilde{S}(Q')\delta^{(d)}(Q+Q')$ of a linearly divergent graph, with $\tilde{F}(Q)$ denoting Fourier transform of $F(x)$, when converted to configuration space yields $(1/\epsilon)T_i\nabla_j S(\partial x^i\cdot\partial x^j)$ which signals a nonvanishing contribution to the metric β function $\beta_{\mu\nu}$. For the contributions of the various graphs we use supersymmetric dimensional regularization[16] (SDR). In the process of the presentation of the various contributions we omit graphs that have led to either finite results or graphs that when their subdivergences are substracted out lead to no simple pole. Graphs involving tadpoles, for instance, have been found to belong to such a class (see also Ref. 14).

In Fig. 2 we show all three loop graphs giving rise to a simple pole with external (∂x) lines. These contribute only to the metric β function. Each of the graphs shown contains at least a fermion loop and is accompanied by its countergraph(s) which involves its lower-order subdiver-

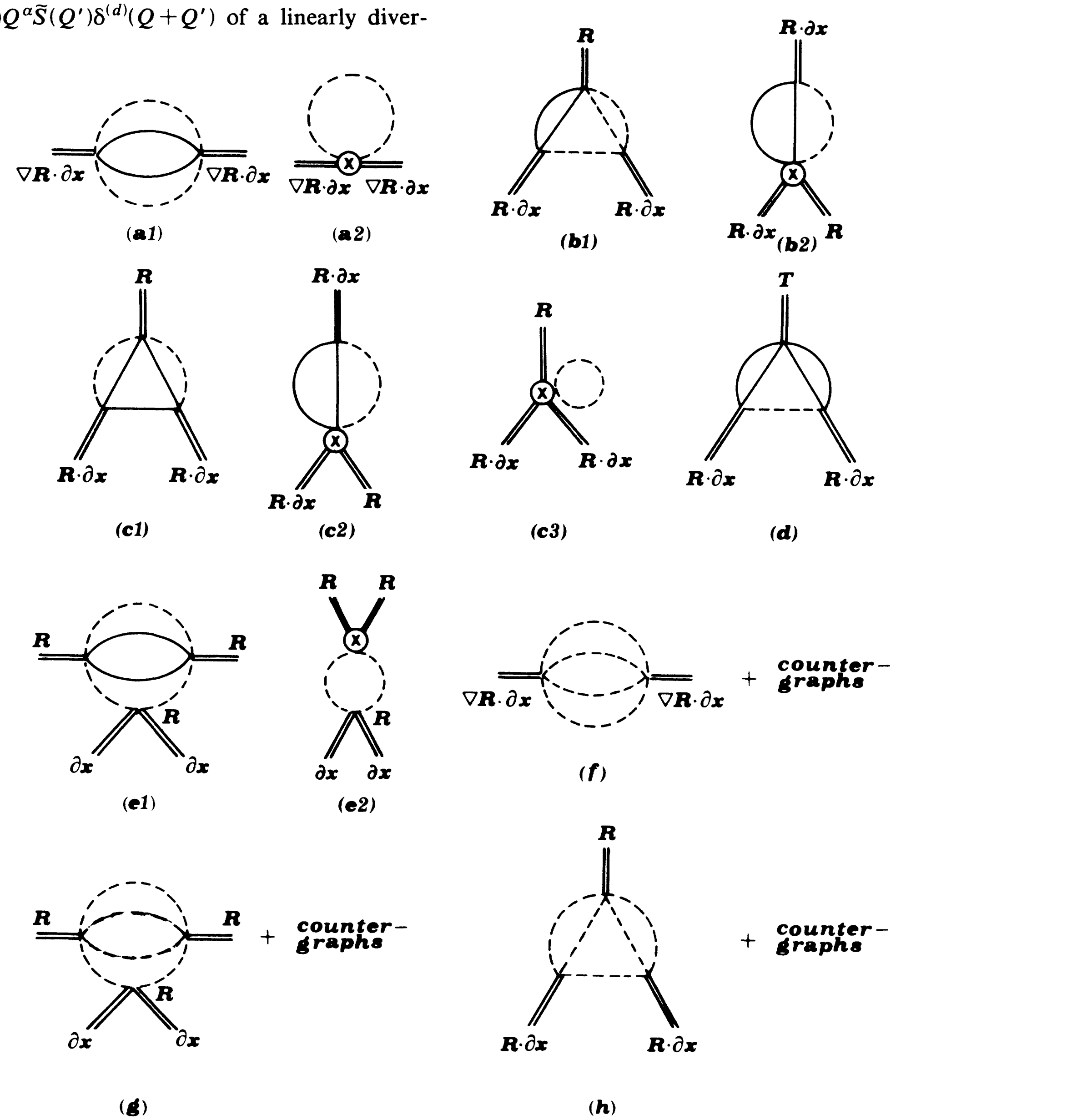

FIG. 2. Three-loop graphs contributing to $\beta_{\mu\nu}$ with external ∂x lines. Solid (dashed) lines denote fermions (bosons). Graphs (f), (g), and (h) are purely bosonic. Countergraphs with the lower-order counterterms marked by ⊗, whenever, contribute stand to the right of the main graph labeled by the same letter.

gences. However the last three graphs, Figs. 2(f)–2(h), are purely bosonic and are there for demonstrating how the supersymmetric cancellation takes place. The contributions of all graphs of Fig. 2 are displayed in Table I. Particularly for the first set of graphs of Fig. 2, namely, a_1, a_2 we displayed explicitly their separate contributions in the conformal gauge to show that their sum a_1+a_2 can be put in a covariant form (see Table I). We remind the reader that finite σ contributions, which in order to save space we do not exhibit here, were necessary for making all the expressions shown in Table I covariant. Figures 2(f)–2(h) are purely bosonic graphs yielding simple poles canceling all but the dilaton-dependent part of the graph of Fig. 2(d) (see Table I). The mass m appearing within the logarithms in the expressions of Table I is an infrared regulator. Actually this mass regulator can be inserted only in one of the Feynman propagators since two of the momentum integrations can be made to be free of infrared singularities (Foakes and Mohammedi in Ref. 8). Note the cancellation of these infrared singularities between graphs a_1 and a_2. Cancellation of the infrared singularities takes place in exactly the same manner between all graphs and their corresponding countergraphs of Fig. 2. Linearly divergent graphs with only one (∂x) at the external leg exist but they yield double poles and hence are not shown. Quadratically divergent graphs, without (∂x)'s, which may contribute to $\beta_{\mu\nu}$ will be considered later when dealing with the dilaton β function. All graphs shown in Fig. 2 yield a vanishing contribution to $\beta_{\mu\nu}$ in the absence of the dilaton field in which case the dilaton-dependent four-fermion coupling is absent, too. As we shall see this also holds for the additional contributions to $\beta_{\mu\nu}$ arising from the three-loop quadratically divergent graphs entailing to vanishing $\beta_{\mu\nu}$, at this order, when the dilaton is absent. All this is of course known from earlier publications (Ref. 11, Ketov in Ref. 10). The reason for presenting the details of our calculation is in order to show that the particular methodology we follow in momentum space and in terms of field components works perfectly well for the calculation of the metric β function and soon we intend to employ these techniques for a calculation of the dilaton β function. The presence of the dilaton field due to its mixing with the two-dimensional geometry complicates the situation and perhaps working with superfields may prove not to be as easy a task as in the dilaton-free case.

The dilaton β-function β_ϕ in the $N=1$ supersymmetric σ model, with dilaton present, has not been explicitly calculated beyond the two-loop order, although in the ordinary σ model β_ϕ is known up to the four-loop order (α'^3).[8] Graphs giving rise to nonvanishing nth-order contribution to β_ϕ can be divided into two main classes. In the first we include the quadratically divergent graphs involving $e^{\kappa\epsilon\sigma}$ factors from the vertices and also those involving an $\epsilon R^{(2)}\xi\xi$ insertion in a bosonic line [see Eq. (5)]. In both cases we need to have a graph carrying an $1/\epsilon^2$ pole as has been explained earlier. Also in that case we need to consider an $(n+1)$th-loop graph for an $O(\alpha'^n)$ contribution to β_ϕ. The second class of graphs is that involving only one vertex of the type $\alpha'(\nabla\cdots\nabla\phi)\xi\cdots\xi R^{(2)}$ stemming from the normal coordinate expansion of the dilaton term which already carries a factor α'. Therefore for the $O(\alpha'^n)$ contribution to β_ϕ we need to consider an nth-loop graph having $1/\epsilon$ pole. However this latter class of graphs needs no calculation at all using results from the three-loop calculations of the metric β function in the absence of a dilaton field. The reason is that a graph with a $\alpha'(\nabla\cdots\nabla\phi)\xi\cdots\xi R^{(2)}$

TABLE I. Contributions of graphs displayed in Fig. 2. All terms should be multiplied by $1/4\pi\alpha'$.

Graph in Fig. 2	Contribution
(a1)	$\alpha'^3\left[\frac{1}{12\epsilon^2}+\frac{1}{24\epsilon}-\frac{\ln m^2}{8\epsilon}\right]e^{2\epsilon\sigma}R_{abm(c;d)}R^{ab}{}_n{}^{(c;d)}\vartheta_\alpha\chi^m\vartheta^\alpha\chi^n$
(a2)	$\alpha'^3\left[-\frac{1}{4\epsilon^2}+\frac{\ln m^2}{8\epsilon}\right]R_{abm(c;d)}R^{ab}{}_n{}^{(c;d)}\vartheta_\alpha\chi^m\vartheta^\alpha\chi^n$
(a1+a2)	$\alpha'^3\left[-\frac{1}{6\epsilon^2}+\frac{1}{24\epsilon}\right]\sqrt{-g}\; g^{\alpha\beta}R_{abm(c;d)}R^{ab}{}_n{}^{(c;d)}\vartheta_\alpha\chi^m\vartheta_\beta\chi^n$
(b1+b2)	$\alpha'^3\left[\frac{1}{12\epsilon^2}-\frac{1}{8\epsilon}\right]\sqrt{-g}\; g^{\alpha\beta}R_{abcd}R_{mf}{}^{ab}R_n{}^{fcd}\vartheta_\alpha\chi^m\vartheta_\beta\chi^n$
(c1+c2+c3)	$\alpha'^3\left[-\frac{1}{6\epsilon^2}+\frac{1}{6\epsilon}\right]\sqrt{-g}\; g^{\alpha\beta}R_{abcd}R_m{}^{caf}R_n{}^{db}{}_f\vartheta_\alpha\chi^m\vartheta_\beta\chi^n$
(d)	$\alpha'^3\left[-\frac{1}{\epsilon}\right]\sqrt{-g}\; g^{\alpha\beta}T_{abcd}R_{mf}{}^{ab}R_n{}^{fcd}\vartheta_\alpha\chi^m\vartheta_\beta\chi^n$
(e1+e2)	$\alpha'^3\left[-\frac{1}{6\epsilon^2}+\frac{1}{8\epsilon}\right]\sqrt{-g}\; g^{\alpha\beta}R^a{}_{fcd}R^{bfcd}R_{mabn}\vartheta_\alpha\chi^m\vartheta_\beta\chi^n$
(f+g+h)	Their $1/\epsilon$ pole cancels the corresponding pole of graphs (a1)–(e2) of Fig. 2 except the dilaton-dependent piece of the contribution of graph (d)

tex exhibits $1/\epsilon$ pole if and only if the same graph but with the external $\alpha'\nabla_{k_1}\cdots\nabla_{k_n}\phi R^{(2)}$ replaced by $F_{ijk_1\cdots k_n}(R)\partial x^i\cdot\partial x^j$ has also a simple pole. In this expression $F_{ij,\cdots k_n}(R)$ stands for a function of Riemann tensor and its derivatives. For instance when $n=3$ we have a dilaton term

$$\frac{\alpha'}{6}\sqrt{-g}\;R^{(2)}\nabla_{\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\phi\xi^{\mu_1}\xi^{\mu_2}\xi^{\mu_3}$$

and also a vertex

$$\frac{\sqrt{-g}}{6}R_{i\mu_1 j\mu_2;\mu_3}\partial x^i\cdot\partial x^j\xi^{\mu_1}\xi^{\mu_2}\xi^{\mu_3}\;.$$

Previous calculations in the dilaton-free case,[11] as well as our previous considerations (see Fig. 2), have shown that such a simple pole for these graphs does not exist up to the three-loop order. This proves therefore that the aforementioned vertices in the supersymmetric σ model yield vanishing contributions to β_ϕ up to this loop order and hence are not considered. Thus we will consider only graphs of the first class, that is, with $e^{\kappa\epsilon\sigma}$ and $R^{(2)}\tilde{\xi}\tilde{\xi}$ insertions.

In Figs. 3(a1)–3(c) we present three-loop graphs involving at least one fermion loop yielding nonvanishing $O(\alpha'^2)$ contributions to β_ϕ and $O(\alpha'^3)$ to $\beta_{\mu\nu}$. The results from the calculation of these graphs are shown in Table II. All expressions in this table have been put in a covariant form as was done for the graphs of Fig. 2. As regards the dilaton β function at this loop order we have found a contribution, from graphs 3(a1)–3(c), that cancels the corresponding bosonic one of Figs. 3(d) and 3(e) which is (see Ref. 8)

$$\frac{1}{4\pi\alpha'}\frac{\alpha'^2}{48}\sqrt{-g}\;R^{(2)}R^2_{abcd}\;.$$

Thus we have found explicitly that the $O(\alpha'^2)$ contribution to β_ϕ vanishes in the $N=1$ supersymmetric σ model, a result not unexpected.[17] Note that graphs 3(a1), 3(a2), and 3(c) contribute to the metric β function. However their $\beta_{\mu\nu}$ contributions which depend only on the

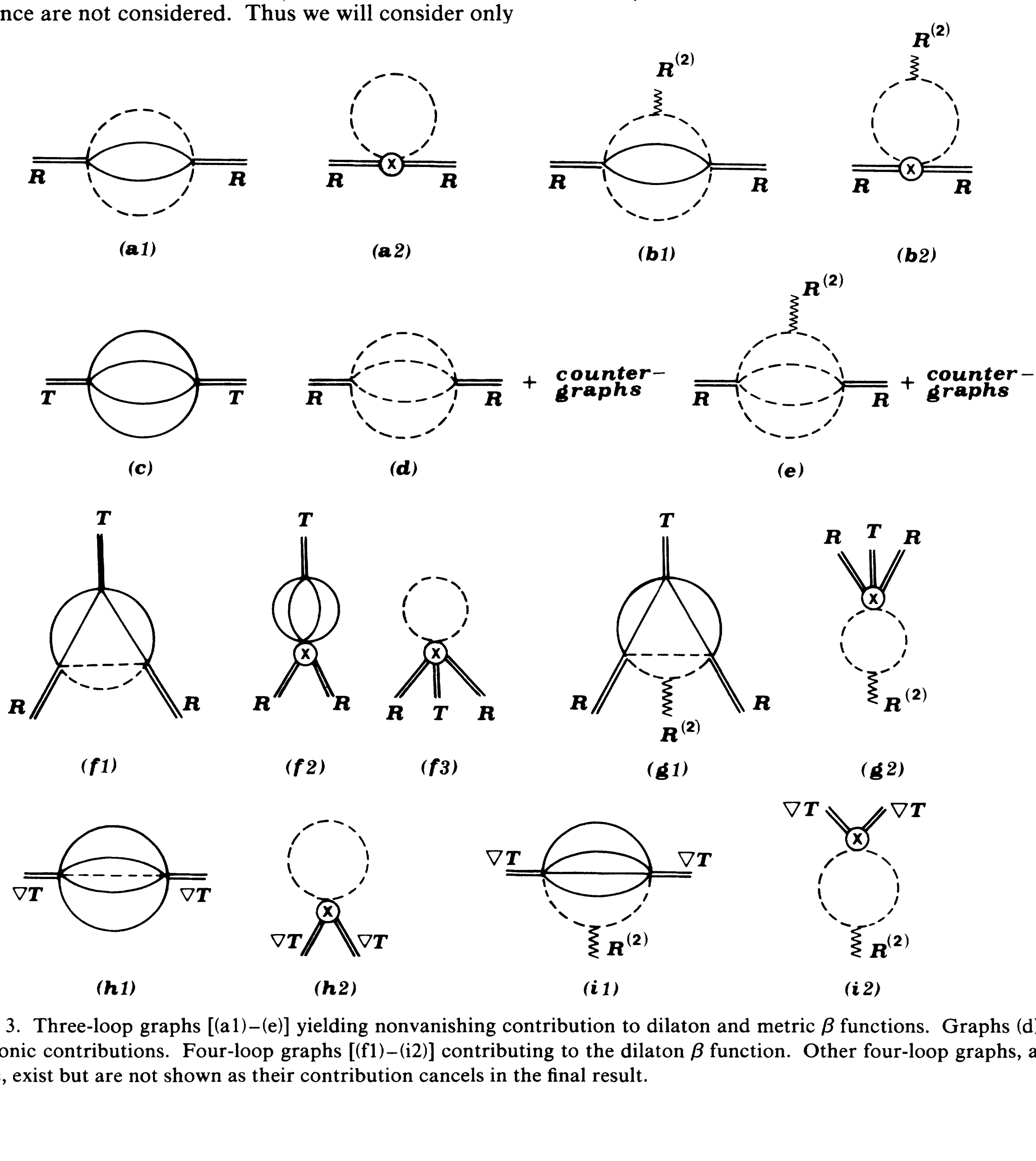

FIG. 3. Three-loop graphs [(a1)–(e)] yielding nonvanishing contribution to dilaton and metric β functions. Graphs (d) and (e) are the bosonic contributions. Four-loop graphs [(f1)–(i2)] contributing to the dilaton β function. Other four-loop graphs, among them bosonic, exist but are not shown as their contribution cancels in the final result.

geometric elements of the target space cancel against the corresponding bosonic ones of graph 3(d) leaving out a nonvanishing contribution to $\beta_{\mu\nu}$ through the four-fermion coupling $A_{\mu\nu\kappa\lambda}(\phi)$.

Our next task is to proceed to the four-loop $O(\alpha'^3)$ calculation of the dilaton β- function $\beta_\phi^{(3)}$. To this effect we need all four-loop graphs having $1/\epsilon^2$ singularities with $e^{\kappa\epsilon\sigma}$ at the vertices or $R^{(2)}$ insertions. The vertices $\alpha'\nabla\cdots\nabla\phi\tilde{\xi}\cdots\tilde{\xi}R^{(2)}$ do not contribute to β_ϕ for reasons that have already been stated. In Figs. 3(f1)–3(i2) we give all graphs with at least a fermion loop which contribute to $\beta_\phi^{(3)}$. Their contributions are listed in Table II. Notice that all four-loop graphs of Fig. 3 depend on the dilaton field ϕ through the coupling $A_{\mu\nu\kappa\lambda}(\phi)$. Other four-loop graphs not depending on ϕ do indeed exist; however their contribution to β_ϕ cancels against the corresponding bosonic one exactly as in the three-loop case and hence are not shown. Because of the nontriviality of the calculation we wrote separately the contribution of the three first loop graphs Figs. 3(f1)–3(f3), and as well as their sum put in a covariant form (see Table II). This is in order to reveal the mechanism of the cancellation of the infrared singularities $\ln m^2/\epsilon$ and also to show how the various contributions are added to yield a covariant result. Collecting everything together we find the following result for the three-loop contributions to the renormalization-group β functions:

$$\beta_{\mu\nu}^{(3)}=\alpha'^3\{3A_{abcd}R_{\mu m}{}^{ab}R_\nu{}^{mcd}-9A_{abcd;\mu}A^{abcd}{}_{;\nu}+\tfrac{3}{4}[A_{abcd;\mu}R^{abcd}{}_{;\nu}+(\mu\leftrightarrow\nu)]\}\ , \tag{6}$$

$$\beta_\phi^{(3)}=-\tfrac{1}{6}G^{\mu\nu}\beta_{\mu\nu}^{(3)}\ .$$

Both β_ϕ and $\beta_{\mu\nu}$ are nonvanishing at this loop order since $A_{\mu\nu\kappa\lambda}(\phi)\neq 0$ in general. They only vanish when $A_{\mu\nu\kappa\lambda}=0$ which is obviously satisfied if the dilaton obeys $\nabla_\mu\nabla_\nu\phi=0$. Such a relation yields that $\xi_\mu\equiv\nabla_\mu\phi$ must be a

TABLE II. Contributions of graphs displayed in Fig. 3. All terms should be multiplied by $1/4\pi\alpha'$.

Graph in Fig. 3	Contribution
(a1+a2)	$\alpha'^3\left[-\frac{1}{24}\left[\frac{1}{\epsilon^2}+\frac{3}{4\epsilon}\right]\sqrt{-g}\,g^{\alpha\beta}R_{abcd;m}R^{abcd}{}_{;n}\vartheta_\alpha\chi^m\vartheta_\beta\chi^n+\frac{1}{48\epsilon}\sqrt{-g}\,R_{abcd}R^{abcd}R^{(2)}\right]$
(b1+b2)	$\alpha'^3\left[-\frac{1}{24\epsilon}\right]\sqrt{-g}\,R_{abcd}R^{abcd}R^{(2)}$
(c)	$\alpha'^3\frac{3}{\epsilon}\sqrt{-g}\,g^{\alpha\beta}T_{abcd;m}T^{abcd}{}_{;n}\vartheta_\alpha\chi^m\vartheta_\beta\chi^n$
(d+e)	They cancel the $R^{(2)}RR$ and $R;R;\ \vartheta\chi\vartheta\chi$ contributions of graphs (a1)–(c) depicted in Fig. 3
(f1)	$\alpha'^4\left[\left[-\frac{1}{16\epsilon^2}+\frac{\ln m^2}{8\epsilon}\right]e^{\epsilon\sigma}\vartheta_\alpha(e^{\epsilon\sigma}R^{ablm})\vartheta^\alpha(e^{\epsilon\sigma}R^{cd}{}_{lm})T_{abcd}+\left[-\frac{3}{16\epsilon^2}+\frac{3\ln m^2}{8\epsilon}\right]e^{2\epsilon\sigma}R^{ablm}R^{cd}{}_{lm}\vartheta^2(e^{\epsilon\sigma}T_{abcd})\right]$
(f2)	$\alpha'^4\left[\frac{1}{4\epsilon^2}-\frac{3\ln m^2}{8\epsilon}\right]e^{\epsilon\sigma}R^{ablm}R^{cd}{}_{lm}\vartheta^2(e^{\epsilon\sigma}T_{abcd})$
(f3)	$\alpha'^4\left[\frac{1}{4\epsilon^2}-\frac{\ln m^2}{8\epsilon}\right][\vartheta_\alpha R^{ablm}\vartheta^\alpha R^{cd}{}_{lm}+2\epsilon(\vartheta^\alpha R^{ablm})R^{cd}{}_{lm}\vartheta_\alpha\sigma]T_{abcd}$
(f1+f2+f3)	$\alpha'^4\left[-\frac{1}{16\epsilon}\right]\sqrt{-g}\,R^{ablm}R^{cd}{}_{lm}T_{abcd}R^{(2)}$ + terms not contributing to $\beta_\phi^{(3)}$
(g1+g2)	$\alpha'^4\frac{3}{16\epsilon}\sqrt{-g}\,R^{ablm}R^{cd}{}_{lm}T_{abcd}R^{(2)}$
(h1+h2)	$\alpha'^4\frac{3}{16\epsilon}\sqrt{-g}\,T^{abcd,m}T_{abcd;m}R^{(2)}$ + terms not contributing to $\beta_\phi^{(3)}$
(i1+i2)	$\alpha'^4\left[-\frac{9}{16\epsilon}\right]\sqrt{-g}\,T^{abcd;m}T_{abcd;m}R^{(2)}$

Killing vector of the D-dimensional target space. We shall come to this point later.

III. CONFORMAL INVARIANCE AT THE THIRD-LOOP ORDER

As stated in the beginning the dilaton dependence of the β functions is solely due to the auxiliary fields of the matter and zweibein multiplets which get mixed in the supersymmetric Lagrangian of the σ model when the dilaton field is introduced. The auxiliary fields being nondynamical can be replaced via their classical equations of motion and the resulting theory is described only by the dynamical degrees of freedom. After imposing the superconformal gauge $g_{\alpha\beta}=\eta_{\alpha\beta}e^{2\sigma}$, $\psi_\alpha=i\gamma_\alpha\chi$ (ψ_α denotes the gravitino field) the conformal factor σ and the Majorana spinor χ transform as members of the same supermultiplet under the special world-sheet supersymmetry, remnant of the general coordinate transformations (GCT) and supersymmetry which the original Lagrangian possesses.[18] The auxiliary field H appearing in Eq. (1) completes the σ multiplet so that the special two-dimensional supersymmetry algebra closes. As far as β functions and conformal invariance considerations are concerned, σ is not considered quantized but is rather treated as an external classical field exactly as in the ordinary σ model; therefore its superpartner χ in the supersymmetric model should be handled in the same manner. However one can pose the question whether correct treatment of the conformal invariance conditions requires that both σ and χ fluctuate about some background values σ_0,χ_0. If it turns out that this is indeed the case, it would mean that the quantum effects of the two-dimensional world-sheet geometry should be taken into account. The only case that this ceases to take place and the quantum effects of the world-sheet geometry do not affect the β functions, and therefore the conformal invariance conditions, is when $\nabla_\mu\nabla_\nu\phi=0$. At first sight this condition while making the expression of Eq. (6) vanish does not, at the same time, make σ and χ decouple in the classical Lagrangian. Therefore even in this case one may fear that they alter the results for β functions. Starting, for instance, with the field χ there exists in the Lagrangian a mixing term

$$\Delta\mathcal{L}=\alpha'(\nabla_\mu\phi)\bar{\lambda}^\mu\partial\!\!\!/\chi \tag{7}$$

which seems to contribute if χ is considered quantized. One then uses the special supersymmetry of the world sheet to impose the gauge fixing $\chi=0$ which is implemented by a set of Majorana commuting fermions c,d whose Lagrangian is

$$\mathcal{L}_{\text{ghost}}=e^{(2-\epsilon)\sigma}[i\bar{c}(\partial\!\!\!/+\partial\!\!\!/\sigma)d-H\bar{c}d]\ . \tag{8}$$

This gauge choice makes the mixing term (7) disappear; the field H appearing in (8) is the auxiliary field of the zweibein multiplet. Eliminating H as before but taking into account the ghost term $H\bar{c}d$ above yields a ghost Lagrangian which when $\nabla_\mu\nabla_\nu\phi=0$ receives the simple form

$$\mathcal{L}'_{\text{ghost}}=e^{(2-\epsilon)\sigma}[i\bar{c}(\partial\!\!\!/+\partial\!\!\!/\sigma)d+\Lambda(\bar{c}d)^2]\ ,$$
$$\Lambda\equiv 1/(\nabla_\mu\phi)^2\ . \tag{8'}$$

The additional coupling in (8′) where four ghosts participate was explicitly found not to contribute to the $\beta_{\mu\nu}$, β_ϕ beta functions. In fact all relevant graphs were found to involve at least one derivative of Λ which is zero because of the condition $\nabla_\mu\nabla_\nu\phi=0$. The only graph not involving derivatives of Λ but Λ itself is a four-loop graph with only ghost lines connected at three different four ghost vertices which may in principle contribute to $\beta_\phi^{(3)}$. However explicit calculation of this and its associated countergraph, involving the lower-order counterterms, yielded a vanishing result. Therefore the gravitino mode χ indeed decouples and does not contribute to the β functions at this order. An analogous type of behavior is expected for the conformal factor σ when it is considered quantized. In fact there is a mixing term $\alpha'(\nabla_\mu\phi)\xi^\mu\Box\sigma$ in the Lagrangian which is actually the supersymmetric part of the coupling (7). Therefore we conclude that dilaton backgrounds satisfying $\nabla_\mu\nabla_\nu\phi=0$ have the virtue that the quantum effects of the world-sheet geometry do not interfere with the quantum effects of the target-space geometry as far as conformal invariance is concerned and can be safely ignored. Notice that it is only in this case that the renormalizations of the target-space metric $G_{\mu\nu}$ do not depend on the dilaton field in the supersymmetric σ model. This has been assumed in works where the Curci-Paffuti[6] equation for the supersymmetric σ model was derived[19] and probably has to be reconsidered in case ϕ is an arbitrary background not satisfying $\nabla_\mu\nabla_\nu\phi=0$.

Note that $\nabla_\mu\nabla_\nu\phi=0$ along with $R_{\mu\nu}=0$ are fixed points of the renormalization-group β functions up to $O(\alpha'^3)$ and also satisfy the conformal invariance conditions to order α'^2. In order to check whether these are consistent with conformal invariance to order α'^3 one has to take into account the diffeomorphisms whose contributions at this order are in general nonvanishing. To find the effect of diffeomorphisms one may use the Curci-Paffuti equation[5,6] which is certainly valid in the supersymmetric case when ϕ satisfies $\nabla_\mu\nabla_\nu\phi=0$ as we have remarked. In fact in this case the Curci-Paffuti equation is identical in form with that of the ordinary σ model. Then following a procedure similar to that described in Ref. 14, and using the fact that the two- and the three-loop contributions to the β functions vanish one can find in a straightforward manner that at this loop order the diffeomorphisms vanish too. This was perhaps expected on the grounds that at this loop order the β functions receive contributions from only the one-loop effects and the diffeomorphisms are known to vanish at the one-loop order. Therefore $R_{\mu\nu}=0$, $\nabla_\mu\nabla_\nu\phi=0$ are backgrounds consistent with superconformal invariance up to the third order in the string slope parameter α'. In addition, owing to the nontrivial character of the β-function dependence [see Eq. (6)] on the dilaton we may argue that these are the simplest possible background fields up to this order. A simple solution to these equations is obtained if the space is flat, $G_{\mu\nu}=\eta_{\mu\nu}$, and the dilaton depends linearly on the space-time components, that is, $\phi=\eta^\mu\chi_\mu$. Solutions of this type satisfy the conformal invariance conditions and have been proposed as backgrounds bearing special features which may also be important for cosmology.[20,21] However Minkowskian background for the target space is not the only possible solution available. The dilaton condition $\nabla_\mu \nabla_\nu \phi = 0$ yields $\nabla_\mu \xi_\nu = 0$ for the vector $\xi_\mu \equiv \nabla_\mu \phi$ and hence $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$. Therefore ξ_μ, derived from the potential ϕ, is a Killing vector of vanishing covariant derivative. Nontrivial spaces adopting such Killing vectors do indeed exist. In fact according to a theorem due to Bochner the particular class of compact and orientable Riemannian manifolds having vanishing or negative Ricci tensor can adopt Killing vectors of only this kind.[22] Therefore product spaces $M^d \times S$ with M^d the d-dimensional Minkowski space and S a Ricci-flat compact and orientable manifold, naturally emerge as backgrounds satisfying the conformal invariance conditions up to $O(\alpha'^3)$.

In the next section we shall check whether the special backgrounds under discussion are consistent with Robertson-Walker cosmologies and comment on their cosmological implications.

IV. COSMOLOGICAL IMPLICATIONS

The equations

$$R_{\mu\nu} = 0, \quad \nabla_\mu \nabla_\nu \phi = 0 \ , \tag{9}$$

as we have shown are consistent with conformal invariance of the supersymmetric σ model in the presence of the dilaton up to the third order in the slope parameter α'. Backgrounds satisfying these equations are the simplest possible and arguments supporting this were given in the previous section. Particular solutions satisfying Eqs. (9) are consistent with Robertson-Walker cosmologies as have been shown elsewhere.[21] In these considerations the dilaton was assumed to depend linearly on the space-time components of the target space whose metric was taken flat satisfying trivially therefore the condition for Ricci flatness $R_{\mu\nu} = 0$. In the absence of an antisymmetric field $B_{\mu\nu}$ such solutions are consistent with conformal invariance, actually to all orders. In fact at any order the β functions depend on the geometric elements of the target space, which vanish in a flat space-time, and on the dilaton field through expressions involving at least the second covariant derivative of it. The latter also vanish when the dilaton is a linear function of the space-time coordinates resulting to vanishing β functions. Backgrounds which satisfy Eqs. (9) do not always lead to a flat target-space metric and the previous all-order argument does not hold any more. Although perfectly legitimate the Minkowskian background $G_{\mu\nu} = \eta_{\mu\nu}$ is not the only solution available and we think that for a more complete study one should exhaust all the possibilities offered by the set of Eqs. (9). After all the initial aim was to study the motion of the string in nontrivial backgrounds $G_{\mu\nu} \neq \eta_{\mu\nu}$. It is our intention therefore in this section to examine all the solutions offered by the system of Eqs. (9) without committing ourselves to a necessarily flat target space assuming however that the four-dimensional metric is of the Robertson-Walker form. We will not assume either any particular dependence of the dilaton on the space-time coordinates and in this respect we take into account all solutions consistent with Robertson-Walker cosmologies.

The σ-model metric $G_{\mu\nu}$ is related to the physical metric through the rescaling

$$G_{\mu\nu} = e^{\phi} g_{\mu\nu} \ ,$$

where we have rescaled in this section the dilaton as $\phi \to \phi/2$ in such a way that the string-loop expansion parameter is e^{ϕ}. Using this, Eqs. (9) take on the form

$$R_{\mu\nu}(g) = -\tfrac{1}{2}(D_\mu \phi D_\nu \phi + g_{\mu\nu} D^2 \phi) \ , \tag{10a}$$

$$D_\mu D_\nu \phi = D_\mu \phi D_\nu \phi - \frac{g_{\mu\nu}}{2} D_\lambda \phi D^\lambda \phi \ , \tag{10b}$$

where D_μ refers to the metric $g_{\mu\nu}$ and D^2 is its four-dimensional D'Alambertian. Equation (10b) when contracted by $g^{\mu\nu}$ yields

$$D^2 \phi = -D_\lambda \phi D^\lambda \phi \ . \tag{10c}$$

In a Robertson-Walker metric the line element ds can be expressed as

$$ds^2 = dt^2 - a^2(t) \tilde{g}_{ij}(x) dx^i dx^j \ , \tag{11}$$

where $a(t)$ is the cosmic scale factor. The second of Eqs. (10), which actually stems from the dilaton condition $\nabla_\mu \nabla_\nu \phi = 0$ yields the following set of ten partial differential equations when the Robertson-Walker metric of Eq. (11) is adopted:

$$\partial_i \dot{\phi} = \left[\dot{\phi} + \frac{\dot{a}}{a} \right] \partial_i \phi \ , \tag{12a}$$

$$\tilde{g}^{ij} \partial_i \phi \partial_j \phi = a^2 (2\ddot{\phi} - \dot{\phi}^2) \ , \tag{12b}$$

$$\tilde{\nabla}_i \tilde{\nabla}_j \phi - \partial_i \phi \partial_j \phi = a^2 \tilde{g}_{ij} \left[-\ddot{\phi} + \dot{\phi}^2 + \frac{\dot{a}}{a} \dot{\phi} \right] . \tag{12c}$$

The indices i,j run from 1 to 3 and the overdot(s) stands as usual for differentiation with respect to time. Equation (12a) is solved immediately giving

$$\partial_i \phi = a e^{\phi} K_i(x) \ ,$$

with $K_i(x)$ a function of the space variables x, which in turn yields

$$\phi = -\ln[a(t) f(x) + b(t)] \ , \tag{13}$$

where $K_i = -\partial_i f$. Plugging Eq. (13) into (12b) and taking the time derivative of both sides of it we arrive at

$$(e^{-\phi})^{\cdots} = 0 \tag{14}$$

from which it follows that

$$\begin{aligned} a(t) &= a_2 t^2 + a_1 t + a_0 \ , \\ b(t) &= b_2 t^2 + b_1 t + b_0 \ , \end{aligned} \tag{15}$$

provided that $f(x) \neq 0$. The case $f(x) = 0$ will be considered separately in the sequel. The demand that ϕ as given by (13), with a and b having the form given by Eqs. (15), satisfies Eqs. (12b) and (12c) restricts the form of the functions a, b, and f as we shall see.

In order to proceed to our analysis we distinguish two cases.

A. $f=$const

This covers the $f=0$ case, left out from our considerations so far, and the dilaton turns out to be a function of the time only. The set of Eqs. (12) yields then

$$\ddot{\phi}=\frac{\dot{\phi}^2}{2}\ , \tag{16a}$$

$$-\ddot{\phi}+\dot{\phi}+(\ln a)^{\cdot}\dot{\phi}=0\ . \tag{16b}$$

An immediate solution of these equations is the trivial solution $\phi=$const which we do not consider. Actually this has been considered in Ref. 21, and leads to a static universe since the scale factor $a(t)$ turns out to be a constant. In this $\phi=$const case it is not possible to infer the form of $a(t)$ by the second of Eqs. (16) but we have to appeal to the first of Eqs. (10). When $\phi\neq$const Eq. (16a) is solved giving

$$\phi=-2\ln(t+c)+\lambda \tag{17a}$$

from which in view of (16b) it follows that

$$a(t)=\text{const}\times t\ . \tag{17b}$$

We are allowed throughout to a shifting of the time coordinate and thus the constant appearing within the logarithm in (17a) is unimportant. The solution given by Eq. (17a) has been already obtained in Ref. 21 by assuming however that the dilaton has only a time dependence. Here it was derived as one of the solutions of the equation $\nabla_\mu\nabla_\nu\phi=0$, which in principle may provide us with additional solutions. The time-dependent dilaton given by Eq. (17a) is consistent with Eq. (10a) provided that the three-dimensional curvature k is either vanishing or positive depending on whether one has a vanishing or a nonvanishing background value, respectively, for the antisymmetric tensor field $H_{\mu\nu\lambda}$.

We next proceed to consider the second case.

B. $f\neq$const

In this case ϕ is a function of both space and time. To decide on the existence of solutions consistent with the Robertson-Walker metric we require a systematic analysis of the system of Eqs. (12). Using Eq. (13), Eqs. (12b) and (12c) receive the forms

$$\tilde{g}^{\,ij}\partial_i f\partial_j f=[(\dot{a}f+\dot{b})^2-2(\ddot{a}f+\ddot{b})(af+b)]\ , \tag{18a}$$

$$\tilde{\nabla}_i\tilde{\nabla}_j f=\tilde{g}_{ij}[(\dot{a}^{\,2}-a\ddot{a})f+(\dot{a}\dot{b}-a\ddot{b})]\ . \tag{18b}$$

The first of these does not have any time dependence as expected unlike the second whose right-hand side (RHS) depends on time. Since its LHS is t independent so must be the RHS. By using Eq. (15) this implies

$$a_2(a_2f+b_2)=0\ .$$

Therefore we see that a_2 must be necessarily zero since otherwise f would be a constant. Thus the cosmic scale factor is a linear function of the time corresponding to either an expanding (contracting) universe, when $a_1\neq0$, or to a static universe in the case that $a_1=0$. We will consider these two distinct cases separately.

1. $a_1\neq0$

In this case by a shift in the time variable we can always put $a_0=0$. Then by defining

$$\Psi=a_1f+b_1$$

the system of Eqs. (18) takes the form

$$\tilde{g}^{\,ij}\partial_i\Psi\partial_j\Psi=a_1^2\Psi^2-4a_1^2b_0b_2\ , \tag{19a}$$

$$\tilde{\nabla}_i\tilde{\nabla}_j\Psi=a_1^2\tilde{g}_{ij}\Psi\ . \tag{19b}$$

We shall work in a coordinate system in which the three-metric tensor has the well-known form

$$\tilde{g}_{rr}=(1-kr^2)^{-1},\quad \tilde{g}_{\theta\theta}=r^2,\quad \tilde{g}_{\phi\phi}=r^2\sin^2\theta\ , \tag{20}$$

where the constant k is the space curvature. In a straightforward manner one can verify that the $r\theta$ and $r\varphi$ components of Eq. (19b) give

$$\Psi=rV(\theta,\theta)+\sigma(r)\ , \tag{21}$$

where V depends on the angles θ,φ and $\sigma(r)$ is a function of the radius r. In view of this from the rr part of (19b) it follows that

$$\sigma''-\frac{\lambda'}{2}\sigma'-a_1^2e^\lambda\sigma=\left[\frac{\lambda'}{2}+a_1^2e^\lambda r\right]V\quad (e^\lambda\equiv\tilde{g}_{rr})\ . \tag{22}$$

This gives the result V is a constant and thus Ψ is a function of r only, unless $a_1^2+k=0$, in which case the coefficient of V in Eq. (22) vanishes. It is only in this case that V can have a nontrivial θ,φ dependence. It is an easy task for one to verify that the $\Psi=\Psi(r)$ case is incompatible with the $\theta\theta$ component of Eq. (19b) and also Eq. (19a) so that it only remains to consider the case of having $a_1^2+k=0$. Since $a_1\neq0$ this is only relevant for spaces having negative curvature $k<0$. It is not difficult to verify that with $a_1^2+k=0$ one has a unique solution to the system of Eqs. (19) resulting in a dilaton

$$\begin{aligned}&\phi(\mathbf{r},t)=-\ln\{t[\mathbf{a}\cdot\mathbf{r}+C(1-kr^2)^{1/2}]+bt^2+b'\}\ ,\\ &a(t)=a_1t\quad (k=-a_1^2<0)\ ,\end{aligned} \tag{23}$$

where the constants b,b',C appearing in (23) as well as the constant vector $\mathbf{a}$ are constrained by

$$C^2=4bb'-\frac{\mathbf{a}^2}{k}\ .$$

Other equivalent solutions are obtained by a shift in time. Whether this is compatible with Eqs. (10a) is an issue that will be discussed later. At the moment we can only remark that the equation $\nabla_\mu\nabla_\nu\phi=0$ for the dilaton field does not give only the solution shown in (17a).

We next pass to consider the case of a static universe $a_1=0$.

2. $a_1=0$

By a shift in the time variable we can now put, for convenience, the constant $b_0=0$. Then the system of Eqs. (18) gets the form

$$\tilde{g}^{ij}\partial_i\Psi\partial_j\Psi=2\lambda_0\Psi+b_1^2 , \quad (24a)$$

$$\tilde{\nabla}_i\tilde{\nabla}_j\Psi=\lambda_0\tilde{g}_{ij} , \quad (24b)$$

where $\lambda_0=-2a_0^2b_2$ with $\Psi=a_0f$.

In this case too, with Ψ defined as above, it can be shown in exactly the same manner as before that Ψ is of the form given by Eq. (21) while $\sigma(r)$ now satisfies the equation

$$\sigma''-\frac{\lambda'}{2}\sigma'-\lambda_0e^{\lambda}=\frac{\lambda'}{2}V \quad (25)$$

as can be seen by considering the rr component of Eq. (24b). When the three-dimensional curvature k is different from zero, V has to be constant in which case Ψ turns out to be only a function of the variable r. Then the $\theta\theta$ component of (24b) and Eq. (24a) can be satisfied only if Ψ is a constant which cannot happen since we have assumed $f\neq$const. Therefore it only remains to explore the $k=0$ case. Skipping all the irrelevant details one can verify that the system of Eqs. (24) gives rise to a dilaton

$$\phi(\mathbf{r},t)=-\ln\left[\mathbf{a}\cdot\mathbf{r}+Cr^2-\frac{C}{a_0^2}t^2+Bt+A\right]$$

with

$$a(t)=a_0,\quad a_0\neq0 \quad\text{and}\quad k=0,\quad \mathbf{a}^2=B^2+4AC \quad (26)$$

and all that follows from this by a shift in time.

Therefore we have seen that in addition to the trivial solution $\phi=$const and the time-dependent dilaton solution in Eq. (17a) the condition $\nabla_\mu\nabla_\nu\phi=0$ in a Robertson-Walker four-dimensional background yields two additional space- and time-dependent solutions given by Eqs. (23) and (26). To be acceptable as solutions these should also satisfy Eqs. (10a). However in a Robertson-Walker metric the Ricci tensor components R_{0i} vanish ($i=1,2,3$) and this in view of Eq. (10a) gives $\dot{\phi}\partial_i\phi=0$. Therefore either ϕ is a function of time or ϕ is constant. Thus in the absence of the antisymmetric tensor field $B_{\mu\nu}$ the solutions in Eqs. (23) and (26) are excluded. This means that when $B_{\mu\nu}=0$ necessarily the dilaton must be a function of time, only having the form given by Eq. (17a) and the curvature k should vanish. Thus the assumption of a dilaton field which depends only on time is mandatory when $B_{\mu\nu}=0$. However when $B_{\mu\nu}\neq0$ Eq. (10a) relating the four-dimensional Ricci tensor to the dilaton gets modified and its RHS receives contributions dependent on the $H_{\mu\nu\lambda}$ field which are nonvanishing.

Therefore in order to proceed to a complete investigation of this issue we require knowledge of the β functions up to third-loop order in the presence of the antisymmetric tensor field $B_{\mu\nu}$. Lacking such a calculation in the supersymmetric σ model we cannot give a firm statement concerning the compatibility of the solution given by Eqs. (23) and (26) with Robertson-Walker cosmologies in conjunction with conformal invariance requirement. However if we rely on the one-loop results for the β functions we can show that the solutions under discussion are inconsistent with Eq. (10a). In fact the condition ${\beta^B}_{\mu\nu}=0$ for the antisymmetric field β function is solved by the duality transformation $H_{\mu\nu\rho}\sim\epsilon_{\mu\nu\rho}{}^{\lambda}\partial_\lambda b$ and Eq. (10a) is modified by receiving contributions on its RHS depending on the axion field b. The dilaton backgrounds given by Eqs. (23) and (26) always lead to a Ricci-flat four-dimensional space $[R_{\mu\nu}(g)=0]$ because of the fact that in both cases we have $a_1^2+k=0$. Then the system of Eq. (10a) is easily proved to be incompatible with having a real axion field. Based on the one-loop results therefore we can say that even with an antisymmetric field present the dilaton condition $\nabla_\mu\nabla_\nu\phi=0$ in a Robertson-Walker four-dimensional background is consistent only with a time-dependent dilaton of the form $-2\ln t+\lambda$, and non-negative space curvature. Therefore the ansatz of a time-dependent dilaton used in the literature[21] comes out naturally from the condition $\nabla_\mu\nabla_\nu\phi=0$ which, following arguments presented in the previous section, gives the simplest and most plausible dilaton backgrounds consistent with conformal invariance.

We find it unlikely that these results will be modified by higher-loop effects although a rigorous proof of this assertion has to await until the full three-loop results for the β functions, when $B_{\mu\nu}$ is present, are known. At any rate one can observe that even with the additional solutions we have found, the situation is not altered as far as the expansion rate of the universe is concerned and we are led again to an either linearly expanding (contracting) or a static universe.

V. SUMMARY

In the $N=1$ supersymmetric σ model with a dilaton field present we have calculated the $O(\alpha'^3)$ contribution to the renormalization-group β functions. We have found that both metric ${\beta^G}_{\mu\nu}$ and dilaton β^ϕ beta functions at this order depend nontrivially on the dilaton field, being vanishing if the dilaton satisfies $\nabla_\mu\nabla_\nu\phi=0$. As a byproduct of our calculation we have explicitly found that the $O(\alpha'^2)$ corrections to the dilaton β function vanish as has been previously claimed in the literature by employing other indirect arguments. This along with the fact that ${\beta^G}_{\mu\nu}$ receives no $O(\alpha'^2)$ contributions indicates that backgrounds respecting local scale invariance to first and thus to second order in the Regge slope α' do not necessarily satisfy the conformal consistency conditions beyond that order. The special backgrounds $R_{\mu\nu}=0$, $\nabla_\mu\nabla_\nu\phi=0$ satisfy the conformal invariance conditions up to two-loop order and are shown to be also consistent with conformal invariance to next loop order. A basic tool in proving this is the Curci-Paffuti relation which can be safely used in its well-known form in the supersymmetric model as well as when $\nabla_\mu\nabla_\nu\phi=0$. Because of the nontrivial character of the dilaton dependences of the β functions we argue that the aforementioned backgrounds are the simplest and most plausible ones consistent with local scale invariance. The particular class of Ricci-flat compact and orientable Riemannian manifolds naturally offer as internal spaces for string theories.

We have furthermore explored the consistency of these special backgrounds with Robertson-Walker cosmologies. In a four-dimensional Robertson-Walker background we found all dilaton solutions satisfying $\nabla_\mu\nabla_\nu\phi=0$. Except the trivial constant solution and the time-dependent solution of the form $-2\ln t+\lambda$ whose cosmological implications have been previously discussed in the literature, additional solutions exist which require a nonvanishing background value for the antisymmetric tensor field $B_{\mu\nu}$. From the reality of the axion field it follows that these are inconsistent with conformal invariance to leading order in the slope parameter α' and may be relevant only if higher-loop effects are taken into account. This gives further support to conclusions reached in the literature where the dilaton is *a priori* assumed to depend only on time. In any event these solutions do not alter the situation regarding the behavior of the cosmic scale factor yielding either a static or a linearly expanding (contracting) universe.

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