Spectral flow and supersymmetry operators in coset construction of $N=2$ superconformal field theory

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For $N=2$ superconformal coset G/H models, supersymmetry operators are constructed in accordance with the spectral flow. Gauge representation operators are also discussed. These are very powerful in the analysis of spectra in string models. In four-dimensional string theories, which are constructed via compactifications using $N=2$ superconformal models, one can use these operators to obtain all the other massless spectra, once canonical states are calculated.

I. INTRODUCTION AND SUPERSYMMETRY OPERATOR IN $N=2$ MINIMAL SUPERCONFORMAL FIELD THEORY

With the increasing amount of knowledge gained on conformal field theories in the past few years, it is becoming rather clear that the $N=2$ superconformal theory (SCFT) is a special class of conformal theory. In particular, in the application to four-dimensional string models, $N=2$ supercomformal symmetry turns out to be a very powerful tool. It was shown by Gepner that $N=2$ minimal SCFT can be used for both right and left movers as building blocks to achieve compactifications [(2,2) compactification].[1] The central charges of the $N=2$ minimal series are given by $c=3k/(k+2)$ and these models can be combined together to get $c_{\text{tot}}=9$ for six-dimensional compactification. String models thus constructed have $N=1$ spacetime supersymmetry and are solvable. Namely, not only massless spectra (which belong to the E_6 gauge group) of string models but also Yukawa couplings and some higher-point correlation functions can be studied. These are examined in detail for the three-generation model,[2-5] the so-called 1×16^3 model that is obtained from one copy of $k=1$ and three copies of exceptional $(E_7)k=16$ sectors.

Subsequently a new class of $N=2$ SCFT, G/H coset models has been constructed by Kazama and Suzuki (KS) by applying the Goddard-Kent-Olive (GKO) method to the super Kac-Moody algebra.[6] It is found that the condition for having $N=2$ superconformal symmetry is that H contains a single U(1) factor. The Hermitian symmetric spaces, which are mathematically classified, are found to give rise to these $N=2$ SCFT's. The central charges of these models can be greater than 3 and some of them are regarded as different types of extension of $N=2$ minimal series. Upon applying Gepner's method, one expects that $N=2$ SCFT can provide quite rich vacua of four-dimensional string models. From the phenomenological point of view, it would be very interesting if different three-generation models are found.

A characteristic feature of $N=2$ superconformal symmetry is the existence of the η algebra, which represents the automorphism of the superconformal algebra.[7] Under this automorphism, the Neveu-Schwarz (NS) and the Ramond (R) states can be shown to correspond one to one as η is continued from 0 to $\frac{1}{2}$ (spectral flow). In order to see this aspect, we define the twisted $N=2$ superconformal algebra by introducing the following boundary conditions on the supersymmetry generators $G^{\pm}$:

$$G^{\pm}(z)=e^{\pm 2\pi i\eta}G^{\pm}(e^{2\pi i}z) .$$

Then the Fourier transforms of the energy-momentum tensor (L_n), supersymmetry generators ($G_n^{\pm}$), and U(1) charge generator (J_n) satisfy the following relations under the spectral flow:[7]

$$L_n'=L_n+\eta J_n+\frac{c}{6}\eta^2\delta_{n,0} ,$$

$$J_n'=J_n+\frac{c}{3}\eta\delta_{n,0} ,$$

$$(G_r^{\pm})'=G_{r\pm\eta}^{\pm} ,$$

where the operators on the left-hand sides are the ones twisted by η. The $n=0$ parts, especially, dictate how the conformal weight h and the U(1) charge Q of a state transform under the spectral flow:

$$h'=h+\eta Q+\frac{c}{6}\eta^2 , \tag{1}$$

$$Q'=Q+\frac{c}{3}\eta . \tag{2}$$

Since the NS and the R states are specified by certain values of conformal weights and U(1) charges, these two formulas allow one to obtain the R state from the NS state and vice versa. This is nothing but a supersymmetry transformation in string models with $N=1$ spacetime supersymmetry.

Let us consider this supersymmetry operator in the case of $N=2$ minimal models, which has been studied well.[8,2] We are interested in the states that are the representations of the gauge group E_6 of models, $\mathbf{27}, \overline{\mathbf{27}}$ and singlets. Notice, however, that representations of E_6 are actually specified by those of SO(10)×U(1), which are linearly realized. Therefore, $\mathbf{27}$ is decomposed as $\mathbf{10}_{-1}+\mathbf{16}_{1/2}+\mathbf{1}_2$ and similarily for $\overline{\mathbf{27}}$. In $N=2$ minimal conformal models, these states are described as primary states of level k, specified by sets of integers (l,q,s) and $(\bar{l},\bar{q},\bar{s})$. For a given level k, the principle quantum number l is defined to take values in the range $0 \le l \le k$ and charge q defined by modulo $2(k+2)$, respectively. s is the quantum number that distinguishes NS and R sectors and is defined modulo 4; $s=0,2$ for NS and $s=1,3$ for R states. For the computations of massless spectra the standard range $|q-s| \le l$ is employed. The conformal weight h and the U(1) charge Q of such right-moving primary states are given by

$$h=\frac{l(l+2)-q^2}{4(k+2)}+\frac{s^2}{8}\ , \tag{3}$$

$$Q=-\frac{q}{k+2}+\frac{s}{2}\ , \tag{4}$$

and similarly for the left mover. Let us assume a compactified theory constructed by tensoring r minimal models, so that

$$c_{\text{tot}}=\sum_{i=1}^{r} 3k_i/(k_i+2)=9\ .$$

The massless states then are given by r sets of integers (l_i,q_i,s_i) and $(\bar{l}_i,\bar{q}_i,\bar{s}_i)$, denoted by

$$\otimes \Phi^{l_i;\bar{l}_i}_{q_i,s_i;\bar{q}_i,\bar{s}_i}=\begin{bmatrix} l_i & q_i & s_i \\ \bar{l}_i & \bar{q}_i & \bar{s}_i \end{bmatrix}^r .$$

Suppose that a massless NS state is given by

$$\begin{bmatrix} l_i & q_i & s_i \\ \bar{l}_i & \bar{q}_i & \bar{s}_i \end{bmatrix}^r ,$$

which has $h=\bar{h}=\frac{1}{2}$ and $Q=\bar{Q}=-1$, the scalar $\mathbf{10}_{-1}$ of $\mathbf{27}$ under the decomposition. One can then show that the holomorphic operator

$$S(z)=\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}^r ,$$

which has conformal weight $h=\frac{3}{8}$, creates a R state when acting on this NS state. By considering the operator-product expansion of

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} l_i & q_i & s_i \\ \bar{l}_i & \bar{q}_i & \bar{s}_i \end{bmatrix} ,$$

one can derive that the effect of $S(z)$ on a primary state is to shift the quantum numbers q_i and s_i as $q_i \to q_i+1$ and $s_i \to s_i+1$. This is easily seen from the fact that the primary fields of $N=2$ minimal models are represented by the product of a Z_k parafermion field and a free boson with charge

$$\alpha_{qs}=\frac{1}{\sqrt{k(k+2)}}[-q+\tfrac{1}{2}(k+2)s]\ .$$

The parafermion part of the operator $S(z)$ is the identity; hence, $S(z)$ is simply realized by the free boson. Therefore, in the product of $S(z)$ with a primary field, the free boson charge is given by the sum of the charge of each boson. States obtained by these shifts of q and s have the conformal weight $h=\frac{3}{8}$, which is the correct value for the R states. In this way all superpartners, which belong to the representation of E_6, are obtained by this supersymmetry operator.

On the other hand, it is an easy matter to show how the shifts of q and s are understood in terms of the spectral flow. Regarding the conformal weight and U(1) charge to be h' and Q' for the primary state given by

$$\begin{bmatrix} l_i & q_i+1 & s_i+1 \\ \bar{l}_i & \bar{q}_i & \bar{s}_i \end{bmatrix}^r ,$$

one can derive the right-hand sides of Eqs. (1) and (2) for $\eta=\frac{1}{2}$ and $s=0$ with the use of $\sum 3k_i/(k_i+2)=9$. Especially, for those states with $h=\frac{1}{2}$ and $Q=-1$, the sum of the first two terms in Eq. (1) is vanishing, leaving $h=\frac{3}{8}$, which is the required conformal weight for massless states in the R sector. From this argument, one can regard the primary states with $s=0$ as the canonical ones from which all other states are obtained consistently with the spectral flow. This choice of the canonical states turns out to be convenient, for example, in the study of higher point correlation functions in terms of selection rules. In Ref. 2, scalar **10**'s are computed explicitly with the use of the β method.

One may wonder that this supersymmetry transformation can also be derived from the modular transformation property of the character $\chi^l_{q,s}(\tau,z)$ of a primary state (l,q,s), where τ is the torus parameter and z is a variable that measures the U(1) charge. [Notice that $\chi^l_{q,s}$ is nonvanishing for $l+q+s=0 \pmod 2$, so the above shifts are consistent with this condition.] Since the modular transformations change U(1) charges of states as well as boundary conditions, NS and R states interchange with each other. It may not be, however, straightforward to show these shifts of q and s in terms of the character. Instead, one may find it easier to check the consistency of this supersymmetry transformation using the character $\chi^l_m(\tau,z)$ of the corresponding SU(2) Wess-Zumino-Witten (WZW) model. The WZW model is related to the $N=2$ minimal models through the parafermion and another free boson, and this relation allows one to calculate correlation functions in $N=2$ minimal models. The index m is related to those of minimal models by $m=(q-s)/2$. Therefore, the changes of q and s in the supersymmetry transformation leave the index m unchanged in the partition function of the WZW model.

In the analysis of spectra in four-dimensional string models compactified via coset models G/H, it is obvious that the supersymmetry operators are also very useful. As is expected, these operators reflect the structures of coset G/H. In the next section, we construct the supersymmetry operators for coset models obtained by KS. In

Sec. III, we also discuss that the gauge representation transformations can be constructed in a similar way. Since we are specifying the representations of E_6 in terms of those of SO(10)×U(1), it is necessary to have such operators that relate $\mathbf{10}_{-1}$, $\mathbf{16}_{1/2}$, and $\mathbf{1}_2$ of SO(10). Such operators are antiholomorphic, namely constructed from the left-moving (bosonic string) sector, which is responsible for the generation of gauge groups. Section IV is devoted to discussions.

II. SUPERSYMMETRY OPERATORS IN $N=2$ COSET MODELS

$N=2$ coset models G/H are shown to be associated to a special kind of Kähler manifold called a Hermitian symmetric space. The models in this coset construction are given in Table I with the central charges for a level k. In order to study spectra of compactified string models one needs to construct primary states that are specified by the conformal weight h and the U(1) charge Q. For this purpose, it is noted that the super Kac-Moody algebra associated with a coset G/H are regarded as the ordinary Kac-Moody algebra of $G\times\mathrm{SO}(\dim G/H)/H$ where the Kac-Moody system SO(dimG/H) is represented by the fermions at the level $k=1$. The primary states of $N=2$ SCFT are then given in terms of those of Kac-Moody algebra of the groups G, H, and SO(dimG/H). The conformal weight h and U(1) charge Q are expressed as

$$h=\frac{1}{2(k+g)}[(\Lambda\cdot\Lambda+2\rho_G\cdot\Lambda)-(\lambda\cdot\lambda+2\rho_H\cdot\lambda)]+\tfrac{1}{2}\tilde{\Lambda}\cdot\tilde{\Lambda}\ , \tag{5}$$

$$Q=-\frac{1}{k+g}(\rho_G-\rho_H)\cdot\lambda+\sum_{l=1}^{(1/2)\dim G/H}\tilde{\Lambda}^l\ , \tag{6}$$

where Λ is the highest weight of a highest-weight state $|\Lambda\rangle$ of the affine algebra of G, and similarly for λ. For the level k, Λ is given in the form

$$\Lambda=\sum_{i=0} n_i\Lambda_i\ , \tag{7}$$

where $\sum n_i=k$. ρ_G (ρ_H) is the half-sum of the positive roots of G (H), $\rho_G=\frac{1}{2}\sum_{\alpha>0}\alpha$. $\tilde{\Lambda}^l$ denotes the highest weights of SO(dimG/H) at level one, and are either singlet ($\tilde{\Lambda}_0$), vector ($\tilde{\Lambda}_v$), spinor ($\tilde{\Lambda}_s$), or antispinor ($\tilde{\Lambda}_{\bar{s}}$). The spacetime property of a primary state is determined by the highest weight $\tilde{\Lambda}$, namely, singlet and vector for the NS sector and spinor and antispinor for the R sector, respectively. This is the generalization of the quantum number s in the case of $N=2$ minimal models.

We first remind the reader that the class of coset models G/H under consideration is characterized by the following set of algebraic identities, which are written in terms of the quantities in the definitions of h and Q:

$$\alpha_+\cdot\rho_H=0\ , \tag{8}$$

$$\alpha_+\cdot\alpha_+=\alpha_+\cdot\rho_G=g\tilde{\Lambda}_s\cdot\tilde{\Lambda}_s\ , \tag{9}$$

where g is the dual Coxeter number of G and $\alpha_+=\alpha_G-\alpha_H$. Furthermore, the central charge c of each model is simply expressed in terms of the spinor weight of SO(dimG/H) as

$$c=\frac{12k\tilde{\Lambda}_s^2}{k+g}\ . \tag{10}$$

The first identity implies that the direction of the charge Q is orthogonal to that of ρ_H (orthogonality condition). In the second indentity, the appearance of the spinor weight of SO(dimG/H) is due to the existence of the $N=1$ spacetime supersymmetry where the spinor weight is necessary to obtain R states from NS states, as we discuss below (supersymmetry condition). These two identities are explicitly checked for all the coset models. It is, however, not proved in general that these two identities are also the sufficient conditions for having $N=2$ superconformal symmetry. See Ref. 9 for some more discussions.

In the construction of spacetime supersymmetry operators, we restrict ourselves to the class of operators regarded as the generalization of that of $N=2$ minimal models. Namely, we assume that the supersymmetry operators are realized as transformations achieved by the appropriate shifts of the U(1) charge and the change of level of one highest weight of SO(dimG/H) in the right-moving sector. Under this circumstance, the spectral flow, Eqs. (1) and (2), are considered as two conditions for two parameters: the shift of the U(1) charge $q=2\alpha_+\cdot\lambda$ and the coefficient of it in the contribution to the conformal weight. Hence they are uniquely determined. We show that the supersymmetry operator can be expressed in a model-independent way for the $N=2$ coset models.

Let us first consider the spectral flow of the charge Q of a primary state. As η changes from 0 to $\frac{1}{2}$ (the flow from NS to R sector), suppose that the charge changes from q to $q+\delta q$ as well as that the singlet weight $\tilde{\Lambda}_0$ of SO(dimG/H) is replaced by the spinor weight $\tilde{\Lambda}_s$. The charge thus obtained should be identified with Q' in Eq. (2):

TABLE I. Hermitian symmetric spaces and central charges c (level k).

$SU(m+n)/SU(m)\times SU(n)\times U(1)$	$c=3kmn/(k+m+n)$
$SO(n+2)/SO(n)\times U(1)$	$c=3kn/(k+n)$
$SP(2n)/SU(n)\times U(1)$	$c=3kn(n+1)/2(k+n+1)$
$SO(2n)/SU(n)\times U(1)$	$c=3kn(n-1)/2(k+2n-2)$
$E_6/SO(10)\times U(1)$	$c=48k/(k+12)$
$E_7/E_6\times U(1)$	$c=81k/(k+18)$

$$Q' = -\frac{q+\delta q}{k+g} + \sum_l \tilde{\Lambda}_s^l$$
$$= \frac{-q}{k+g} + \frac{k\sum\tilde{\Lambda}_s^l}{k+g} + \frac{-\delta q + g\sum\tilde{\Lambda}_s^l}{k+g}$$
$$= \frac{-q}{k+g} + \frac{c}{6} ,$$

if the shift of q is given by $\delta q = 2\alpha_+ \cdot \alpha_+$. Note that, in the last equality, we used the identity Eq. (9) and the fact that for the spinor weight of SO($2n$) (dimG/H always turns out to be an even integer), $2\tilde{\Lambda}_s^2 = \sum \tilde{\Lambda}_s^l$, since the quantity just depends on the number of the components.

Next in order to determine the contribution of the U(1) charge q to the conformal weight h, let us separate the U(1) part from H to write $H = H' \times U(1)$ and express h as

$$h = \frac{1}{2(k+g)} [C_2(\Lambda) - C_2(\lambda') - \gamma q^2] , \quad (11)$$

where $C_2(\Lambda)$ is the quadratic Casimir constant for G, $C_2(\lambda')$ for the H' and γ is the coefficient of the U(1) part which is to be determined. Regarding h as the conformal weight for a primary state in the NS sector, we perform the shift q as determined above as well as add the spinor weight $\frac{1}{2}\tilde{\Lambda}_s^2$, with the result being considered as h':

$$h' = \frac{1}{2(k+g)} [C_2(\Lambda) - C_2(\lambda') - \gamma q^2] + \frac{-2\gamma q(\delta q)}{2(k+g)} + \frac{-\gamma(\delta q)^2}{2(k+g)} + \tfrac{1}{2}\tilde{\Lambda}_s^2 .$$

Upon choosing $\gamma = 1/(2\delta q) = 1/(4\alpha_+ \cdot \alpha_+)$, the second term becomes $Q/2$, which is identified with the second term in Eq. (1) for $\eta = \frac{1}{2}$. With the use of Eq. (10), the last two terms are combined to give

$$\frac{k\tilde{\Lambda}_s^2}{2(k+g)} + \frac{1}{2(k+g)}(g\tilde{\Lambda}_s^2 - \alpha_+ \cdot \alpha_+) = \frac{c}{24} .$$

Therefore, we understand that the group-theory identity Eq. (9) arises as the consequence of the requirement of the $N=1$ spacetime supersymmetry. It would be rather obvious that, by repeating this procedure, one is able to get all of the superpartners of a multiplet, as in the case of the $N=2$ minimal models.

Summarizing, the supersymmetry operator is given by the following two operations: the change of the highest weight

$$\tilde{\Lambda}_0 \to \tilde{\Lambda}_s \quad (12)$$

and

$$q \to q + 2\alpha_+ \cdot \alpha_+ . \quad (13)$$

And the conformal weight is given as

$$h = \frac{1}{2(k+g)} \left[C_2(\Lambda) - C_2(\lambda') - \frac{q^2}{(4\alpha_+^2)} \right] + \tfrac{1}{2}\tilde{\Lambda}^2 . \quad (14)$$

Actually, the U(1) charge $q = 2\alpha_+ \cdot \lambda$ is determined once the highest-weight state $\lambda = \sum n_i \lambda_i$ is specified. In order to calculate q in terms of n_i, one needs to examine each model separately.

We now consider some examples. Let us take the Grassmannian model

$$G/H = \mathrm{SU}(m+n)/\mathrm{SU}(m) \times \mathrm{SU}(n) \times \mathrm{U}(1) ,$$

which is one of the extensions of the $N=2$ minimal models, and determine the supersymmetry operator. First, in order to calculate the sum of the positive roots of SU($m+n$), we introduce the ($m+n$)-dimensional orthonormal basis e_i:

$$e_i = (0,0,\ldots,1,0,\ldots,0) ,$$

where "1" is in the ith position. The simple roots of SU($m+n$) are then given by $\alpha_i = e_i - e_{i+1}$ ($i = 1,\ldots,m+n-1$) and are normalized as $\alpha_i^2 = 2$. The half-sum of the positive roots, ρ_G, which has the Dynkin coefficient $(1,1,\ldots,1)$, is given in the basis e_i as

$$\rho_G = \left[\frac{m+n}{2}, \frac{m+n}{2} - 1, \ldots, 1 - \frac{m+n}{2}, \frac{m+n}{2} \right] .$$

Next in computing the half-sum of the positive roots of the subgroup H, ρ_H, we have to assign the simple roots such that the Cartan subalgebras of SU(m) and SU(n) are commuting. In the Dynkin diagram of SU($m+n$), we can simply choose the first $m-1$ roots for those of SU(m) and the last $n-1$ ones for SU(n), and the remaining mth root obviously replaced by U(1). ρ_H and then α_+, in this choice of roots, are calculated in this basis e_i to be

$$\rho_H = \left[\frac{m-1}{2}, \frac{m-1}{2} - 1, \ldots, -\frac{m-1}{2}, \frac{n-1}{2}, \frac{n-1}{2} - 1, \ldots, 1 - \frac{n-1}{2}, \frac{n-1}{2} \right] ,$$

$$\alpha_+ = \rho_G - \rho_H$$
$$= \left[\frac{n}{2}, \ldots, \frac{n}{2}, -\frac{m}{2}, \ldots, -\frac{m}{2} \right] ,$$

where, in the expression of α_+, $n/2$ for the first m entries and $-m/2$ for the last n. The Kac-Moody system SO(dimG/H) is actually SO($2mn$) and the spinor weight is the mn component vector $\tilde{\Lambda}_s = \frac{1}{2}(1,1,\ldots,1)$; hence, $\tilde{\Lambda}_s^2 = \frac{1}{4}mn$. With the dual Coxeter number $g = m+n$ for SU($m+n$), the identity Eq. (9) is easily checked,

$$\alpha_+ \cdot \alpha_+ = \tfrac{1}{4}mn(m+n) = g \times \tilde{\Lambda}_s^2 ,$$

as well as Eq. (8),

$$\alpha_+ \cdot \rho_H = \frac{n}{2} \sum_{i=1}^{m} \left[\frac{m-1}{2} + 1 - i \right] - \frac{m}{2} \sum_{i=1}^{n} \left[\frac{n-1}{2} + 1 - i \right]$$
$$= 0 .$$

Note that the summation is vanishing separately for SU(m) and SU(n).

From the expression of α_+, the amount of shift δq and γ are obtained as

$$\delta q = 2\alpha_+ \cdot \alpha_+ = \tfrac{1}{2} mn(m+n) ,$$
$$\gamma = \frac{1}{mn(m+n)} .$$

One can also be convinced that the central charge is correctly reproduced by the formula Eq. (10).

Now, in order to write the charge q in terms of the highest weight $\lambda = \sum n_i \lambda_i$, let us consider the simple case where $\Lambda = \lambda$. This actually turns out to be an important case. By putting $\Lambda = \lambda$ in the formula of conformal weight (we consider the NS sector, $\tilde{\Lambda} = \tilde{\Lambda}_0$), we simply get $h = -Q/2$. This is nothing but the condition for chiral primary states, which are of the prime interest in studying spectra of models (for example, **27** and $\overline{\mathbf{27}}$ of the E_6 gauge belong to this class). The fundamental weights λ_i are defined by

$$\frac{2(\alpha_i \cdot \lambda_j)}{\alpha_j \cdot \alpha_j} = \delta_{ij} ; \tag{15}$$

then, the charge $q = 2\alpha_+ \cdot \lambda$ is calculated to be

$$q = n \sum_{i=1}^{m} i n_i + m \sum_{i=m+1}^{m+n-1} (m+n-i) n_i . \tag{16}$$

This means that once a NS state is obtained by specifying n_i, q, and singlet weight $\tilde{\Lambda}_0$, the corresponding R state is simply gotten by the set of numbers n_i, $q + \frac{1}{2} mn(m+n)$ and $\tilde{\Lambda}_s$.

A simpler but important subgroup of the Grassmannian manifold is the complex projective spaces

$$G/H = \mathrm{SU}(n+1)/\mathrm{SU}(n) \times \mathrm{U}(1) .$$

The central charges of the series are given by $c = 3kn/(k+n+1)$, and are relatively small; hence, those models have a potential to describe a variety of four-dimensional string vacua. With the choice of the orthonormal basis e_i as before, the half-sum of the positive roots and α_+ are given by

$$\rho_G = \left[\frac{n}{2}, \frac{n}{2} - 1, \ldots, 1 - \frac{n}{2}, -\frac{n}{2} \right] ,$$
$$\alpha_+ = \left[\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{n}{2} \right] .$$

From these, the shift of charge is given by $\delta q = 2\alpha_+ \cdot \alpha_+ = \frac{1}{4} n(n+1)$ and the conformal weight h is

$$h = \frac{1}{2(k+g)} \left[C_2(\Lambda) - C_2(\lambda') - \frac{q^2}{n(n+1)} \right] + \tfrac{1}{2} \tilde{\Lambda}^2 .$$

For $n = 1$ we easily see that the supersymmetry operator is gotten as $\delta q = 1$ and the expression of the conformal weight h becomes that of $N = 2$ minimal modes [hence deleting $C_2(\lambda')$].

For other coset models the supersymmetry operators are calculated in a similar way, and ρ_G and α_+ are summarized in the appendix.

III. GAUGE GROUP REPRESENTATION OPERATORS

In this section we consider the left-moving (bosonic) sector, from which the structure of gauge groups arises. Again we concentrate just on the part compactified by the $N = 2$ superconformal models.

We are interested in the spectra that are the representations of the gauge group E_6 although the full group in this compactification scheme is generally given by $E_6 \times U(1)^{r-1} \times E_8$. The states belonging to E_6 are specified in terms of the representations of SO(10)×U(1) as mentioned before; i.e., **27** is decomposed as $\mathbf{10}_{-1} + \mathbf{16}_{1/2} + \mathbf{1}_2$. The purpose is to construct the operators that relate those different states of SO(10) representation.

Let us again consider the $N = 2$ minimal models, since the discussion can be easily extended for coset models. Introduce the antiholomorphic operator $\bar{S}^{(27)}(\bar{z})$ given by

$$\bar{S}^{(27)}(\bar{z}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}^r .$$

One can then show, as in the case of the supersymmetry operator, that the action of $\bar{S}^{(27)}(\bar{z})$ on a state in the $\mathbf{10}_{-1}$ of SO(10), for example, shifts the quantum numbers of left movers as $\bar{q} \to \bar{q} + 1$ and $\bar{s} \to \bar{s} + 1$, giving a state in the $\mathbf{16}_{1/2}$ of SO(10). By applying this operator twice, one gets the $\mathbf{1}_2$ of SO(10). The key point is that the U(1) charge of this operator is $\bar{Q} = \frac{3}{2}$ and $Q = 0$ for tensor-product theories with $c_{\text{tot}} = 9$; therefore, the U(1) charges of the states $\mathbf{10}_{-1}$, $\mathbf{16}_{1/2}$, and $\mathbf{1}_2$ change correctly under this transformation. States of $\overline{\mathbf{27}}$ in representations other than $\mathbf{10}_1$ of SO(10) are obtained by defining an analogous operator:

$$\bar{S}^{\overline{27}}(\bar{z}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}^r .$$

With this preparation, in order to construct the gauge representation operators for the coset models, we need to check whether the shifts of the U(1) charge $\bar{q}$ and the highest weight of SO(dimG/H), in the left mover, reproduce the changes of the U(1) charges of the SO(10) states as required. Suppose that $\mathbf{10}_{-1}$ of SO(10) (we consider **27**) is chosen as the canonical states that are given by some fundamental weights, Λ and λ of G and H and the singlet weight of SO(dimG/H); i.e., the charge $\bar{Q}$ is given by

$$\bar{Q} = -\frac{\bar{q}}{k+g} . \tag{17}$$

Consider the shift of U(1) charge given by $\delta\bar{q} = 2\alpha_+ \cdot \alpha_+$ as well as the replacement of the singlet weight by the spinor weight. Then the change of $\bar{Q}$ of a primary state is computed as

$$\delta\bar{Q} = -\frac{\delta\bar{q}}{k+g} + \sum_l \tilde{\Lambda}_s^l = \frac{2k\tilde{\Lambda}_s^2}{k+g} = \frac{c}{6} = \frac{3}{2} ,$$

where we made use of the previous relation $\sum \tilde{\Lambda}_s^l = 2\tilde{\Lambda}_s^2$ and Eq. (10). Hence this operation can be identified as the group representation transformation by which $\mathbf{16}_{1/2}$ and $\mathbf{1}_2$ are obtained, and similarly for $\overline{\mathbf{27}}$. It is obvious that this construction is also valid for any tensor product theories with $c=9$. The gauge representation operators in the coset models have the same structure as the supersymmetry operators.

IV. DISCUSSION

We have constructed supersymmetry and gauge group representation operators in the context of $N=2$ coset models, using the fact that the models are characterized by group-theory identities. The operators considered here are the generalization of those of the $N=2$ minimal models and obtained in a model-independent way. On the other hand, it may be possible to construct such operators of different types, still consistent with the spectral flow, which in general change n_i in the definitions of the highest weights of G and H. However, one of the aspects that is not fully discussed in the $N=2$ coset models is the field identifications of primary states. Even for the known four-dimensional string models constructed via $N=2$ minimal models, the analysis of the spectrum becomes complicated due to the field identifications derived from the property of the partition function. In the coset models the field identifications originate in the outer-automorphism of Dynkin diagrams, which interchanges the highest weight representations of groups. Therefore, it is not clear whether such supersymmetry operators that also depend on n_i can be constructed consistent with the field identifications.

It is also shown how the characteristic identity Eq. (9) results from the spectral flow, in other words the existence of $N=1$ spacetime supersymmetry. Although all the $N=2$ coset models are found to satisfy this relation explicitly, this requirement of the spectral flow is inherently independent of the construction of Hermitian symmetric spaces. It is not clear yet that these algebraic identities are the necessary and sufficient conditions to have $N=2$ superconformal symmetry. However, from the point of view of general discussion that the $N=1$ spacetime supersymmetry implies the $N=2$ superconformal invariance,[10] these identities are nothing but the conditions for having $N=2$ superconformal symmetry. Hence, the set of equations (8)–(10) may be regarded as the criteria for constructing new $N=2$ coset models. One can choose some G and H to check the identities. It turns out that the orthogonality condition is relatively easy to be satisfied; on the other hand, examples that satisfy the supersymmetry condition are not found yet.

Finally, we note that our result can also be applicable to the Kähler manifold, which can be decomposed into a product of models each of which is based on a Hermitian symmetric space.[11]

APPENDIX

We summarize some results for the coset models that are not considered explicitly in the text. The expressions for ρ_G and α_+ (and ρ_H if necessary) are given in the basis e_i, from which the identities are easily checked.

(1) $G/H = \mathrm{SO}(2n+2)/\mathrm{SO}(2n)\times \mathrm{U}(1)$

$$\rho_G = (n, n-1, n-2, \ldots, 2, 1, 0) ,$$
$$\alpha_+ = (n, 0, 0, \ldots, 0) .$$

(2) $G/H = \mathrm{SO}(2n+1)/\mathrm{SO}(2n-1)\times \mathrm{U}(1)$

$$\rho_G = \left[\frac{n-1}{2}, \frac{n-3}{2}, \frac{n-5}{2}, \ldots, \frac{3}{2}, \frac{1}{2}\right] ,$$
$$\alpha_+ = \left[\frac{n-1}{2}, 0, \ldots, 0\right] .$$

(3) $G/H = \mathrm{SP}(2n)/\mathrm{SU}(n)\times \mathrm{U}(1)$

$$\rho_G = \frac{1}{\sqrt{2}}(n, n-1, n-2, \ldots, 2, 1) ,$$
$$\rho_H = \frac{1}{2\sqrt{2}}(n-1, n-3, n-5, \ldots, 1-n) ,$$
$$\alpha_+ = \frac{1}{2\sqrt{2}}(n+1, \ldots, n+1) .$$

(4) $G/H = \mathrm{SO}(2n)/\mathrm{SU}(n)\times \mathrm{U}(1)$

$$\rho_G = (n-1, n-2, \ldots, 2, 1, 0) ,$$
$$\alpha_+ = \tfrac{1}{2}(n-1, n-1, \ldots, n-1) .$$

For the following cases with exceptional groups, we regard E_6 and E_7 as the subgroups of E_8 and assign the roots appropriately, for convenience. The roots of E_8 are chosen to be

$$\alpha_1 = \tfrac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8) ,$$
$$\alpha_2 = e_7 - e_8 , \quad \alpha_3 = e_6 - e_7 , \quad \alpha_4 = e_5 - e_6 ,$$
$$\alpha_5 = e_4 - e_5 , \quad \alpha_6 = e_3 - e_4 ,$$
$$\alpha_7 = e_2 - e_3 , \quad \alpha_8 = e_7 + e_8 .$$

(5) $G/H = E_6/\mathrm{SO}(10)\times \mathrm{U}(1)$

$$\rho_G = 4(e_1 - e_2 - e_3) + 4e_4 + 3e_5 + 2e_6 + e_7 ,$$
$$\rho_H = 4e_4 + 3e_5 + 2e_6 + e_7 ,$$
$$\alpha_+ = 4(e_1 - e_2 - e_3) .$$

(6) $G/H = E_7/E_6 \times U(1)$

$$\rho_G = \tfrac{17}{2}(e_1 - e_2) + 5e_3 + 4e_4 + 3e_5 + 2e_6 + e_7 ,$$
$$\alpha_+ = \tfrac{9}{2}(e_1 - e_2) + 9e_3 .$$

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