Consistency of Faddeev-Popov ghost statistics with gravitationally induced pair creation

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It has been noted that Bose-Einstein statistics of a scalar field follows from gravitationally induced particle creation. We point out that Fermi-Dirac statistics is allowed when there are two or more scalar fields obeying the same field equation. We determine the general form of allowed anticommutation relations and find that the anticommutation relations satisfied by Faddeev-Popov ghosts fit into this form and, therefore, that they are consistent with particle creation caused by a gravitational background. A similar result is obtained for allowed (bosonic) commutation relations of spinor fields. These opposite spin-statistics relations require negative-norm states.

It is known that the spin-statistics relation can be derived for spin-0 and spin-$\frac{1}{2}$ fields by requiring consistency of the commutation relations in curved spacetime.[1] This result has recently been extended to higher-spin fields.[2] (The connection between spin and statistics and the inner product in curved spacetime has also been noted by Wald.[3] Sorkin[4] has suggested that a spin-statistics correlation will exist whenever the underlying theory incorporates the possibility of pair creation.) Since the Faddeev-Popov (FP) ghosts in covariantly quantized gauge theories are fermions with spin 0, it is natural to ask whether the anticommutation relations for the FP ghosts are consistent with their dynamical evolution in curved spacetime.[5] A brief calculation shows that their anticommutation relations are indeed consistent with their pair creation due to the gravitational background field. In this paper, instead of simply showing this fact, we obtain the most general form of allowed anticommutation relations for scalar fields which are consistent with particle creation in curved spacetime.

Let us first review how Bose-Einstein statistics follows from gravitationally induced particle creation for a scalar field as shown in Refs. 1 and 2. Assume that the spacetime is flat for $t<T_1$ and $t>T_2$, and is nonflat for $T_1<t<T_2$. Let a scalar field ϕ satisfy a Klein-Gordon field equation

$$(\Box+\xi R+m^2)\phi=0 \ , \tag{1}$$

throughout the whole region of the spacetime including $T_1<t<T_2$. Let us expand the field ϕ as

$$\phi=\sum_k (u_k a_k + u_k^* a_k^\dagger) \ , \tag{2}$$

where the u_k are positive-frequency solutions in the region $t<T_1$ and

$$\phi=\sum_k (\bar{u}_k \bar{a}_k + \bar{u}_k^* \bar{a}_k^\dagger) \ , \tag{3}$$

where the $\bar{u}_k$ are positive-frequency solutions in the region $t>T_2$. The subscript k refers to momentum. (The summation symbol here represents integration over k.)

If f and g are two solutions of Eq. (1), the Klein-Gordon inner product

$$\langle f,g\rangle = i\int d\Sigma^\mu (f^*\partial_\mu g - g\partial_\mu f^*) \tag{4}$$

is conserved, where $d\Sigma^\mu$ is the surface element of spacelike hypersurface. We normalize the solutions u_k and $\bar{u}_k$ as

$$\langle u_k,u_l\rangle = \langle \bar{u}_k,\bar{u}_l\rangle = \delta_{kl} \ . \tag{5}$$

The δ_{kl} is a shorthand notation for $\delta^3(k-l)$. We find that positive- and negative-frequency solutions are orthogonal to each other. That is,

$$\langle u_k,u_l^*\rangle = \langle \bar{u}_k,\bar{u}_l^*\rangle = 0 \ . \tag{6}$$

The solutions in the two regions $t<T_1$ and $T_2<t$ are related in general as[6]

$$\bar{u}_k = u_l\alpha_{lk} + u_l^*\beta_{lk} \ , \tag{7}$$

$$\bar{u}_k^* = u_l^*\alpha_{lk}^* + u_l\beta_{lk}^* \ . \tag{8}$$

If some of the coefficients β_{lk} are nonzero, there is particle creation. This relation is known as a Bogolubov transformation. Let us define a matrix M as

$$M \equiv \begin{bmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{bmatrix} , \tag{9}$$

where α and β are matrices themselves defined by the components α_{lk} and β_{lk}. Let us further define a row vector f as

$$f \equiv (u, u^*) , \tag{10}$$

where u and u^* are row vectors themselves defined by the components u_k and u_k^*, and similarly for $\bar{f}$. Then the above relations between two complete sets of solutions are concisely written as

$$\bar{f} = fM \tag{11}$$

and Eqs. (5) and (6) are written as

$$\langle f^T, f \rangle = \langle \bar{f}^T, \bar{f} \rangle = \Sigma , \tag{12}$$

where the superscript T indicates transposition and the matrix Σ is defined by

$$\Sigma \equiv \begin{bmatrix} \delta_{lk} & 0 \\ 0 & -\delta_{lk} \end{bmatrix} . \tag{13}$$

The matrix M depends on the portion of the spacetime with $T_1 < t < T_2$. However, the conservation of the scalar product implies

$$\langle \bar{f}^T, \bar{f} \rangle = M^\dagger \langle f^T, f \rangle M . \tag{14}$$

Using Eq. (12), we find that the matrix M satisfies

$$M^\dagger \Sigma M = \Sigma . \tag{15}$$

First let us impose the usual commutation relations for $t < T_1$ as follows:

$$[a_k, a_l^\dagger] = \delta_{kl}, \quad [a_k, a_l] = 0 . \tag{16}$$

Let us define a column vector b by

$$b \equiv \begin{bmatrix} a \\ a^\dagger \end{bmatrix} , \tag{17}$$

where a and $a^\dagger$ are column vectors themselves defined by the components a_k and $a_k^\dagger$. Let us similarly define $\bar{b}$, with components $\bar{a}_k$ and $\bar{a}_k^\dagger$. The above commutation relations can be written as

$$[b, b^\dagger] = \Sigma . \tag{18}$$

Here, a dagger indicates the combined operation of Hermitian conjugation of the component operators and transposition. Equation (11) implies that

$$\bar{b} = M^{-1} b . \tag{19}$$

Hence we find

$$[\bar{b}, \bar{b}^\dagger] = M^{-1}[b, b^\dagger](M^\dagger)^{-1} = (M^\dagger \Sigma M)^{-1} = \Sigma . \tag{20}$$

Thus, the commutation relations for $t > T_2$ take the same form as those for $t < T_1$. That is, the commutation relations (18) are consistent with gravitationally induced particle creation.

Next suppose we imposed anticommutation relations

$$\{a_k, a_l^\dagger\} = \delta_{kl}, \quad \{a_k, a_l\} = 0 , \tag{21}$$

which can be written as

$$\{b, b^\dagger\} = I , \tag{22}$$

where I is the unit matrix. Then Eq. (19) would imply that

$$\{\bar{b}, \bar{b}^\dagger\} = (M^\dagger M)^{-1} . \tag{23}$$

These anticommutation relations differ from those given by Eq. (22) unless $\beta_{kl} = 0$ for all k and l. Thus, the mixing between positive- and negative-frequency solutions, or equivalently, the existence of pair creation, implies that one cannot impose the anticommutation relations (22).

Now let us show that if there are two or more real scalar fields satisfying the same Klein-Gordon equation, then anticommutation relations can be consistently imposed. Let $\phi^{(n)}$ $(n = 1, \ldots, N)$ satisfy the same Klein-Gordon field equation (1). Then one can expand $\phi^{(n)}$ as

$$\begin{aligned} \phi^{(n)} &= \sum_k (u_k a_k^{(n)} + u_k^* a_k^{(n)\dagger}) \\ &= \sum_k (\bar{u}_k \bar{a}_k^{(n)} + \bar{u}_k^* \bar{a}_k^{(n)\dagger}) . \end{aligned} \tag{24}$$

We define the column vectors $b^{(n)}$ ($\bar{b}^{(n)}$) of creation and annihilation operators, $a_k^{(n)\dagger}$ and $a_k^{(n)}$ ($\bar{a}_k^{(n)\dagger}$ and $\bar{a}_k^{(n)}$), in the same way as we defined b and $\bar{b}$. The operators $b^{(n)}$ and $\bar{b}^{(n)}$ for each field satisfy Eq. (19) with the same matrix M. Let us impose the following *anti*commutation relations at early times $t < T_1$:

$$\{b^{(m)}, b^{(n)\dagger}\} = C_{mn} \Sigma , \tag{25}$$

where C_{mn} is a matrix to be determined.[7] Relations analogous to (15) and (19) imply that the anticommutation relations of Eq. (25) are invariant under the Bogolubov transformation induced by the curvature of the spacetime. That is,

$$\begin{aligned} \{\bar{b}^{(m)}, \bar{b}^{(n)\dagger}\} &= M^{-1}\{b^{(m)}, b^{(n)\dagger}\}(M^\dagger)^{-1} \\ &= C_{mn}(M^\dagger \Sigma M)^{-1} = C_{mn} \Sigma . \end{aligned} \tag{26}$$

Hence the anticommutators at late times take the same form as those at early times. In fact, Eq. (25) gives all possible anticommutation relations consistent with pair creation, under the assumption of Poincaré invariance at early times. (The Poincaré invariance requires that $\{b^{(m)}, b^{(n)\dagger}\} = C_{mn}\Sigma + C'_{mn} I$. However, if C'_{mn} were nonzero, then we would get the same contradiction with $\beta_{kl} \neq 0$ as we did before.) The fact that Eq. (25) describes anticommutators imposes the following conditions on the matrix C_{mn}:

$$C_{mn} = -C_{nm} = C_{nm}^* . \tag{27}$$

These conditions follow by writing out (25) in terms of the $a_k^{(n)}$ and $a_k^{(n)\dagger}$ and then using the relations

$$\{a_k^{(m)}, a_l^{(n)\dagger}\} = \{a_l^{(n)\dagger}, a_k^{(m)}\} = \{a_k^{(m)\dagger}, a_l^{(n)}\}^\dagger . \tag{28}$$

Equation (27) implies that the matrix C_{mn} must be pure imaginary and antisymmetric. With this condition on C_{mn}, Eq. (25) gives the possible anticommutators which are consistently propagated from the initial to the final flat spacetime.

These anticommutators can be put into the canonical form[8]

$$\{b^{(2n)},b^{(2n-1)\dagger}\}=-\{b^{(2n-1)},b^{(2n)\dagger}\}=i\Sigma \quad (1\le n\le N')\ , \tag{29}$$

where $N'\le N/2$ and the other anticommutators are zero. With $N'=1$, we have the anticommutation relations satisfied by the FP ghosts in QED obtained from the Lagrangian

$$\mathcal{L}_{\rm gh}=-i\partial^\mu c_*\partial_\mu c\ , \tag{30}$$

where c and c_* are the ghost and the antighost, which are *Hermitian* fields.[9] The asymptotic fields of the ghosts in non-Abelian gauge theories satisfy the same anticommutation relations. We note that the fields which satisfy the "wrong" statistics operate in a state-vector space which includes negative-norm states, as follows from Eq. (29).

The conditions on C_{mn} can also be obtained by imposing the following local anticommutation relations on the real fields:

$$\{\phi^{(m)}(\mathbf{x},t),\dot\phi^{(n)}(\mathbf{y},t)\}=iC_{mn}\delta^3(\mathbf{x}-\mathbf{y})\ , \tag{31}$$

$$\{\phi^{(m)}(\mathbf{x},t),\phi^{(n)}(\mathbf{y},t)\}=\{\dot\phi^{(m)}(\mathbf{x},t),\dot\phi^{(n)}(\mathbf{y},t)\}=0\ . \tag{32}$$

One again finds that C_{mn} has to be pure imaginary and antisymmetric and that these conditions on C_{mn} are sufficient for consistency.

The situation is similar when we impose bosonic commutation relations on spinor fields satisfying the same Dirac equation. Since free fields may all be considered Majorana fields, we study the case where there are two or more Majorana spinor fields $\psi^{(n)}$ satisfying the same Dirac equation. These fields can be expanded in the Majorana representation as

$$\psi^{(n)}=\sum_k (u_k a_k^{(n)}+u_k^* a_k^{(n)\dagger})\ , \tag{33}$$

where the u_k are positive-frequency solutions in the region $t<T_1$. Here the subscript k represents the spin angular momentum as well as the momentum of the solution. The conserved inner product is

$$\langle f,g\rangle=\int d\Sigma^\mu \bar f\gamma_\mu g\ , \tag{34}$$

for two solutions f and g of the Dirac equation. We have

$$\langle u_k,u_l^*\rangle=0\ . \tag{35}$$

Let us normalize the solutions u_k as

$$\langle u_k,u_l\rangle=\delta_{kl}\ . \tag{36}$$

Now we have

$$\langle u_k^*,u_l^*\rangle=\delta_{kl}\ . \tag{37}$$

Then, instead of Eq. (15), we have

$$M^\dagger M=I\ . \tag{38}$$

Suppose that we impose commutation relations

$$[b^{(m)},b^{(n)\dagger}]=C_{mn}I\ . \tag{39}$$

This is invariant under Bogolubov transformations satisfying Eq. (38). As in the case of scalar fields discussed before, Eq. (39) gives the most general commutators consistent with gravitationally induced particle creation, under the assumption of Poincaré invariance at early times. The fact that these are commutation relations implies, again, that C_{mn} has to be pure imaginary and antisymmetric. (This also results from imposing local commutation relations on the spinor fields.) One can redefine the fields so that they satisfy

$$[b^{(2n)},b^{(2n-1)\dagger}]=-[b^{(2n-1)},b^{(2n)\dagger}]=iI \quad (1\le n\le N')\ , \tag{40}$$

where $N'\le N/2$ and the other commutators are zero. (The fields satisfying these commutation relations act in a space involving negative-norm states for the same reason as in the case of scalar fields.) These are the commutation relations satisfied by spinor ghosts with the Lagrangian

$$\mathcal{L}_{\rm gh}=\bar c_* \not{D} c\ , \tag{41}$$

where c and c_* are Majorana spinor fields. This is the conventional spinor-ghost Lagrangian in supergravity.[10]

In summary, we have determined the general form of possible anticommutation relations allowed for sets of scalar fields when their consistency with pair creation due to the background gravitational field is imposed. They are essentially the ones satisfied by the FP ghosts in gauge theories. We also did the same analysis for sets of spinor fields and obtained similar results.

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[1] L. Parker, Phys. Rev. **183**, 1057 (1969); Phys. Rev. D **3**, 346 (1971); in *Asymptotic Structure of Space-Time*, edited by F. P. Esposito and L. Witten (Plenum, New York, 1977).

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[3] R. M. Wald, Ann. Phys. (N.Y.) **118**, 490 (1979).

[4] R. Sorkin, Phys. Rev. D **27**, 1787 (1983); Commun. Math. Phys. **115**, 421 (1988).

[5] We emphasize here that the FP ghosts are essential even in (covariantly quantized) quantum electrodynamics (QED) for cancellation of unphysical particles. This is because the usual Gupta-Bleuler condition $\nabla^\mu A_\mu^{(+)}|\text{phys}\rangle=0$, where $A_\mu^{(+)}$ is the positive-frequency part, cannot be used in curved spacetime due to the absence of a unique definition of the positive-frequency part. Thus, one must use the condition $Q_B|\text{phys}\rangle=0$, where Q_B is the Becchi-Rouet-Stora-Tyutin charge. This necessitates introduction of the FP ghosts even in (covariantly quantized) QED.

[6] Our notation here is the same as in N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University, Cambridge, England, 1982), except that the matrices α_{kl}

and β_{kl} here are the transposed matrices of those in that textbook.

[7]We could consistently impose the usual bosonic commutation relations, but our point is to show that certain anticommutation relations are also consistent.

[8]One can go from Eq. (25) to Eq. (29) by means of the orthogonal transformation O_{mn}, which block diagonalizes the matrix C_{mn}. The orthogonal matrix O is given by

$$O \equiv \left[\frac{e_1+e_1^*}{\sqrt{2}}, \frac{e_1-e_1^*}{\sqrt{2}i}, \ldots, \frac{e_{N'}+e_{N'}^*}{\sqrt{2}}, \frac{e_{N'}-e_{N'}^*}{\sqrt{2}i}, \tilde{e}_1, \ldots, \tilde{e}_{N-2N'} \right],$$

where e_i is an eigenvector satisfying $Ce_i=\lambda_i e_i$ and the $\tilde{e}_j$ satisfy $C\tilde{e}_j=0$. Here the λ_i are real and positive and C is the matrix with elements C_{mn}. Then the transformation O^TCO gives a block-diagonal pure imaginary matrix as follows:

$$O^TCO = i\,\mathrm{diag}\left[\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \lambda_{N'} \\ -\lambda_{N'} & 0 \end{pmatrix}, 0, \ldots, 0 \right].$$

A further rescaling of the field then puts the anticommutation relations into the form of Eq. (29).

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[10]D. Z. Freedman and A. Das, Nucl. Phys. **B114**, 271 (1976). See H. Hata and T. Kugo, *ibid.* **B158**, 357 (1979), for another possible ghost Lagrangian.