General covariance, new variables, and dynamics without dynamics

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We give an example of a generally covariant geometric field theory which leads to the standard Gaussian and supermomentum constraints but which lacks the super-Hamiltonian constraint. This theory is closely linked to Ashtekar's canonical formulation of general relativity. We discuss the Dirac constraint quantization of our model and comment on the issues which it raises regarding the quantization of generally covariant systems.

I. INTRODUCTION

While the construction of a consistent quantum theory of gravity remains an elusive and difficult task, much effort has been spent in quantizing toy geometric models. These include, in the order of increasing complexity, the topological field theories,[1] lower-dimensional gravity,[2] minisuperspace homogeneous cosmological models,[3] minisuperspace inhomogeneous systems,[4] and bosonic strings.[5] The first three of these models are not field theories but finite-dimensional quantum-mechanical systems, which unduly simplifies (one is tempted to say trivializes) the quantization process. As a result, when the work is done, one may not be any wiser with regard to the conceptual or technical problems associated with quantizing a geometric field theory. The remaining systems, so far as they are tractable, often reduce to a linear field theory. Quantization of the remaining models, in spite of all simplifications, still presents conceptual and technical problems.

In the canonical quantization program, many such problems center about the question of how to handle the super-Hamiltonian constraint. Unlike the gauge constraints (associated with spatial diffeomorphisms and whatever internal gauge group the system in question may have), the super-Hamiltonian generates the dynamics of the system. Its role in the quantization process and reconciliation with gauge constraints remains highly controversial. The generally covariant geometric field theory we are going to describe has a surprising feature that while the gauge constraints are still there, the super-Hamiltonian constraint is missing. Whereas all the model theories we have mentioned have a fewer number of degrees of freedom than general relativity, our model theory actually has more: The state functional of our system can be prescribed as an arbitrary functional of a Riemannian three-geometry. Moreover, because the super-Hamiltonian of the system vanishes, this functional is the same on every hypersurface. The dynamics of the system is best described by saying that there is no dynamics. The virtue of the model (in addition to its capacity to shatter cherished prejudices about the canonical structure of covariant systems) is this clear separation of gauge from dynamics. It can help us to realize why and when and how the super-Hamiltonian constraint is to be incorporated into quantum theory. The model itself is closely related to Ashtekar's formulation of general relativity in terms of the new canonical variables;[6] its action is a slight but significant modification of the Samuel action[7] for the gravitational field. As such, the model is particularly suitable for clarifying those issues which are specific to Ashtekar's formulation, notably, the problem of taking into account the super-Hamiltonian constraint in the loop space representation.

We start by deriving the canonical equations of motion from the covariant spacetime action and showing that while the supermomentum and Gaussian constraints are present, the super-Hamiltonian is missing. We explain this seemingly paradoxical feature by analyzing the geometric structure of our spacetime. We clarify the concept of an observable in our theory and proceed with the Dirac constraint quantization of the model. We discuss three different representations of the quantum states: the metric, connection, and loop space representations. We conclude with a comparison of our model with quantum gravity in Ashtekar's variables.

II. CANONICAL EQUATIONS OF MOTION

The field variables of our model are SU(2)-valued frame one-forms $e^i_\alpha(X)$ and connection one-forms $A^i_\alpha(X)$. The internal indices i,j,k,l run from 1 to 3, $i=1,2,3$, the spacetime indices $\alpha,\beta,\gamma,\delta$ run from 0 to 3, $\alpha=0,1,2,3$. We assume that the frame is regular, i.e., that the three one-forms e^i_α are linearly independent. (Such an assumption is analogous to the regularity of the metric in Einstein's theory of gravitation.) The group manifold carries the SU(2)-invariant metric δ_{ij} and the volume element given by the Levi-Civita tensor ϵ_{ijk}. The group metric is used for lowering and raising the internal indices. The spacetime manifold M does not possess any *a priori* geometric structure. The only spacetime objects at our disposal are the alternating symbols $\tilde{\eta}^{\alpha\beta\gamma\delta}$ and $\underset{\sim}{\eta}_{\alpha\beta\gamma\delta}$. We use the Ashtekar convention: the tilde above a symbol denotes a density of weight 1, under a symbol a density of weight -1. The connection one-forms help us to form covariant derivatives

$$D_\alpha \omega^i = \partial_\alpha \omega^i + \epsilon^i{}_{jk} A^j_\alpha \omega^k \ , \tag{2.1}$$

whose commutator leads to the curvature two-form

$$F^i{}_{\alpha\beta} = \partial_{[\alpha} A^i_{\beta]} + \epsilon^i{}_{jk} A^j_\alpha A^k_\beta \ . \tag{2.2}$$

When we rotate the internal vectors w^i by an angle $\Lambda = (\delta_{ij}\Lambda^i\Lambda^j)^{1/2}$ about an axis Λ^i/Λ,

$$w^i \to w^i + \epsilon^i{}_{jk}\Lambda^j w^k \ , \tag{2.3}$$

the connection forms change by

$$A^i_\alpha \to A^i_\alpha + D_\alpha \Lambda^i \tag{2.4}$$

for $D_\alpha w^i$ to transform as an internal vector.

We shall study the field equations which follow from the action

$$S[e^i_\alpha, A^i_\alpha] = \tfrac{1}{4}\int_M d^4X\, \tilde{\eta}^{\alpha\beta\gamma\delta} e^i_\alpha e^j_\beta F^k{}_{\gamma\delta}[A]\epsilon_{ijk} \ . \tag{2.5}$$

The action (2.5) is invariant under spacetime diffeomorphisms Diff$M$ as well as under the internal gauge group SU(2). Its choice is motivated by its close analogy with the Samuel form[7] of the action for Einstein's theory of gravitation in the Ashtekar variables.[6] Indeed, if we replace the SU(2)-valued frame fields e_α by the SO(3,1)-valued frame fields (i.e., the triad by a tetrad) and the curvature $F_{\alpha\beta}$ by the curvature of the *self-dual* part of the spin connection [which makes $F_{\alpha\beta}$ complex and effectively valued in SU(2)], the action (2.5) becomes the Samuel action.

We shall analyze the spacetime field equations in the next section. Here we are concerned with their canonical form. We cast the action (2.5) into a canonical form by performing its 3+1 decomposition. We assume that M has the topology $R\times\Sigma$, where Σ is a compact space. We choose the spacetime coordinates $X^\alpha = x^\alpha = (t, x^a)$ such that $t \in R$ and the leaves $t =$const of the time foliation have the topology of Σ. (Unlike general relativity, M does not carry a Minkowskian metric and hence it does not make any sense to stipulate that the leaves be spacelike.) Next, we introduce a number of auxiliary quantities. Let

$$\tilde{\eta}^{abc} = \tilde{\eta}^{0abc} \tag{2.6}$$

be an alternating symbol in Σ and

$$\tilde{e} = \frac{1}{3!}\tilde{\eta}^{abc} e^i_a e^j_b e^k_c \epsilon_{ijk} \tag{2.7}$$

the determinant of the triad e^i_a. Because the frame e^i_α is assumed to be regular, $\tilde{e} \neq 0$. The triad

$$\tilde{e}^a_i = \tfrac{1}{2}\tilde{\eta}^{abc}\epsilon_{ijk} e^j_b e^k_c \tag{2.8}$$

represents a frame which is dual to e^i_a. Indeed,

$$\tilde{e}^a_i = \tilde{e} e^a_i, \quad \text{with } e^a_i e^i_b = \delta^a_b \ \text{ and } e^a_i e^j_a = \delta^j_i \ . \tag{2.9}$$

Further, we introduce

$$N^a := e^i_0 e^a_i \tag{2.10}$$

and

$$\Lambda^i := A^i_0 - N^a A^i_a \ , \tag{2.11}$$

and call

$$\tilde{H}_a := F^i{}_{ab}\tilde{e}^b_i - A^i_a \tilde{G}_i = \partial_{[a} A^i_{b]}\tilde{e}^b_i - A^i_a \partial_b \tilde{e}^b_i \ , \tag{2.12}$$

$$\tilde{G}_i := -D_a \tilde{e}^a_i = -\partial_a \tilde{e}^a_i - \epsilon_{ij}{}^k A^j_a e^a_k \ . \tag{2.13}$$

With all these abbreviations, it is easy to check that the action (2.5) assumes the form

$$s[A^i_a, \tilde{e}^a_i; N^a, \Lambda^i] = \int_R dt \int_\Sigma d^3x\, (\tilde{e}^a_i \dot{A}^i_a - N^a \tilde{H}_a - \Lambda^i \tilde{G}_i) \ . \tag{2.14}$$

Because the frame e^i_α is regular, one can freely vary the newly introduced variables A^i_a, $\tilde{e}^a_i$, N^a, and Λ^i instead of the old field variables A^i_α and e^i_α. We see that the action (2.14) has the canonical form if we identify $\tilde{e}^a_i$ with the momentum canonically conjugate to A^i_a. The variation of the multipliers N^a and Λ^i yields the constraints

$$\tilde{H}_a = 0 = \tilde{G}_i \ , \tag{2.15}$$

while the variation of the canonical variables A^i_a and $\tilde{e}^a_i$ leads to the Hamiltonian equations of motion

$$\dot{\tilde{e}}^a_i(x) = \{\tilde{e}^a_i(x), h\}, \quad \dot{A}^i_a(x) = \{A^i_a(x), h\} \ . \tag{2.16}$$

The Hamiltonian

$$h = H_N + G_\Lambda = \int_\Sigma d^3x\, N^a(x)\tilde{H}_a(x) + \int_\Sigma d^3x\, \Lambda^i(x)\tilde{G}_i(x) \tag{2.17}$$

consists of the constraints smeared by the corresponding multipliers.

While the 3+1 decomposition of the action is most easily performed in the adopted system of coordinates $X^\alpha = (t, x^a)$, the equations hold on an arbitrary one-parameter family of embeddings:

$$R\times\Sigma \to M : (t,x) \to X^\alpha = X^\alpha(t,x) \ . \tag{2.18}$$

By inverting the mapping (2.18), we obtain t and x^a as scalar functions on the spacetime manifold M:

$$t = t(X), \quad x^a = x^a(X) \ . \tag{2.19}$$

The hypersurfaces $t =$const are the leaves of the time foliation and the curves $x^a =$const are the world lines of the reference frame. All our equations remain valid if we interpret the variables as appropriate projections with respect to the mappings (2.18) and (2.19). Thus, $\tilde{\eta}^{abc}$ of Eq. (2.6) becomes

$$\tilde{\eta}^{abc} = t_{,\delta}\tilde{\eta}^{\delta\alpha\beta\gamma} x^a{}_{,\alpha} x^b{}_{,\beta} x^c{}_{,\gamma} \ , \tag{2.20}$$

A^i_a and e^i_a are defined as the projections

$$A^i_a := A^i_\alpha X^\alpha{}_{,a}, \quad e^i_a = e^i_\alpha X^\alpha{}_{,a} \ , \tag{2.21}$$

the momentum $\tilde{e}^a_i$ is given by Eq. (2.8), and the determinant $\tilde{e}$ by Eq. (2.7). The multiplier

$$N^a = e^a_i e^i_\alpha \dot{X}^\alpha \tag{2.22}$$

is a projection of the vector $\dot{X}^\alpha = \partial X^\alpha/\partial t$ which is tangent to the reference line $x^a =$const. The geometric meaning of the second multiplier Λ^i is best revealed when we introduce the vector density

$$\tilde{u}^{\alpha}=\frac{1}{3!}\tilde{\eta}^{\alpha\beta\gamma\delta}e^{i}_{\beta}e^{j}_{\gamma}e^{k}_{\delta}\epsilon_{ijk} \tag{2.23}$$

orthogonal to the triad e^{i}_{α}. This vector density does not depend on the foliation (2.18). However, when we want to turn $\tilde{u}^{\alpha}$ into a vector, we must rely on the foliation as an auxiliary element. By projecting $\tilde{u}^{\alpha}$ into the normal $t_{,\alpha}$ to the foliation, we get the scalar density

$$\tilde{e}=t_{,\alpha}\tilde{u}^{\alpha} \tag{2.24}$$

by which we can normalize $\tilde{u}^{\alpha}$:

$$u^{\alpha}:=\tilde{u}^{\alpha}/\tilde{e}, \quad t_{,\alpha}u^{\alpha}=1 \ . \tag{2.25}$$

We can then write

$$\Lambda^{i}=A^{i}_{\alpha}u^{\alpha} \ . \tag{2.26}$$

It is easy to check that, in the adopted system of coordinates $X^{\alpha}=(t,x^{a})$, Eq. (2.24) reduces back to Eq. (2.7), and Eq. (2.26) to Eq. (2.11). The rest of our equation can be checked in exactly the same way. The geometric insight obtained by the explicit introduction of the mappings (2.18) and (2.19) will be helpful when comparing the canonical equations with their spacetime counterparts.

Let us continue with the discussion of canonical formalism. The H_N piece of the Hamiltonian generates, through the Poisson bracket (2.16), the change of the canonical variables A^{i}_{a} and $\tilde{e}^{a}_{i}$ under infinitesimal diffeomorphisms $\mathbf{N}\in$ Diff Σ. Indeed, Eq. (2.12) assigns to $\tilde{H}_a$ the form which is needed to generate diffeomorphisms of the three covector fields A^{i}_{a} and their canonical conjugate vector field densities $\tilde{e}^{a}_{i}$. Similarly, the G_{Λ} piece of the Hamiltonian generates, again through the Poisson bracket (2.16), the rotation (2.3) of the triad $\tilde{e}^{a}_{i}$ and the accompanying inhomogeneous change (2.4) of the connection A^{i}_{a}. The constraints (2.12) and (2.13) mean that we are dealing with a gauge theory under the transformations DiffΣ and SU(2). The constraints (2.15) are first class because

$$\{\tilde{G}_i(x),\tilde{H}_a(x')\}=\tilde{G}_i(x)\delta_{,a}(x,x')\approx 0 \ . \tag{2.27}$$

A (classical) observable in such a theory is any functional $O[A^{i}_{a},\tilde{e}^{i}_{a}]$ of the canonical variables which is invariant under DiffΣ and SU(2):

$$\{O,\tilde{H}_a(x)\}\approx 0\approx\{O,\tilde{G}_i(x)\} \ . \tag{2.28}$$

We shall discuss examples of observables in Sec. IV. Since there are six first-class constraints and nine configuration variables per space point, naive counting reveals three local degrees of freedom.

Other than $\tilde{H}_a$ and $\tilde{G}_i$, there are no constraints. In particular, there is no super-Hamiltonian which would evolve canonical variables from one hypersurface to another. Our theory has infinitely many degrees of freedom, plenty of gauge, but no dynamics! The phase space and the constraints are actually the *same* as those for 3+1 gravity in the Ashtekar variables except that the Hamiltonian constraint is missing. This comes both as a surprise and a puzzle since one expects that there will be a first-class constraint associated with every continuous symmetry of the action. Because the spacetime action (2.5) is invariant not only under *spatial* diffeomorphisms DiffΣ, but under all *spacetime* diffeomorphisms Diff$M$, one expects to find a Hamiltonian constraint side by side with the supermomentum constraints $\tilde{H}_a=0$. Its absence requires an explanation which we shall give by studying the spacetime field equations in the next section.

III. SPACETIME FIELD EQUATIONS

By varying the action (2.5) with respect to the field variables e^{i}_{α} and A^{i}_{α}, we obtain the field equations

$$\epsilon_{ijk}(e^{j}_{\alpha}F^{k}{}_{\beta\gamma}+e^{j}_{\beta}F^{k}{}_{\gamma\alpha}+e^{j}_{\gamma}F^{k}{}_{\alpha\beta})=0 \tag{3.1}$$

and

$$\epsilon_{ijk}(e^{j}_{\alpha}D_{[\beta}e^{k}_{\gamma]}+e^{j}_{\beta}D_{[\gamma}e^{k}_{\alpha]}+e^{i}_{\gamma}D_{[\alpha}e^{k}_{\beta]})=0 \ . \tag{3.2}$$

Their projection into Σ by means of $X^{\alpha}{}_{,a}X^{\beta}{}_{,b}X^{\gamma}{}_{,c}$ yields the constraints (2.13). Because we are interested in the dynamics rather than in the constraints, we should study a projection transverse to Σ. We thus multiply Eq. (3.1) by the vector density (2.23) orthogonal to the triad e^{i}_{α}:

$$\epsilon_{ijk}e^{j}_{[\alpha}\tilde{F}^{k}{}_{\beta]}=0, \quad \text{with } \tilde{F}^{k}{}_{\beta}:=F^{k}{}_{\beta\gamma}\tilde{u}^{\gamma} \ . \tag{3.3}$$

Because $F^{k}{}_{\beta\gamma}$ is antisymmetric in $\beta\gamma$, $\tilde{F}^{k}{}_{\beta}$ is orthogonal to $\tilde{u}^{\beta}$, which means that it necessarily lies in the space spanned by the triad e^{k}_{β}:

$$\exists\tilde{F}^{k}{}_{l}:\tilde{F}^{k}{}_{\beta}=\tilde{F}^{k}{}_{l}e^{l}_{\beta} \ . \tag{3.4}$$

Equation (3.3) thus amounts to

$$\epsilon_{ijk}\tilde{F}^{k}{}_{l}e^{j}_{[\alpha}e^{l}_{\beta]}=0 \ . \tag{3.5}$$

The basis $e^{j}_{[\alpha}e^{l}_{\beta]}$ is antisymmetric in jl and hence Eq. (3.5) can be satisfied only if $\tilde{F}_{ijl}:=\epsilon_{ijk}\tilde{F}^{k}{}_{l}$ is symmetric in jl. However, there is no tensor $\tilde{F}_{ijl}$ which is antisymmetric in the first pair of indices and symmetric in the second pair of indices. As a result, $\tilde{F}_{ijl}$ and therefore $\tilde{F}^{k}{}_{l}$ and $\tilde{F}^{k}{}_{\beta}$ must all vanish. The field equations (3.1) thus imply that

$$F^{k}{}_{\alpha\beta}\tilde{u}^{\beta}=0 \ . \tag{3.6}$$

By the same reasoning, Eq. (3.2) implies that

$$D_{[\alpha}e^{k}_{\beta]}\tilde{u}^{\beta}=0 \ . \tag{3.7}$$

The direction $\tilde{u}^{\alpha}$ plays a prominent role in our theory. Its flow lines define a privileged frame of reference. They thread the spacetime and traverse the embeddings $X^{\alpha}=X^{\alpha}(x)$. In the $\tilde{u}^{\alpha}$ reference frame, those points on different embeddings which are connected by the same flow line count as the same point of Σ. Keep the foliation $t(X)$ fixed and relate an $\dot{X}^{\alpha}$ frame of reference to the $\tilde{u}^{\alpha}$ frame. This is done by decomposing the deformation vector $\dot{X}^{\alpha}$ into a component u^{α} in the direction $\tilde{u}^{\alpha}$ and a component $N^{\alpha}=N^{a}X^{\alpha}{}_{,a}$ along a leaf of the foliation:

$$\dot{X}^{\alpha}=u^{\alpha}+N^{a}X^{\alpha}{}_{,a} \ . \tag{3.8}$$

The component u^{α} is given by Eq. (2.25), and the spatial vector N^{a} by Eq. (2.22). To check the first statement, multiply Eq. (3.8) by $t_{,\alpha}$. To check the second statement, multiply Eq. (3.8) by $e^i_\alpha e^a_i$. For $N^a=0$, the $\dot{X}^\alpha$ reference frame coincides with the $\tilde{u}^\alpha$ reference frame.

The decomposition (3.8) is the closest we can get to the standard lapse-shift decomposition of a pseudo-Riemannian spacetime. The triad field e^i_α endows our spacetime manifold with a covariant metric

$$g_{\alpha\beta}:=\delta_{ij}e^i_\alpha e^j_\beta \ . \tag{3.9}$$

This metric stays the same when the triad is rotated. However, because it is constructed from a triad rather than from a tetrad, the metric (3.9) is degenerate. The direction in which the metric becomes degenerate is orthogonal to the triad:

$$g_{\alpha\beta}\tilde{u}^\beta=0 \ . \tag{3.10}$$

Because the triad itself is regular, $g_{\alpha\beta}$ does not have any further degeneracy and its signature is $(0,+,+,+)$.[8]

The vector u^α is orthogonal to the hypersurface $X^\alpha=X^\alpha(x)$ because

$$g_{\alpha\beta}X^\alpha{}_{,a}u^\beta=0 \ . \tag{3.11}$$

Indeed, it is orthogonal to *every* hypersurface which passes through the point X because u^α is the degeneracy direction. With respect to the foliation $t(X)$, u^α is normalized by the condition

$$t_{,\alpha}u^\alpha=1 \ . \tag{3.12}$$

The normal component u^α of the deformation vector $\dot{X}^\alpha$ in a pseudo-Riemannian spacetime is defined by the same equations, (3.11) and (3.12). In such a spacetime, one can further split u^α into the lapse function N and the unit vector n^α: $u^\alpha=Nn^\alpha$. It is this last step which becomes impossible in our spacetime, because u^α has zero length with respect to the degenerate metric (3.9). The lapse function has no place in our scheme, while the shift vector N^a is still well defined.

With this geometric picture in mind, we ask how the triad e^i_α and the connection A^i_α changes along the flow lines of u^α. Equations (2.1), (2.2), (2.26), (3.7), and (3.12) help us to evaluate the relevant Lie derivatives. We get

$$L_u e^i_\alpha=\epsilon^i{}_{jk}\Lambda^j e^k_\alpha \tag{3.13}$$

and

$$L_u A^i_\alpha=D_\alpha\Lambda^i \ . \tag{3.14}$$

The Lie transport along u^α thus amounts to a rotation by Λ^i of Eq. (2.26). Any SU(2)-invariant quantity constructed out of the field variables e^i_α and A^i_α remains the same along the flow lines of u^α. In particular, the spacetime metric (3.9) does not change:

$$L_u g_{\alpha\beta}=0 \ . \tag{3.15}$$

The intrinsic metric

$$g_{ab}=g_{\alpha\beta}X^\alpha{}_{,a}X^\beta{}_{,b} \tag{3.16}$$

in the comoving frame of the observer $\tilde{u}^\alpha$ is thus the same on *every* transverse hypersurface. In other words, every transverse hypersurface has the same intrinsic geometry. There is no dynamics of geometry in our spacetime.

Moreover, we can choose the gauge in which even the field variables e^i_α, A^i_α themselves will not change along the flowlines of $\tilde{u}^\alpha$. Equations (3.13) and (3.14) determine the change of e^i_α and A^i_α when Λ^j is given, but they leave the choice of $\Lambda^j=A^j_\alpha u^\alpha$ itself arbitrary. By choosing $\Lambda^j=0$, we ensure that e^i_α and A^i_α are Lie propagated along the flow lines of $\tilde{u}^\alpha$. Their projections on an arbitrary hypersurface which is parametrized by the comoving coordinates of the $\tilde{u}^\alpha$ reference frame are thus always the same. This is exactly what the Hamilton equations (2.15)–(2.17) predict: when we put $\Lambda^j=0$ and stay in the comoving coordinates of $\tilde{u}^\alpha$ (i.e., put $N^a=0$), $\tilde{e}^a_i$ and A^i_a are constants of motion. When we choose a different Λ^j, we subject $\tilde{e}^a_i$ and A^i_a to a rotation, and when we choose a different reference frame, we subject $\tilde{e}^a_i$ and A^i_a to a spatial diffeomorphism.

This resolves the mystery of the missing super-Hamiltonian. The super-Hamiltonian is missing not because the hypersurface is stuck and cannot move from a spot, but because the field equations predict that the SU(2)-invariant dynamical variables must be the same everywhere along the flow line of $\tilde{u}^\alpha$. The super-Hamiltonian is thus expected to produce no change; the super-Hamiltonian which definitely does not produce any change is one which is equal to zero. It is more accurate to say that the super-Hamiltonian vanishes than to say that it is missing.

IV. OBSERVABLES

In Sec. II, we have defined an observable as a dynamical variable $O[A^i_a,\tilde{e}^i_a]$ which is invariant under DiffΣ and SU(2), i.e., which has vanishing Poisson brackets (2.28) with the constraints. Because the super-Hamiltonian of our dynamical system vanishes, any observable O is also a constant of motion; i.e., its value is the same on any transverse hypersurface. This distinguishes our theory from a truly dynamical theory, as Einstein's theory of gravitation.[9]

It is easy to give examples of observables. Let us first discuss such observables which depend only on the field momenta $\tilde{e}^a_i$. The simplest of them is the volume

$$O[\tilde{e}^a_i]=\int_\Sigma d^3x\,\tilde{e}=\int_\Sigma d^3x\,(\underset{\sim}{\eta}_{abc}\tilde{e}^a_i\tilde{e}^b_j\tilde{e}^c_k\epsilon^{ijk})^{1/2} \tag{4.1}$$

of the hypersurface Σ. Another class of such observables is obtained by taking the metric

$$g_{ab}(x)=\delta_{ij}e^i_a e^j_b \ , \tag{4.2}$$

finding its Riemann curvature tensor $R_{abcd}(x)$, differentiating it covariantly on Σ, forming any scalar $\Phi(x)$ out of g_{ab}, R_{abcd}, and covariant derivatives of R_{abcd} up to finite order, and integrating this scalar with respect to the volume element $\tilde{e}$:

$$O[\tilde{e}^a_i]=\int_\Sigma d^3x\,\tilde{e}(x)\Phi(x) \ . \tag{4.3}$$

One can conceive of more complicated observables, e.g., those formed by a double integration of scalars or tensors connected by a kernel, such as the world function or a product of parallel propagators. Thus, if $\Omega(x,x')$ is the world function on Σ, i.e., the integral of $g_{ab}dx^a dx^b$ along the geodesic connecting x with x', and $\Phi(x)$ an arbitrary scalar as in Eq. (4.3), the double integral

$$O[\tilde{e}_i^a]=\int_\Sigma d^3x\int_\Sigma d^3x'\tilde{e}(x)\Phi(x)\Omega(x,x')\tilde{e}(x')\Phi(x') \tag{4.4}$$

is an observable. Of course, on a compact Σ there may be more than one geodesic connecting the points x and x' and we run into difficulties of specifying which one is to be taken when defining the world function.

Second, let us discuss observables which are constructed solely from the field coordinates A_a^i. One such observable is provided by the Chern-Simons functional

$$O[A_a^i]=\int_\Sigma d^3x\,\tilde{\eta}^{abc}(\delta_{ij}A_a^iF^j{}_{bc}+\tfrac{2}{3}\epsilon_{ijk}A_a^iA_b^jA_c^k) . \tag{4.5}$$

Indeed, the dual to the curvature two form, i.e., the "magnetic component"

$$\tilde{B}^{ia}:=\tfrac{1}{2}\tilde{\eta}^{abc}F^i{}_{bc} \tag{4.6}$$

of the field helps us to construct other examples of observables which are closely parallel to those constructed from the "electric component" $\tilde{e}_i^a$ of the field. There is a "magnetic volume"

$$O[A_a^i]=\int_\Sigma d^3x\,\tilde{B}=\int_\Sigma d^3x\,(\underset{\sim}{\eta}_{abc}\tilde{B}^{ia}\tilde{B}^{jb}\tilde{B}^{kc}\epsilon_{ijk})^{1/2} . \tag{4.7}$$

Further, if the "magnetic metric"

$$h^{ab}=(\tilde{B})^{-1}\delta_{ij}\tilde{B}^{ia}\tilde{B}^{jb} \tag{4.8}$$

turns out to be nondegenerate, we can go through the familiar steps of forming its curvature tensor, the world function, etc., and defining the counterparts of the "electric" observables such as (4.3) or (4.4).

Finally, we can write observables which depend both on $\tilde{e}_i^a$ and A_a^i, as

$$O[A_a^i,\tilde{e}_i^a]=\int_\Sigma d^3x\,\underset{\sim}{e}\tilde{B}^2 . \tag{4.9}$$

While the constraints (2.12) and (2.13) make it difficult to construct nontrivial scalars out of $\tilde{e}_i^a$ and $F^i{}_{ab}$ because they imply

$$F^i{}_{ab}\tilde{e}_i^b=0 , \tag{4.10}$$

once we pass to the level of curvature tensors of g_{ab} and h_{ab}, we can easily construct observables such as

$$O[A_a^i,\tilde{e}_i^a]=\int_\Sigma d^3x\,\underset{\sim}{e}R_{ab}[g]h^{ab} . \tag{4.11}$$

These examples can be proliferated ad infinitum. The problem which remains unsolved, however, is a construction of a complete commuting set of observables.

There is another set of functionals on the phase space that are invariant under the SU(2) gauge transformations but not under spatial diffeomorphisms DiffΣ. These are the T variables based on loops that were introduced for 3+1 gravity[10] and which have exactly the same form in our theory.

Let γ be a curve in Σ with end points x_0, x_1, and $\tau_{iA}{}^B$ be the Pauli matrices, where A,B are the matrix (spinor) indices. The holonomy $U[\gamma]$ of the connection A_a^i is an SU(2) matrix defined by

$$U[\gamma]_A{}^B(x_0,x_1)=\left[P\exp\left[\int_{x_0}^{x_1}dx^aA_a^i(x)\tau_i\right]\right]_A{}^B , \tag{4.12}$$

where P denotes path ordering. When γ is a closed curve, one can take the trace of this matrix and define the first of the loop variables:

$$T^0[\gamma]=\mathrm{Tr}(U[\gamma]) . \tag{4.13}$$

Further, one can start inserting $\tilde{e}^a=\tilde{e}_i^a\tau^i{}_A{}^B$ at selected points $x_1,x_2,\ldots$ on the loop γ and define the variables

$$T^a[\gamma](x_1)=\mathrm{Tr}\{U[\gamma]\tilde{e}^a(x_1)\} , \tag{4.14}$$

$$T^{ab}[\gamma](x_1,x_2)=\mathrm{Tr}\{U[\gamma](x_0,x_1)\tilde{e}^a(x_1)U[\gamma](x_1,x_0)\times\tilde{e}^a(x_2)U[\gamma](x_2,x_0)\} , \tag{4.15}$$

with obvious generalization to T's containing any number of $\tilde{e}^a$ insertions. These variables form a closed algebra under Poisson brackets and are invariant under the Gaussian constraint. A particular representation of this algebra has been used in quantum gravity.

V. QUANTIZATION

To quantize our system, we turn the canonical variables $A_a^i(x)$ and $\tilde{e}_i^a(x)$ into operators satisfying the Dirac commutation relations

$$\frac{1}{i}[\hat{A}_a^i(x),\hat{\tilde{e}}_j^b(x')]=\delta_j^i\delta_a^b\delta(x,x') . \tag{5.1}$$

By substituting these operators into the classical expressions (2.12) and (2.13), the supermomentum H_a and the Gauss variable G_i are also turned into operators:

$$\hat{\tilde{H}}_a(x)=\tilde{H}_a(x;\hat{A}_b^i,\hat{\tilde{e}}_i^b],\quad \hat{\tilde{G}}_i(x)=\tilde{G}_i(x;\hat{A}_b^i,\hat{\tilde{e}}_i^b] . \tag{5.2}$$

By the standard algorithm of the Dirac constraint quantization, the classical constraints (2.15) are then imposed as restrictions on the physical states Ψ of the theory:

$$\hat{\tilde{H}}_a(x)\Psi=0=\hat{\tilde{G}}_i(x)\Psi . \tag{5.3}$$

The states Ψ in the Schrödinger picture refer to a given embedding $X^\alpha(x)$, i.e., to an instant of time. Because the Hamiltonian (2.17) is a linear combination of the constraints, the Schrödinger equation tells us that the state function in the Schrödinger picture actually does not depend on the embedding. Because the classical theory does not predict any dynamics, this is quite understandable and it is not puzzling as in theories with nontrivial super-Hamiltonian. Whatever inner product is chosen in the space of physical states (5.3), this product is conserved in the dynamical evolutions. The virtue of our model is that by making the dynamics trivial, it focuses attention on those aspects of quantum gravity which are purely kinematic. It enables one to pinpoint those aspects of quantum gravity which the inclusion of a super-Hamiltonian constraint sets apart from the quantum theory of our spacetime, with its degenerate metric and lack of dynamics.

We shall describe now three different representations of the quantum theory (5.1)–(5.3). These are the metric and connection representations (which correspond to the momentum and position representations in ordinary quantum mechanics), and the loop space representation. The last representation will be constructed along the lines used for the 3+1 Einstein's theory of gravitation by Rovelli and Smolin.[10]

A. Metric representation

As contrasted with the metric representation in the standard Dirac-ADM (Arnowitt-Deser-Misner) approach, where the spatial metric on a spacelike hypersurface is a coordinate variable on the phase space, the metric (or triad) representation in our model (as in Ashtekar's version of canonical gravity) is the momentum representation. To satisfy Eq. (5.1), the momentum variables $\tilde{e}_j^b(x)$ are turned into multiplication operators and the coordinate variables $A_a^i(x)$ into variational derivatives acting on the space of functionals $\Psi[\tilde{e}_k^c]$:

$$\hat{\tilde{e}}_j^b(x)=\tilde{e}_j^b(x)\times, \quad \hat{A}_a^i(x)=i\delta/\delta\tilde{e}_i^a(x) \ . \tag{5.4}$$

The constraint operators (5.2) are factor ordered so that the action of $\tilde{A}_a^i(x)$ precedes the action of $\hat{\tilde{e}}_j^b(x)$. With this factor ordering, the quantum constraints (5.3) are equivalent to the statement that the physical states are invariant functionals of $\tilde{e}_i^a(x)$ under the action of SU(2) and DiffΣ groups on the arguments $\tilde{e}_i^a(x)$. Any SU(2)-invariant functional of $\tilde{e}_i^a(x)$ can depend on $\tilde{e}_i^a(x)$ only through the metric $g_{ab}(x)$:

$$\hat{\tilde{G}}_i(x)\Psi[\tilde{e}_j^b(x)]=0 \Longrightarrow \Psi=\Psi[g_{ab}(x)] \ . \tag{5.5}$$

The Gauss constraint thus ensures that the triad representation is actually a metric representation. The supermomentum constraint then ensures that the state functional $\Psi[g_{ab}]$ is the same for all metrics connected by a spatial diffeomorphism, i.e., that it depends on the three-geometry $\mathbf{g}$ rather than on the metric: $\Psi=\Psi[\mathbf{g}]$. There is no other restriction on the physical states. The functionals (4.1), (4.3), and (4.4) which were given as examples of classical observables are thus simultaneously examples of physical states in the metric representation.

The Schrödinger representation automatically carries with it an appropriate concept of an inner product. Such a product should be defined only for the states which satisfy the constraints (5.3), i.e., for the physical states $\Psi[\mathbf{g}]$. It is given by the functional integral

$$\langle\Psi_1|\Psi_2\rangle=\int D\mathbf{g}\,\Psi_1^*[\mathbf{g}]\Psi_2[\mathbf{g}] \tag{5.6}$$

over all three-geometries. All classical observables should be represented by operators which are self-adjoint with respect to the inner product (5.6). In a theory with a vanishing super-Hamiltonian, the conservation of the inner product in time is no issue.

B. Connection representation

In the connection representation, Eq. (5.1) is satisfied by defining $\hat{A}\,_a^i(x)$ and $\hat{\tilde{e}}\,_j^b(x)$ as the operators

$$\hat{A}_a^i(x)=A_a^i(x), \quad \hat{\tilde{e}}_j^b(x)=-i\delta/\delta A_b^j(x) \tag{5.7}$$

acting on the functionals $\Psi[A_a^i]$ of the connection. The constraint operators are now factor ordered so that the action of $\hat{\tilde{e}}_j(x)$ precedes the action of $\hat{A}_a^i(x)$. With this factor ordering, the quantum constraints (5.3) are again equivalent to the statement that the physical states $\Psi[A_a^i]$ are unchanged by the action of the SU(2) and DiffΣ groups on the arguments $A_a^i(x)$. The functionals (4.5) or (4.7) are examples of such functionals.

A well-known set of solutions $\Psi[A_a^i]$ of the Gauss constraints are the traces of the Wilson loops formed from the connections. These solutions are parametrized by closed curves in Σ. Unfortunately, their transformation properties under diffeomorphisms are nontrivial and we do not know how to construct out of them, combinations which are invariant under DiffΣ.

When we denote the equivalence classes of connections under SU(2) and DiffΣ groups by $\mathbf{A}$, the inner product of two physical states $\Psi_1[\mathbf{A}]$ associated with the Schrödinger representation (5.7) can be formally written as

$$\langle\Psi_1|\Psi_2\rangle=\int D\,\mathbf{A}\Psi_1^*[\mathbf{A}]\Psi_2[\mathbf{A}] \ . \tag{5.8}$$

Again, its conservation in time is not an issue. Classical observables O must be represented by operators $\hat{O}$ which are self-adjoint with respect to this inner product.

C. Loop space representation

In the previous two subsections, the first step in obtaining a quantum theory was to find a realization of the fundamental canonical commutation rules (5.1) in a suitable representation. The approach used in this section involves finding a linear representation of a noncanonical algebra. Such an algebra is available on the classical phase space of our theory and it is identical to that used for general relativity. This is the algebra of the (Gauss-law-invariant) T variables defined in the preceding section.

The steps involved in constructing a quantum theory are (1) obtain a linear representation of this algebra that reduces in the limit $\hbar\rightarrow 0$ to the classical Poisson algebra on the phase space, (2) define operators on this space that represent the DiffΣ constraint, and (3) obtain the kernal of this constraint to extract the physical states (according to the Dirac prescription). Note that we need not define the Gaussian constraint on the space since the representation basis is, by construction, invariant under this constraint.

The above steps have already been carried out for general relativity and so we only summarize the results here and refer the reader to Ref. 10 for details.

A linear representation of the T algebra is provided by considering complex-valued functions $a[\gamma]$ $(=a[\gamma^{-1}])$ of loops γ, and defining the action of the operators $\hat{T}$ on this basis. As a simple illustration, consider two loops β,γ that have a common point. The definitions are

$$\hat{T}^{0}[\gamma]a[\beta]:=a[\gamma\cup\beta]\ , \tag{5.9}$$

$$\hat{T}^{a}[\gamma](x)a[\beta]:=\hbar\Delta^{a}[\gamma,\beta](x)\times(a[\gamma\circ\beta]-a[\gamma\circ\beta^{-1}])\ . \tag{5.10}$$

Here

$$\Delta^{a}[\gamma,\beta](x)=\int dt\,\dot{\beta}^{a}(t)\delta^{3}(\gamma(s),\beta(t))\ ,$$

where s,t are parameters that run along the loop, $x=\gamma(s)$, and $\dot{\beta}^{a}(t)$ is the tangent vector to β at the parameter value t. The composite loop $\gamma\circ\beta$ is obtained by first transversing γ and then β. The function a evaluated on the union of γ and β is $a[\gamma\cup\beta]$. It satisfies the relation $a[\gamma\cup\beta]=a[\gamma\circ\beta]+a[\gamma\circ\beta^{-1}]$ which reflects a relation between SU(2) holonomies. It is straightforward to check that the commutator algebra that follows from these definitions reduces to the Poisson algebra in the classical limit.

These definitions are generalized in a straightforward way to higher T's, and to loops with multiple intersections.

There is a natural action of DiffΣ on $a[\gamma]$. For $\phi\in$ DiffΣ, the operator $U(\phi)$ is defined by

$$U(\phi)a[\gamma]=a[\phi^{-1}\cdot\gamma]\ , \tag{5.11}$$

where $\phi^{-1}\cdot\gamma$ is the image of γ under ϕ. The states invariant under DiffΣ must satisfy

$$U(\phi)a[\gamma]=a[\gamma]\ . \tag{5.12}$$

Since the diffeomorphism invariant information in a loop is the way that it is knotted, the solutions to (5.12) are functions of the knot classes of the loops and are denoted $a[K(\gamma)]$.

The physical states for the theory in this representation are therefore these functions of knot classes, $a[K(\gamma)]$, and their generalizations to multiloops, the link classes. The issue of the inner product in the loop representation is not clear to us. Since the knot classes form a countable basis, one could formally postulate that the inner product be one that makes states corresponding to different knot classes orthogonal. This issue will be discussed further in the following section.

VI. DISCUSSION

Having completed the discussion of the quantum theory for our model, we turn now to a comparison of the metric, connection and loop representations. Following this, we compare the loop space representations for general relativity and the present model.

The metric and connection representations may be thought of as being related by a formal functional transform, analogous to the ordinary Fourier transform between the coordinate and momentum representations.

Since we have obtained the physical states in the metric and the loop representations, the questions of their equivalence and transformations between them arises. Is there a well-defined functional transform that maps states of one space into states on the other? Another associated question is what is the inner product in the loop representation associated with (5.6)? The answers to these questions may provide a natural association of three-geometries with knot classes of loops and possibly provide an inner product for the physical states of general relativity in the loop representation.

The physical states for general relativity form a subset of the states for our model. This subset is selected by imposition of the super-Hamiltonian constraint of general relativity, and is expected to consist of specific linear combinations of states based on intersecting loops.[11,12] Those states of quantum gravity that are based on smooth nonintersecting loops occur in our model as well. This suggests that such states, by themselves, cannot distinguish the presence or absence of the super-Hamiltonian constraint.

A further difference in the phase-space variables of our model and general relativity is that the coordinate variable for the latter is a complex SU(2) connection, and the Ashtekar formalism includes an associated reality condition. This information has to be fed into the loop space representation in order to distinguish it further from our model (in which A_a^i is real). A proposal is that this be accomplished in the choice of inner product on the loop space, since the issue is one of specifying the hermiticity properties of operators.[13]

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[9] People often define observables in general relativity as those dynamical variables which commute with *all* the constraints, including the super-Hamiltonian. Such a definition implies that an observable necessarily is a constant of motion. This certainly does not capture the ordinary meaning of the term "observable" in unconstrained quantum theory. We believe that in Einstein's theory of gravitation, observables should be defined as those dynamical variables which are gauge invariant under DiffΣ [and, in Ashtekar's formulation, also under SU(2)], but do not necessarily commute with the super-Hamiltonian.

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