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Geometrizing the dynamics of Bianchi cosmology

Claes Uggla and Kjell Rosquist*
Department of Physics, University of Stockholm, Vanadisvägen 9, S-113 46 Stockholm, Sweden

Robert T. Jantzen
Department of Mathematical Sciences, Villanova University, Villanova, Pennsylvania 19085
and International Center for Relativistic Astrophysics, University of Rome, I-00185 Rome, Italy
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The reparametrization freedom in the choice of time variable in the dynamics of spatially homogeneous cosmological models is used to reformulate the field equations as a geodesic flow for a "Jacobi geometry" in a particular time gauge called the Jacobi time gauge. For the diagonalizable models this Jacobi geometry is a conformally flat Lorentzian geometry. By choosing variables which are adapted to the symmetries of the Jacobi geometry, considerable simplification of the field equations is achieved, and one can explain the existence of all known exact solutions in terms of this analysis, as well as simplify the study of the qualitative behavior of the dynamics. In addition, certain "hidden symmetries" which arise in the Jacobi formulation lead to a class of new exact solutions.

I. INTRODUCTION

DeWitt[1] has shown that one can reformulate the vacuum Einstein field equations as an infinite-dimensional system of geodesic equations with respect to a certain Lorentzian metric on the space of three-dimensional Riemannian metrics. He was motivated by the hope that this reformulation might be a useful model for exploring various ideas in gravitational theory. Misner[2] then applied this technique to vacuum cosmological models of Bianchi types I and IX and to the Kantowski-Sachs model. The present paper extends this reformulation to those vacuum and orthogonal perfect-fluid spatially homogeneous (SH) Bianchi models (excluding the general Bianchi type-$\mathrm{VI}_{-1/9}$ case) for which a purely Hamiltonian description exists involving only the metric variables, and by so doing achieves an elegant geometrical interpretation of the dynamics of these models.

The geometrization of a Hamiltonian system by transforming it to a geodesic flow is a well-known technique of classical mechanics associated with the name of Jacobi.[3,4] Examples of how this type of geometric reformulation of a mechanical system can be fruitful were discussed by Pin.[5] By computing the curvature of the Jacobi metric, he obtained a geometrical description of certain properties of the solutions of various mechanical problems. The Jacobi reformulation is associated with a particular choice of time which will be called Jacobi time and in the cosmological case the Jacobi metric is obtained by a conformal rescaling of the finite-dimensional DeWitt metric using a special choice of the lapse function.

The reformulation of dynamics in terms of a geodesic problem allows the application of a wide range of well-known geometrical techniques in the investigation of the solution space and properties of the equations of motion. The advantage of the particular geometric formulation described here for the diagonal cosmological models under consideration is that the Jacobi metric is always conformally flat and therefore necessarily implies the existence of a large family of symmetries to exploit in simplifying the field equations, namely, the whole conformal group of flat spacetime. In contrast the Ashtekar variables[6] for the same class of models apparently lead to a geodesic formulation where the metric does not admit any conformal symmetries in general.

In two previous papers[7,8] the scale invariance of the Einstein equations and the symmetries arising from the automorphisms of the homogeneity group of certain Bianchi models were studied with the aim of describing those time gauges which allow those symmetries to be used in simplifying somewhat the Hamiltonian equations of motion. The Jacobi time gauge naturally arose in this context, where it was found that those symmetries which are effective in simplifying the dynamics lead to both Killing vectors and homothetic Killing vectors of the Jacobi metric.

The power of the Jacobi reformulation is that all of the dynamical information is collected into a single geometric object in which all the available manifest symmetries are retained. The existence of a wide class of already known exact solutions can be explained by these obvious symmetries. However, certain "hidden symmetries" arise which are not apparent in the usual approach to the problem, and these not only explain the existence of the remaining known solutions but lead to entirely new solutions as well. The search for these hidden symmetries is in fact synonymous with the search for new exact solutions, a task that has met little success over recent decades. In the Jacobi formulation these hidden symmetries are associated with both vector symmetries (Killing and homothetic Killing vectors) and Killing tensor symmetries of the Jacobi metric.

Roughly speaking, exact solutions can be divided into those which arise as particular solutions of special cases and those which come from special cases which are fully solvable. The known particular solutions are related to the existence of a timelike homothetic Killing vector, arising from the scale/automorphism symmetry, and certain properties of the conformal factor. The solutions representing fully solvable cases are associated with the existence of Killing symmetries which give rise to enough constants of the motion to completely integrate the geodesic equations. They also lead to a reduction of the dimension of the dynamical problem even when they are insufficient to lead to a complete integration of the geodesic equations. Homothetic symmetries, on the other hand, lead to constants of the motion which are explicitly time dependent[7,8] and are therefore not as effective in reducing the system.

The known exact solutions of the Killing symmetry class can be broadly divided into two subclasses, the vector and tensor solutions, depending on whether they arise from Killing vectors or Killing tensors in the Jacobi geometry. All previously known solutions of this class except the Bianchi type-VIII and -XI vacuum locally rotationally symmetric (LRS) solutions are vector solutions. It appears that all solutions of the vector class have been found. However, new exact solutions of the tensor class will be described in a subsequent series of papers. Similarly, the homothetic symmetries can be divided into vector and tensor cases, lending the same category names to the corresponding solutions. It is still unclear whether homothetic Killing tensors arise in this problem.

The present study examines the models under consideration on a case-by-case basis, finding new metric variables which are adapted to the symmetries of the Jacobi metric. For the Bianchi types I through VII the symmetries which will be examined are the isometries and homothetic symmetries discussed in previous work,[7,8] while for the locally rotationally symmetric models of Bianchi types VIII and IX, the choice of new variables is determined by the Killing tensor which exists there. In all cases these new variables lead to considerable simplification of the field equations.

However, it should be emphasized that although the Jacobi time gauge is extremely useful for analyzing the SH field equations, other time gauges also prove useful for many purposes. In fact, the freedom to choose different time gauges is in itself a powerful tool in the investigation of dynamical systems and it is important to keep an open mind and use any time gauge which simplifies the analysis of a given problem. When adapting the dependent variables to the available symmetries of the Jacobi metric, one is also led to other interesting time gauges. In particular, the Killing tensor symmetries suggest new time gauges in which a separation of variables occurs. Similarly, a timelike homothetic Killing vector leads to the existence of a monotonic function, and adapting the time gauge to this function leads to a simplified qualitative description of the dynamics. An example of this is given in Appendix B. Therefore the Jacobi formulation provides a vehicle both for finding a useful independent variable as well as in suggesting useful dependent variables.

II. THE JACOBI METRIC

Given a metric

$$ds^2=g_{\alpha\beta}dq^\alpha dq^\beta \ , \tag{2.1}$$

a Hamiltonian H of the form

$$H=\tfrac{1}{2}g^{\alpha\beta}(q)p_\alpha p_\beta+U(q) \tag{2.2}$$

describes the trajectory of a particle moving in the geometry of this metric under the influence of a force field corresponding to the potential energy U. If this potential U is constant, i.e., no force is present, then subtracting this constant leads to a purely kinetic Hamiltonian with the same equations of motion. Their solutions represent affinely parametrized geodesics of the metric. When the potential is not a constant, one can convert the Hamiltonian to a purely kinetic form by scaling the Hamiltonian so that the potential term becomes a constant. One then has a geodesic flow with respect to the corresponding conformally rescaled metric.[4] This process is equivalent to a certain reparametrization of the independent variable.

Let $V=U-E$, where the constant parameter E may be interpreted as the constant of energy for the original Hamiltonian system. Introduce the new Hamiltonian $\mathcal{H}=H-E$ which satisfies the constraint $\mathcal{H}=0$ and a second rescaled Hamiltonian $\mathcal{H}_N=N\mathcal{H}$, where N is some function on the phase space. This last Hamiltonian gives the following equations of motion on the constraint surface:

$$\begin{aligned}dq^\alpha/d\lambda&=\partial\mathcal{H}_N/\partial p_\alpha=N\partial\mathcal{H}/\partial p_\alpha+\mathcal{H}\partial N/\partial p_\alpha\\&=N\partial\mathcal{H}/\partial p_\alpha=N\partial H/\partial p_\alpha\ ,\\dp_\alpha/d\lambda&=-\partial\mathcal{H}_N/\partial q^\alpha=-N\partial\mathcal{H}/\partial q^\alpha-\mathcal{H}\partial N/\partial q^\alpha\\&=-N\partial\mathcal{H}/\partial q^\alpha=-N\partial H/\partial q^\alpha\ .\end{aligned} \tag{2.3}$$

Thus the Hamiltonian $\mathcal{H}_N$ gives the same equations of motion as H but expressed in a new time variable λ related to the old time variable t by $dt=N\,d\lambda$. The scaling factor N describes the relative rate of change of the two independent variables.

For the case in which the original metric is positive definite, V must be negative for the energy constraint to have solutions. The choice $N=(-2V)^{-1}$ in the above discussion leads to

$$\mathcal{H}_N=NT+NV=NT-\tfrac{1}{2} \tag{2.4}$$

which must be constrained to vanish. We introduce an effective Hamiltonian, called the Jacobi Hamiltonian, by

$$\begin{aligned}H_J\equiv\mathcal{H}_N+\tfrac{1}{2}&=NT=(-2V)^{-1}T\\&=-\tfrac{1}{4}V^{-1}g^{\alpha\beta}p_\alpha p_\beta\equiv\tfrac{1}{2}J^{\alpha\beta}p_\alpha p_\beta\ ,\end{aligned} \tag{2.5}$$

which defines the contravariant components $J^{\alpha\beta}=Ng^{\alpha\beta}$ of the Jacobi metric

$$ds_J^2=J_{\alpha\beta}dq^\alpha dq^\beta=N^{-1}g_{\alpha\beta}dq^\alpha dq^\beta\ . \tag{2.6}$$

It follows that the constant of energy of the Jacobi Hamiltonian is $\frac{1}{2}$, giving a reformulation of the problem so that the reparametrized solution curves are arclength-parametrized geodesics of the Jacobi metric. For a pseudo-Riemannian metric there is also the possibility that V may be positive, in which case the choice $N=(2V)^{-1}$ leads to the Jacobi Hamiltonian

$$H_J \equiv \mathcal{H}_N - \tfrac{1}{2} = \tfrac{1}{4} V^{-1} g^{\alpha\beta} p_\alpha p_\beta \equiv \tfrac{1}{2} J^{\alpha\beta} p_\alpha p_\beta \ , \qquad (2.7)$$

with energy constant $-\frac{1}{2}$, again corresponding to a unit arclength parametrization. If V is identically zero in the pseudo-Riemannian case, then one obtains null geodesics with an affine parametrization. For a Lorentz metric with the signature $-++\cdots+$, positive V corresponds to timelike geodesics and negative V to spacelike geodesics. In all cases, zeroes of V lead to singularities of the Jacobi metric which must be dealt with by considering the original metric. Since the energy E enters the Jacobi metric through the potential V, in general there will be a one-parameter set of Jacobi geometries.

Two-dimensional Lorentz Jacobi metrics turn out to play a key role in the exactly solvable cases in cosmological dynamics. An arbitrary two-dimensional Lorentz metric is always locally conformally flat[9] and can be put into the form

$$ds^2 = e^{2F(t,x)}(-dt^2 + dx^2) \ . \qquad (2.8)$$

The single independent coordinate component of the curvature for this metric is given by

$$R_{txtx} = -e^{2F}(F_{,tt} - F_{,xx}) \ . \qquad (2.9)$$

Thus the metric will be flat if F satisfies the wave equation. A number of two-dimensional cosmological Jacobi metrics turn out to have this form with F linear in the natural logarithmic metric variables, therefore automatically satisfying the wave equation. These correspond to cosmological models where the Jacobi time gauge coincides with a power-law-lapse time gauge.[10] On the other hand, if we express the above flatness condition in terms of null variables we obtain the following general form for the flat metric:

$$ds^2 = -f(u)g(v)du\, dv, \quad u = t + x, \quad v = t - x \ , \qquad (2.10)$$

where f and g are arbitrary functions. This too will turn out to be relevant for the cosmological models to be considered.

III. HAMILTONIAN COSMOLOGY

The Jacobi reformulation is directly applicable to those classes of spatially homogeneous cosmological models where the complete set of field equations are equivalent to a purely Hamiltonian system, either because the source equations also follow from the Hamiltonian in general, or because special initial data reduces the presence of the source in the Einstein equations to the appearance of constant parameters, resulting in a Hamiltonian system involving only the metric variables. The Hamiltonian itself comes from the Arnowitt-Deser-Misner (ADM) Hamiltonian[11,12] apart from complications due to the imposition of the spatial homogeneity which reduces the general theory to a finite-dimensional case. One of these complications is the appearance of a nonpotential force component of the force due to the spatial curvature in the vacuum Hamiltonian in Bianchi models of class B, although for special initial data in certain class-B Bianchi types this force vanishes. A general spatially homogeneous perfect fluid can be added as a source, but its equations of motion are not derivable from the ADM Hamiltonian. However, an orthogonal perfect fluid, namely, one which flows orthogonally to the homogeneous hypersurfaces, which further satisfies the usual equation of state $p=(\gamma-1)\rho$ relating the pressure p to the total energy density ρ, adds a single constant parameter to the system, and hence a completely Hamiltonian picture emerges for the same cases as in vacuum. Only such a source will be considered here.

The metric for the SH Bianchi models assumes the following form in zero-shift spatial gauge:

$${}^{(4)}ds^2 = -N^2 dt^2 + g_{ab}\omega^a \omega^b \ , \qquad (3.1)$$

where $\{\omega^a\}$ $(a,b=1,2,3)$ are SH one-forms dual to a SH spatial frame which is comoving along the normal congruence to the family of geodesically parallel SH hypersurfaces. The lapse function N allows one to change from the usual proper time τ $(N=1)$ parametrizing the family of SH orbits to any convenient time t. The equation of state parameter γ for the orthogonal perfect-fluid source if present is assumed to satisfy $1 \le \gamma \le 2$. However, the $\gamma=2$ models can be easily obtained from the corresponding vacuum models[13] and have a very different behavior compared with the remaining perfect-fluid equations of state so only the case $\gamma \ne 2$ will be considered here.

The symmetry type of the homogeneity group depends on the structure constants defined by the differentials $d\omega^a = -\frac{1}{2} C^a{}_{bc}\omega^b\omega^c$, which may be reduced to a standard form parametrized by the four constants $(n^{(1)}, n^{(2)}, n^{(3)}, a)$ subject to the constraint $an^{(3)}=0$.[14] The models of class A $(a=0)$ are completely Hamiltonian and diagonalizable, so that one may assume the metric matrix to be diagonal. A convenient exponential parametrization for the diagonal metric variables was introduced by Misner:[15]

$$\begin{aligned} &(g_{ab}) = e^{2\beta} \ , \\ &\boldsymbol{\beta} = \beta^0 \mathbf{1} + \beta^+ \mathrm{diag}(1,1,-2) + \beta^- \mathrm{diag}(\sqrt{3}, -\sqrt{3}, 0) \ . \end{aligned} \qquad (3.2)$$

These logarithmic metric variables are conformally orthonormal coordinates with respect to the conformally flat Lorentz DeWitt metric[16] on the diagonal minisuperspace:

$$\begin{aligned} &\mathcal{G} = 6e^{3\beta^0}\eta_{AB}\, d\beta^A \otimes d\beta^B \ , \\ &(\eta^{AB}) = (\eta_{AB}) = \mathrm{diag}(-1,1,1), \quad (A,B = 0,+,-) \ . \end{aligned} \qquad (3.3)$$

Certain class-B models $(a \ne 0)$ also admit a completely Hamiltonian formulation in special cases where the nonpotential curvature force vanishes. This occurs for the general Bianchi type-V models and the Taub-like symmetric case models characterized by $n^a{}_a \equiv \epsilon^{abc}C_{abc} = 0$, all of which are diagonalizable as well (excluding the general Bianchi type-VI$_{-1/9}$ case which is not diagonalizable and which is not treated here). The Taub-like symmetric case occurs for Bianchi types V, VI_h, and VII_h but it is nontrivial only for Bianchi type VI_h. The Bianchi type-V and -VII_h models belonging to this class are the "open" isotropic models, which will not be considered by themselves.

The cosmological Hamiltonian H expressed in these variables is "already parametrized" in the sense originally discussed by Arnowit, Deser, and Misner,[11] with the lapse function playing the role of the rescaling function of the previous section. In the class-A case with an orthogonal perfect fluid present, this Hamiltonian takes the form[17,18]

$$H=\mathcal{H}_N=N\mathcal{H}=0\ , \qquad (3.4)$$

$$\mathcal{H}=\mathcal{T}+V=\tfrac{1}{24}e^{-3\beta^0}\eta^{AB}p_Ap_B+e^{\beta^0}V^*+2l^\gamma e^{-3(\gamma-1)\beta^0}\ ,$$

where

$$V^*=\tfrac{1}{2}e^{4\beta^+}(h_-)^2-n^{(3)}e^{-2\beta^+}h_+ +\tfrac{1}{2}(n^{(3)})^2e^{-8\beta^+}\ , \qquad (3.5)$$

and

$$h_\pm=n^{(1)}e^{2\sqrt{3}\beta^-}\pm n^{(2)}e^{-2\sqrt{3}\beta^-}\ . \qquad (3.6)$$

The term $e^{\beta^0}V^*$ is the gravitational potential and equals $g^{1/2}R^*$, where R^* is the curvature scalar of the spatially homogeneous three-surfaces. The last term in the Hamiltonian is the matter potential, involving the constant $l=l_0e^{3\beta^0}\rho^{1/\gamma}$, where l_0 is an uninteresting normalization constant. The constant l can be interpreted as the conserved baryon number per comoving volume associated with the choice of comoving frame used to express the metric. In the Bogoyavlensky-Novikov (BN) time gauge[19] defined by the condition $N=e^{3(\gamma-1)\beta^0}$, the fluid potential is a constant $-2l^\gamma$ interpretable as the energy parameter E of the previous discussion.

Since the system is "already parametrized," the Jacobi Hamiltonian can be obtained immediately by the choice $N=|2V|^{-1}$ of the lapse function, resulting in the Jacobi metric

$$\begin{aligned} ds_J^2&=J_{AB}d\beta^Ad\beta^B\\ &=|2V|(12e^{3\beta^0})\eta_{AB}d\beta^Ad\beta^B\ ,\\ V&=e^{\beta^0}V^*+2l^\gamma e^{-3(\gamma-1)\beta^0}\ , \end{aligned} \qquad (3.7)$$

where η is used in this paper to denote the Minkowski metric of arbitrary dimension with the signature $-++\cdots+$. This leads to a one-parameter family of Jacobi metrics. In the semisimple case these are not isometric and represent a one-parameter family of Jacobi geometries. However, in the nonsemisimple case due to the existence of an automorphism acting as a symmetry of the gravitational potential, it is possible to redefine the coordinates by constant translations in the variables β^0 and β^+ to make the two coefficients of the gravitational and matter contributions to V equal any convenient values. In particular, l is not an essential parameter since it may be eliminated (together with the overall constant 12) by the following redefinitions of the coordinates when it is nonzero,

$$(\beta^0+\ln(48l^\gamma)/[3(2-\gamma)],\ \beta^++\tfrac{1}{4}\ln 12-\ln(48l^\gamma)/[3(2-\gamma)],\ \beta^-)\mapsto(\beta^0,\beta^+,\beta^-)\ ,$$

showing that all such nonvacuum Jacobi metrics are isometric to the metric

$$ds_J^2=(e^{4(\beta^0+\beta^+)}h_-^2+e^{3(2-\gamma)\beta^0})\eta_{AB}d\beta^Ad\beta^B\ .$$

The vacuum Jacobi metric can be written for all Bianchi types with a nonzero gravitational potential in the form

$$ds_J^2=24e^{4\beta^0}|V^*(\beta^\pm)|\eta_{AB}d\beta^Ad\beta^B\ . \qquad (3.8)$$

For the Bianchi type-I case where this potential vanishes, the solution curves are just the null geodesics, so any conformally scaled metric can be used, the simplest being the (three-dimensional) Minkowski metric. The null geodesics are straight lines in the space of β variables and correspond to the Kasner solutions. Depending on the choice of the conformal factor, the vector $\partial/\partial\beta^0$ is a homothetic symmetry, a Killing symmetry (Minkowski metric) or no symmetry at all. For all other vacuum cases, $\partial/\partial\beta^0$ is a homothetic vector which is not related to the scale or automorphism symmetries.

The constraint $\mathcal{H}=0$ becomes

$$J_{AB}\frac{d\beta^A}{dt}\frac{d\beta^B}{dt}=-\mathrm{sgn}V\ . \qquad (3.9)$$

Because of the signature $-++$ of the Jacobi metric, the geodesics will either be timelike or spacelike depending on the sign of V. In the Bianchi type-I vacuum case, V is identically zero and the geodesics are null. For the canonical values of the structure constants listed in Table I of Rosquist and Jantzen[18] the potential V is positive in all other cases except for the Bianchi type-VII_0 vacuum case where $\beta^-=0$, which is equivalent to the Bianchi type-I Taub-like case, and the Bianchi type-IX case which has a region where $V<0$, corresponding to spacelike geodesics. The Jacobi metric is singular on the hypersurface $V=0$ in configuration space and is therefore unsuitable to describe the dynamics on and near that hypersurface.

The Bianchi type-VI_h models of the class being considered are also diagonal in the appropriate choice of frame so one can use the same metric variables β^A, but it proves convenient to redefine them by a constant k depending on the symmetry parameters by the formula $k^{-2}=1+3a^2q^{-2}$, where q is related to the usual structure parameters by $q^2=-n^{(1)}n^{(2)}$, $\mathrm{sgn}q=\mathrm{sgn}n^{(1)}$. In terms of the new phase-space variables $\tilde\beta^0=\beta^0,\tilde\beta^+=k^{-1}\beta^+,\tilde p_0=p_0,\ \tilde p_+=k^{-1}p_+$ the correct Hamiltonian on the constraint subspace for the Taub-like symmetric case models is[7,8]

$$H=\mathcal{H}_N=N\mathcal{H} ,$$
$$\mathcal{H}=T+V=\tfrac{1}{24}e^{-3\tilde{\beta}^0}(-\tilde{p}_0^2+\tilde{p}_+^2)+2q^2k^{-2}e^{\tilde{\beta}^0+4k\tilde{\beta}^+}+2l^\gamma e^{-3(\gamma-1)\tilde{\beta}^0} . \tag{3.10}$$

This Hamiltonian leads to the Jacobi metric

$$ds_J^2=48(q^2k^{-2}e^{4(\tilde{\beta}^0+k\tilde{\beta}^+)}+l^\gamma e^{3(2-\gamma)\tilde{\beta}^0})\times[-(d\tilde{\beta}^0)^2+(d\tilde{\beta}^+)^2] , \tag{3.11}$$

but again by a redefinition of the coordinates involving constant translations in the variables $\tilde{\beta}^0$ and $\tilde{\beta}^+$ one can set the two coefficients of the gravitational and matter terms in the conformal factor to any convenient values, i.e., $48q^2k^{-2}\to 1$ and $48l^\gamma\to 1$. The Bianchi type-VI_0 class-A Taub-like case may be obtained by setting $k=1$.

The general orthogonal Bianchi type-V case is similar but simpler. The Hamiltonian in this case is given by

$$H=\mathcal{H}_N=N\mathcal{H} ,$$
$$\mathcal{H}=T+V=\tfrac{1}{24}e^{-3\beta^0}(-p_0^2+p_-^2)+6\bar{a}^2e^{\beta^0}+2l^\gamma e^{-3(\gamma-1)\beta^0} , \tag{3.12}$$

where $\bar{a}\equiv ae^{\beta^+}$. Here β^+ is a constant gauge parameter which just gives a rescaling of a. The corresponding Jacobi metric is given by

$$ds_J^2=48(3\bar{a}^2e^{4\beta^0}+l^\gamma e^{3(2-\gamma)\beta^0})[-(d\beta^0)^2+(d\beta^-)^2] , \tag{3.13}$$

However, in this case only a translation in β^0 is effective in eliminating constants but since $\bar{a}$ can be assumed to have any convenient value, it can be rescaled and so both coefficients in the conformal factor can again be reduced to the value 1.

A time gauge which is closely connected to the Jacobi time is the Taub time gauge,[20,10] also referred to as supertime time gauge by Misner,[2] characterized by the lapse choice $N_T=12e^{3\beta^0}$. This simplifies the kinetic energy as much as possible by reducing it to the trivial value associated with the explicitly flat Minkowski metric in the variables β^A. On the other hand, the Jacobi time gauge simplifies the potential energy as much as possible by reducing it to the trivial value of a constant, thus providing a complementary picture of the dynamics. The Taub time gauge potential $V_T=N_TV$, apart from a factor of 2, is exactly the expression whose sign determines the causality properties of the Jacobi geodesics and whose absolute value gives the conformal factor relating the Jacobi metric to the flat Minkowski metric.

IV. SYMMETRY-ADAPTED VARIABLES

Having reformulated the Einstein equations for the Hamiltonian Bianchi models as geodesic equations of the Jacobi metric, one can investigate and exploit the symmetries of this metric to find new variables which simplify the discussion of the dynamics in any time gauge. In this section a case-by-case study will reveal new insight into the special results that have been obtained in the past and place them into perspective in the context of the more general qualitative behavior of these complicated systems. Although a unified discussion of the Bianchi models is possible on a high level, the details of the various cases depend crucially on both the automorphism structure and on the algebraic structure of the potential. We therefore divide the set of cases into three categories: no potential term, exactly one exponential potential term and two or more exponential potential terms. Each of these categories is subdivided according to the symmetry group of the Jacobi metric which at least in part reflects the underlying automorphism group. An additional complication comes from the momentum constraints. Because of the different structure of the constraints in the Bianchi types of classes A and B we are forced to consider those two classes separately. Also, in Bianchi type-VI_h models the momentum constraint is invariant under the automorphism group while in Bianchi type-V models the automorphism symmetry is broken by the momentum constraint leading to different solution structures. To summarize, it is convenient in a detailed study to have a case-by-case discussion using the Hamiltonian formalism, redefining independent and dependent variables whenever necessary, the latter by point transformations.

Before embarking on such a study, we recall the symmetry of the Jacobi metric which arises from the coupled scale/automorphism symmetry which exists for the nonsemisimple models[7,8] and which may be seen by inspection of the above expressions for the Jacobi metric. For the vacuum case one has the Killing vector field

$$\frac{\partial}{\partial\beta^0}-\frac{\partial}{\partial\beta^+} , \tag{4.1}$$

in the class-A case where it is null, and

$$\frac{\partial}{\partial\tilde{\beta}^0}-k^{-1}\frac{\partial}{\partial\tilde{\beta}^+} \tag{4.2}$$

in the Bianchi type-VI_h case where it is spacelike. In the nonvacuum case one has a timelike homothetic vector field

$$\frac{\partial}{\partial\beta^0}-\frac{1}{4}(3\gamma-2)\frac{\partial}{\partial\beta^+} . \tag{4.3}$$

in the class-A case and the homothetic Killing vector field

$$\frac{\partial}{\partial\tilde{\beta}^0}-\frac{1}{4}k^{-1}(3\gamma-2)\frac{\partial}{\partial\tilde{\beta}^+} , \tag{4.4}$$

in the Bianchi type-VI_h case, where it can be spacelike, timelike, or null (recall that $k^{-2}=1+3a^2q^{-2}$).

The case with no potential term corresponds to the vacuum Bianchi type-I models discussed above and extensively studied elsewhere from the Hamiltonian point of view (see, e.g., Jantzen[21]). (The type-VII_0 LRS case reduces to the LRS type-I case as well.) The cases in which the potential has a single term are (i) type-I perfect fluid, (ii) type-II vacuum, (iii) special type-II perfect-fluid subsystem, (iv) Taub-like type-VI_0 vacuum, (v) Taub-like type-VI_h vacuum, and (vi) type-V vacuum. In all these cases the Jacobi lapse function is a power-law lapse and the problem is easily solvable in the Jacobi time gauge. The simplest solutions are the exact power-law (EPL) metrics which have been classified by Wainwright.[22] The previously known non-EPL solutions will be given. The remaining models have a nonzero potential which consists of a sum of exponential terms. Some of these cases are exactly solvable, but are usually simpler to solve in other time gauges. Therefore no explicit solutions of this type will be listed.

Bianchi type-II perfect-fluid models

The Taub time gauge Hamiltonian for these models is given by

$$H_T=\tfrac{1}{2}\eta^{AB}p_Ap_B+V_T\ ,$$
$$V_T=6(n^{(1)})^2e^{4(\beta^0+\beta^++\sqrt{3}\beta^-)}+24l^\gamma e^{3(2-\gamma)\beta^0}\ . \tag{4.5}$$

The Jacobi metric

$$ds_J^2=2V_T\eta_{AB}d\beta^Ad\beta^B \tag{4.6}$$

admits both the spacelike Killing vector

$$\sqrt{3}\frac{\partial}{\partial\beta^+}-\frac{\partial}{\partial\beta^-}\ ,$$

arising from the additional diagonal unimodular automorphism which exists for this Bianchi type,[7,21] and the timelike homothetic vector listed above. We now proceed to adapt the coordinates (β^A) of the Jacobi geometry to these symmetries. First one can adapt the variables to the Killing vector by performing the following rotation by 120° in the β^+-β^- plane:[21,23]

$$(\beta_1^+,\beta_1^-)=\tfrac{1}{2}(-\beta^+-\sqrt{3}\beta^-,\sqrt{3}\beta^+-\beta^-)\ . \tag{4.7}$$

in terms of which the Killing vector is just $\partial/\partial\beta_1^-$. One can also choose a new homothetic vector which is orthogonal to this Killing vector by adding an appropriate multiple of the latter vector. The result is the timelike vector

$$\frac{\partial}{\partial\beta^0}+\frac{1}{8}(3\gamma-2)\frac{\partial}{\partial\beta_1^+}\ .$$

Next by making a boost with velocity $v=\frac{1}{8}(3\gamma-2)$ in the β_1^+ direction, one obtains new orthogonal coordinates whose time axis is aligned with the new homothetic vector and one of whose spatial axes is aligned with the Killing vector. Finally by a uniform rescaling of the coordinates by a constant factor (dilatation) and translations in β^0 and β_1^+, one can eliminate unsightly factors in the Jacobi metric as above to obtain the result

$$ds_J^2=e^{3(2-\gamma)\bar\beta^0}F(\bar\beta^+)\eta_{AB}d\bar\beta^Ad\bar\beta^B\ ,$$
$$F(\bar\beta^+)=e^{-3(6-\gamma)\bar\beta^+/2}+e^{3(2-\gamma)(3\gamma-2)\bar\beta^+/8}\ , \tag{4.8}$$

in terms of new variables $\bar\beta^A$ which are related to the old variables by

$$\beta^0=\bar\beta^0+v\bar\beta^++k_0\ ,$$
$$\beta^+=\tfrac{1}{2}(-\beta_1^++\sqrt{3}\beta_1^-)$$
$$=\tfrac{1}{2}(-v\bar\beta^0-\bar\beta^++\sqrt{3(1-v^2)}\bar\beta^-)+k_+\ , \tag{4.9}$$
$$\beta^-=-\tfrac{1}{2}(\sqrt{3}\beta_1^++\beta_1^-)$$
$$=-\tfrac{1}{2}(\sqrt{3}v\bar\beta^0+\sqrt{3}\bar\beta^++\sqrt{1-v^2}\bar\beta^-)+k_-\ ,$$

where k_A are suitably chosen constants. In order to keep track of the meaning of various potential terms when the coefficients are scaled away as, e.g., in $F(\bar\beta^+)$ we always maintain the original order of the terms, i.e., with the gravitational terms coming first and the single matter term coming last. Thus the first term in $F(\bar\beta^+)$ comes from the gravitational potential and the second term from the matter potential.

Although the Jacobi time gauge has been instrumental in obtaining these symmetry adapted variables, the system is most easily understood in the "monotonic-function" time gauge in which the function $F(\bar\beta^+)$ is removed from the conformal factor and returned to the potential term position by a rescaling, leaving behind only the exponential $\bar\beta^0$-dependent factor associated with the existence of the homothetic Killing vector field. This is an example of the way in which the Jacobi time gauge often leads to other useful time gauges. The Hamiltonian is then of the form

$$H_{\mathrm{mf}}=\tfrac{1}{2}e^{-3(2-\gamma)\bar\beta^0}\eta^{AB}\bar p_A\bar p_B+\tfrac{1}{2}F(\bar\beta^+)\ , \tag{4.10}$$

with the $\bar p_0$-momentum term playing the role of a monotonic energy function for the remaining variables as discussed in Appendix B. This monotonic function leads to a very clean and concise qualitative discussion of the late stage dynamics which is at once simpler and more intuitive than previous approaches. For those models where this is possible, this gives a natural complement to the moving potential wall picture describing the early stage dynamics.

In the present case the potential $\frac{1}{2}F(\bar\beta^+)$ has the form of a trough in the $\bar\beta^\pm$ coordinates. In the barred variables the Killing symmetry coming from the unimodular automorphism is $\partial/\partial\bar\beta^-$ and the corresponding constant of the motion is $\bar p_-$. The homothetic vector associated with the combined scale and nonunimodular automorphism symmetry is $\partial/\partial\bar\beta^0$. Setting $\bar p_-=0$ gives the locally rotationally symmetric subsystem (see Jantzen,[21] and references therein for a discussion of the various possible specializations which occur for the different Bianchi types). By setting the variable $\bar\beta^+$ to the constant value $\bar\beta^+_{\min}$ at the bottom of the trough one obtains a two-dimensional problem for which the Jacobi and monotonic-function time gauge lapses coincide (modulo a constant factor). This is the exactly solvable power-law lapse case (iii). There are two solutions which "lie" in the bottom of the trough. The first is the LRS EPL solution[24] which is at rest in the trough ($\dot{\bar\beta}^-=$const). The second is Collins non-LRS non-EPL solution[25] which is moving along the bottom of the trough ($\dot{\bar\beta}^-\neq 0$). These solutions will be discussed in more detail below. No exact solution exists with $\bar{\beta}^+ \neq \bar{\beta}^+_{\rm min}$.

Since the orthogonal perfect-fluid Bianchi type-II models admit a non-null cyclic variable ($\bar{\beta}^-$ or β_1^-) with a conserved momentum, it is possible to reduce the problem to a two-dimensional one with an "effective potential" term and a two-dimensional "reduced" Jacobi metric. The reduced Hamiltonian H'_T in the Taub time gauge is given by

$$H'_T = \tfrac{1}{2}(-\bar{p}_0^2+\bar{p}_+^2) + \tfrac{1}{2}e^{3(2-\gamma)\bar{\beta}^0}F(\bar{\beta}^+)$$
$$= -\tfrac{1}{2}\bar{p}_-^2 = -\tfrac{1}{2}c^2 \quad (M,N=0,+) , \tag{4.11}$$

where the constant c is just the conserved momentum $\bar{p}_-$. The LRS case is obtained by setting $c=0$. The explicitly time-dependent constant of the motion is destroyed since it does not commute with the constant of the motion $\bar{p}_-$. Note that $H'_T \neq 0$ except for the LRS case $\bar{p}_-=0$. This leads to the one-parameter family of two-dimensional Jacobi geometries

$$ds_J'^2 = [e^{3(2-\gamma)\bar{\beta}^0}F(\bar{\beta}^+)+c^2]\eta_{MN}d\bar{\beta}^M d\bar{\beta}^N . \tag{4.12}$$

Bianchi type-V perfect-fluid models

The Jacobi metric is given by

$$ds_J^2 = (e^{4\beta^0}+e^{3(2-\gamma)\beta^0})[-(d\beta^0)^2+(d\beta^-)^2] , \tag{4.13}$$

where we have again used the rescaling freedom to simplify the coefficients. This two-dimensional Jacobi metric is already adapted to the Killing vector $\partial/\partial\beta^-$. Since β^- is a cyclic variable in the Hamiltonian the problem reduces to a one-dimensional one, where the explicit solution is most easily found in the β^0-time time gauge or in a power-law-lapse time gauge with lapse depending only on β^0. The open Friedmann-Robertson-Walker (FRW) model of constant negative spatial curvature is obtained by setting the conserved momentum to zero. The general solution has been discussed by Nayak and Bhuyan.[26]

The Bianchi type-VI$_0$ and -VII$_0$ vacuum cases

The Jacobi metric has a null Killing vector suggesting the use of adapted null variables to express the Jacobi metric

$$ds_J^2 = 12e^{4u}(h_-)^2[-du\,dv+(d\beta^-)^2] ,$$
$$(u,v) = (\beta^0+\beta^+,\beta^0-\beta^+) . \tag{4.14}$$

As discussed, all vacuum models exhibit a hidden homothetic vector $\partial/\partial\beta^0$ which is not related to the scale or automorphism symmetries of the field equations. In the case at hand, a certain linear combination of the homothetic vector and the Killing vector $\partial/\partial v$ gives the lightlike homothetic vector $\partial/\partial u$. The Bianchi type-VII$_0$ Jacobi metric has a curvature singularity at $\beta^-=0$ (for canonical structure constant values). That metric is therefore unsuitable to describe the vacuum VII$_0$ dynamics near $\beta^-=0$; the LRS case $\beta^-=0$ coincides with the Bianchi type-I case. There are no nontrivial Bianchi type-VII$_0$ solutions. In the Taub-like Bianchi type-VI$_0$ case the potential has only one term leading to a two-dimensional Jacobi power-law-lapse case which is discussed below.

The Bianchi type-VI$_0$ and -VI$_0$ perfect-fluid models

These cases admit a timelike homothetic vector associated with the coupled scale/automorphism symmetry. The variables can be adapted to it by making a boost with velocity $v=-\tfrac{1}{4}(3\gamma-2)$ in the β^+ direction. By uniformly scaling the variables and making a translation in the β^0 direction, the Jacobi metric can be further simplified. Written in symmetry adapted variables it becomes

$$ds_J^2 = e^{3(2-\gamma)\bar{\beta}^0}(\tfrac{1}{4}e^{3(2-\gamma)\bar{\beta}^+}h_-^2$$
$$+e^{-3(2-\gamma)(3\gamma-2)\bar{\beta}^+/4})\eta_{AB}d\bar{\beta}^A d\bar{\beta}^B , \tag{4.15}$$

where the variables $\bar{\beta}^A$ are related to β^A by

$$\beta^0 = \bar{\beta}^0+v\bar{\beta}^+ + k_0, \quad \beta^+ = v\bar{\beta}^0+\bar{\beta}^+ + k_+ ,$$
$$\beta^- = \sqrt{1-v^2}\,\bar{\beta}^- \tag{4.16}$$

and k_0 and k_+ are three new constants. The factor $\tfrac{1}{4}$ in front of the first term involving h_-^2 makes the coefficient simpler on the Bianchi type-VI$_0$ Taub-like submanifold, where $h_-=2$ for the canonical choice of structure constants ($q=1$). Written in a monotonic-function time gauge the Hamiltonian assumes the form

$$H_{\rm mf} = \tfrac{1}{2}e^{-3(2-\gamma)\bar{\beta}^0}(-\bar{p}_0^2+\bar{p}_+^2)$$
$$+\tfrac{1}{2}(\tfrac{1}{4}e^{3(2-\gamma)\bar{\beta}^+}h_-^2 + e^{-3(2-\gamma)(3\gamma-2)\bar{\beta}^+/4}) . \tag{4.17}$$

In the Bianchi type-VI$_0$ case the potential is then a two-dimensional well. The Hamiltonian (4.17) will be used in Appendix B in a discussion of the late stage dynamics of that case. The canonical Taub-like submanifold $\beta^-=0=\bar{\beta}^-$ occurs as a special case of the more general Taub-like symmetric case in Bianchi type-VI$_h$ to be examined next. In the Bianchi type-VII$_0$ case the potential has no extremum value. The Taub-like case is identical with the Bianchi type-I Taub-like case.[21] As for vacuum there are no nontrivial Bianchi type-VII$_0$ solutions.

The Taub-like symmetric Bianchi type-VI$_h$ perfect-fluid models

The homothetic Killing vector associated with the scale/automorphism symmetry given involves the parameter $k\in(0,1)]$ where the value $k=1$ corresponds to the Bianchi type-VI$_0$ case. This vector is either spacelike $[k<\tfrac{1}{4}(3\gamma-2)]$, null $[k=\tfrac{1}{4}(3\gamma-2)]$, or timelike $[k>\tfrac{1}{4}(3\gamma-2)]$, making it natural to classify the above problem into three types depending on the causal character of the homothetic vector.

For the null case it is natural to use null coordinates $(\tilde{\beta}^0,\tilde{\beta}^+)=\tfrac{1}{2}(u+v,u-v)$ leading to the Jacobi metric

$$ds_J^2 = -48e^{3(2-\gamma)v/2}(q^2k^{-2}e^{(3\gamma+2)u/2} + l^{\gamma}e^{3(2-\gamma)u/2})du\,dv \ . \qquad (4.18)$$

Since this metric is of the form $ds^2 = -f(u)g(v)du\,dv$, it is flat. The new variables $\tilde{t} = \frac{1}{2}[\int f(u)du + \int g(v)dv]$, $\tilde{x} = \frac{1}{2}[\int f(u)du - \int g(v)dv]$ are inertial coordinates, in terms of which the geodesics are straight lines. However, one needs to invert the coordinate transformation in order to obtain the original variables and the underlying spacetime metric, which is not possible in terms of elementary functions. Thus it is better just to use the Jacobi metric to obtain "good" variables in terms of which the problem can be solved explicitly. For example, one can use $V = e^{3(2-\gamma)v/2}$ as one new variable and $U = e^{cu}$ where c is some convenient constant for the other variable. Then one can use a power-law lapse to solve the problem explicitly. An example of such a power-law lapse is the one used by Wainwright[22] in order to give this solution a simple form. The solution was found first by Collins.[25] This is an example where the Jacobi metric can be used as a powerful intermediary tool to find new useful parametrizations of the original space-time metric.

In the timelike case we can make a boost $v = -\frac{1}{4}k^{-1}(3\gamma - 2)$ in the $\bar{\beta}^+$ direction. Once again we can use the conformal subgroup of scalings and translations to simplify the expression for the Jacobi metric even further:

$$\begin{aligned} ds_J^2 &= e^{3(2-\gamma)\bar{\beta}^0}(\exp\{4k^{-1}[k^2 - \tfrac{1}{4}(3\gamma-2)]\bar{\beta}^+\} + \exp[-\tfrac{3}{4}k^{-1}(2-\gamma)(3\gamma-2)\bar{\beta}^+])(-d\bar{\beta}^{02} + d\bar{\beta}^{+2}) \ , \\ \tilde{\beta}^0 &= \bar{\beta}^0 + v\bar{\beta}^+ + k_0, \quad \tilde{\beta}^+ = v\bar{\beta}^0 + \bar{\beta}^+ + k_+ \ . \end{aligned} \qquad (4.19)$$

The class-A Taub-like case occurs as the special case $k = 1$. The $\bar{\beta}^+$-part of the conformal factor has a minimum if $k^2 > \frac{1}{4}(3\gamma - 2)$ leading to an EPL solution.[22] This EPL solution coincides with a $\bar{\beta}^0$ coordinate line in minisuperspace.

In the spacelike case one uses a boost with velocity v given by $v^{-1} = -\frac{1}{4}k^{-1}(3\gamma - 2)$ followed by dilatations and translations to reduce the spacelike homothetic vector to $\partial/\partial\bar{\beta}^+$ and simplify the Jacobi metric to

$$\begin{aligned} ds_J^2 &= e^{3(2-\gamma)\bar{\beta}^+}(\exp\{-4k^{-1}[\tfrac{1}{4}(3\gamma-2) - k^2]\bar{\beta}^0\} + \exp[-\tfrac{3}{4}k^{-1}(2-\gamma)(3\gamma-2)\bar{\beta}^0])\eta_{MN}d\bar{\beta}^M d\bar{\beta}^M \ , \\ \tilde{\beta}^0 &= v^{-1}\bar{\beta}^0 + \bar{\beta}^+ + k_0, \quad \tilde{\beta}^+ = \bar{\beta}^0 + v^{-1}\bar{\beta}^+ + k_+ \ . \end{aligned} \qquad (4.20)$$

In this case the potential in the corresponding Hamiltonian has no critical point. No previously known exact solutions exist for this family. However, new solutions in the spacelike and timelike cases have recently been found using these variables as a starting point. These new solutions[27,28] are associated with a hidden Killing tensor of the Jacobi metric. Adapting the variables to this symmetry yields new exact solutions. However, as in the null case, other time gauges prove more convenient to describe them, illustrating the way in which the Jacobi time gauge serves as an intermediate geometric tool in the analysis of the dynamics.

Bianchi type-VIII and -IX vacuum models

The Taub time gauge Hamiltonian is

$$H_T = \tfrac{1}{2}\eta^{AB}p_A p_B + V_T \ , \qquad (4.21)$$

where $V_T = 12e^{4\beta^0}V^*$. This leads to a Jacobi Hamiltonian and metric of the form

$$\begin{aligned} H_J &= \tfrac{1}{2}|V_T|^{-1}\eta^{AB}p_A p_B \ , \\ ds_J^2 &= 12e^{4\beta^0}|V^*|\eta_{AB}d\beta^A d\beta^B \ . \end{aligned} \qquad (4.22)$$

As remarked above this Jacobi metric has the extra homothetic symmetry $\partial/\partial\beta^0$ which is not related to the scale or automorphism symmetries. In the monotonic-function time gauge the Hamiltonian becomes

$$H_{\text{mf}} = \tfrac{1}{2}e^{-4\beta^0}\eta^{AB}p_A p_B + 12V^* \ . \qquad (4.23)$$

For the sake of definiteness, assume the canonical values of the structure constants $n^{(1)} = n^{(2)} = 1$ and $n^{(3)} = -1$ and $n^{(3)} = 1$, respectively, in the type-VIII and -IX cases. In the Bianchi type-IX case the potential $V^* = V^*_{\text{IX}}$ is a two-dimensional well in the $\beta^{\pm}$ variables. The minimum is at $\beta^{\pm} = 0$. However, this point is excluded by the Hamiltonian constraint since the minimum value of the potential is negative ($V^*_{\min} < 0$) corresponding to spacelike solutions while only the timelike coordinate β^0 is nonzero at the minimum. In the region $V^* < 0$ the dynamics is represented by spacelike geodesics of the Jacobi geometry. The surface $V^* = 0$ is singular and we have two different Jacobi metrics, one for each of the two regions $V^* > 0$ and $V^* < 0$. The Bianchi type-VIII potential V^*_{VIII} is bounded below by $V^*_{\text{VIII}} > 0$ but the minimum value is only attained at infinity ($\beta^- = 0, \beta^+ \to +\infty$, see Jantzen[21] for a pictorial representation of the potentials for all Bianchi types).

The Taub-like (LRS) solutions ($\beta^- = 0$) can be obtained in the Taub time gauge by making a boost with velocity $v = \frac{1}{2}$ along the β^+ direction

$$(\bar{\beta}^0, \bar{\beta}^+) = \frac{1}{\sqrt{3}}(2\beta^0 - \beta^+, -\beta^0 + 2\beta^+) \qquad (4.24)$$

which decouples the Hamiltonian into the sum of two one-dimensional problems. In terms of these variables the Jacobi metric and Hamiltonian are

$$\begin{aligned} ds_J^2 &= 2V_T[-(d\bar{\beta}^0)^2 + (d\bar{\beta}^+)^2] \ , \\ H_J &= \tfrac{1}{4}V_T^{-1}(-\bar{p}_0^2 + \bar{p}_+^2) \ , \\ V_T &= -24n^{(3)}e^{2\sqrt{3}\bar{\beta}^0} + 6n^{(3)2}e^{-4\sqrt{3}\bar{\beta}^+} \ . \end{aligned} \qquad (4.25)$$

As shown by Dietz,[29] a Hamiltonian of this type has a constant of the motion quadratic in the momenta corresponding to a Killing tensor of the Jacobi metric. According to his formulas this quadratic constant of motion is (dropping an unimportant constant factor)

$$K=\frac{n^{(3)}e^{-4\sqrt{3}\beta^{+}}\bar{p}_{0}^{2}-4e^{2\sqrt{3}\beta^{0}}\bar{p}_{+}^{2}}{4e^{2\sqrt{3}\beta^{0}}-n^{(3)}e^{-4\sqrt{3}\beta^{+}}} \qquad (4.26)$$

corresponding to the Killing tensor whose components coincide with the components of the quadratic form in the momenta in this expression. Appendix C gives an example of the way in which a Killing tensor suggests another time gauge in which a decoupling of the Hamiltonian occurs, thus resulting in hidden conserved quantities in that time gauge.

Bianchi type-VIII and -IX perfect-fluid models

The Jacobi metric is given by

$$ds_J^2=2|V_T|\eta_{AB}d\beta^A d\beta^B\ ,$$
$$V_T=12e^{4\beta^0}V^*+24l^{\gamma}e^{3(2-\gamma)\beta^0}\ , \qquad (4.27)$$

The only known solution of this class is the closed (Bianchi type-IX) isotropic model ($\beta^+=\beta^-=0$). This is a one-dimensional problem and can be solved by integrating the Hamiltonian constraint. However, in this case a power-law lapse is preferable to the Jacobi lapse as described by Jantzen.[30] As in the vacuum case there are two regions, one for each sign of V_T.

This is the only case in which the Jacobi metric has no homothetic symmetry. This means that one cannot get rid of the constant l by a coordinate transformation. However, by an appropriate translation in β^0 one can make l^{γ} appear as a scale factor in the Jacobi metric. Therefore the value of l^{γ} is trivial in that it just corresponds to a scaling of the affine parameter in the geodesic equations of the Jacobi geometry. This is as it should be since l is directly associated with the scale invariance.

Jacobi power-law lapses

When the Jacobi time gauge coincides with a power-law-lapse time gauge, the problem is always solvable and the Jacobi metric assumes the simple form

$$ds_J^2=c^2\exp(Q_C\beta^C)\eta_{AB}d\beta^A d\beta^B\ , \qquad (4.28)$$

where c and the Q_C are constants, or the same form with indices ranging only over two variables in the two-dimensional case. It turns out to be natural to classify these metrics into three different categories (power-law types), depending on the causal character of the constant covector Q_C, which can be either timelike (T), spacelike (S), or null (N) with respect to the Jacobi metric or equivalently with respect to the flat metric, since the conformal factor is positive.

These categories each lead to a certain standard form for the space-time metric in the Jacobi power-law-lapse time gauge (see Appendix A where the perfect-fluid Bianchi type-I case is calculated as an example)

$$ds^2=-t^{2\bar{p}_0}A(t)^{2\bar{q}_0}dt^2+\sum_{a=1}^{3}c_a^2t^{2\bar{p}_a}A(t)^{2\bar{q}_a}(\omega^a)^2\ , \qquad (4.29)$$

where the $c_a,\bar{p}_\mu,\bar{q}_\mu$ ($\mu=0,1,2,3$) are constants. The parameters $\bar{p}_\mu$ should not be confused with the momentum variables. They are related to the Kasner parameters p_a by $p_a=\bar{p}_a/(\bar{p}_0+1)$ (see Appendix A). The orthogonal one-forms ω^a are the diagonal gauge one-forms parametrized by the symmetry parameters given by Jantzen[21] for the class A case and for the Bianchi type-V case, while Rosquist, Uggla and Jantzen[7] give them for the Bianchi type-VI$_h$ case. Table I lists these one-forms for canonical values of the structure constants.

The function $A(t)$ depends on the power-law type (i.e., causal character of the vector Q_C) in the following way:

$$T:A(t)=1+t,\quad S:A(t)=1-t,\quad N:A(t)=e^t\ . \qquad (4.30)$$

The scale and automorphism groups have been used to set inessential parameters in $A(t)$ (the coefficients in the linear expressions) and the metric (4.29) to convenient values. In particular, an overall constant factor in that metric (corresponding to the scale invariance) has been omitted.

Only in the timelike case does the Hamiltonian constraint allow the special inequivalent parameter choice $A(t)=t$ leading to exact power-law solutions. The general timelike case instead represents solutions which generalize these power-law solutions. Qualitatively they can be thought of as the result of interpolating between a Kasner EPL solution at early times and the exact EPL solution $A(t)=t$ at late times (see Appendix A for an example and Appendix B for a qualitative discussion of late time behavior). The solutions are determined by their

TABLE I. This table gives the basis one-forms used in (4.29) for canonical values of the structure constants. Note, however, that in the Taub-like Bianchi type-VI$_h$ case, this is the overbarred non-canonical frame which requires the "effective" nonzero barred β^- given by $\bar{\beta}^-=-\sqrt{3}a\beta^+$ (Ref. 7).

Case	ω^1	ω^2	ω^3
Type I	dx^1	dx^2	dx^3
Type II	$dx^1+x^3dx^2$	dx^2	dx^3
Type-VI$_0$ and -VI$_h$ Taub	$e^{-(a+1)x^3}dx^1$	$e^{-(a-1)x^3}dx^2$	dx^3
Type V	$e^{-x^3}dx^1$	$e^{-x^3}dx^2$	dx^3

power-law character (T, S, or N) and the constants $c_a, \bar{p}_\mu, q_\mu$ which are given in Table II.

Three-dimensional Jacobi power-law-lapse cases

The perfect-fluid Bianchi type-I models and the vacuum Bianchi type-II models are of this type. The full isometry group of the Jacobi metric for these models is closely related to the Poincaré group and contains additional symmetries unrelated to the scale invariance or automorphism group.[7,8]

In the Bianchi type-I case the Jacobi power-law lapse belongs to the timelike category T and coincides with the Bogoyavlensky-Novikov lapse already mentioned. The Jacobi metric is

$$ds_J^2 = e^{3(2-\gamma)\beta^0}\eta_{AB}d\beta^A d\beta^B \ , \tag{4.31}$$

which happens to be a three-dimensional Friedmann-Robertson-Walker dust universe. This suggests a transformation of variables leading to the metric

$$\begin{aligned} ds_J^2 &= -dT^2 + T^2(dX^2+dY^2) \ , \\ T^2 &= [\tfrac{3}{2}(2-\gamma)]^{-2}e^{3(2-\gamma)\beta^0} \ , \\ X &= \tfrac{3}{2}(2-\gamma)\beta^+, \quad Y = \tfrac{3}{2}(2-\gamma)\beta^- \ . \end{aligned} \tag{4.32}$$

This variable choice considerably simplifies the calculations which lead to the explicit solutions (see Appendix A) and provides an instructive example where geometrical considerations lead to an efficient means of obtaining solutions.

The Bianchi type-II Jacobi power-law lapse belongs to the class (S) which can be seen by making a Lorentz transformation consisting of a 120° rotation followed by a boost to obtain the variables[21]

$$(\bar{\beta}^0, \bar{\beta}^+, \bar{\beta}^-) = \frac{1}{\sqrt{3}}\left[2\beta^0 + \frac{1}{2}\beta^+ + \frac{\sqrt{3}}{2}\beta^-, -\beta^0 - \beta^+ - \sqrt{3}\beta^-, \tfrac{1}{2}(-\beta^- + \sqrt{3}\beta^+)\right] . \tag{4.33}$$

in terms of which the Jacobi metric becomes (for canonical structure constant values)

$$ds_J^2 = 12e^{-4\sqrt{3}\bar{\beta}^+}\eta_{AB}d\beta^A d\beta^B \ . \tag{4.34}$$

Curiously enough this metric has the same form as a metric describing particle motion at a planetary surface,[31] which allows an interpretation of the dynamics of these models in terms of our knowledge about throwing stones.

The form of this metric is nearly the same as the one describing the nonvacuum Bianchi type-I models except that the variable in the exponential factor now is spacelike instead of timelike. This similarity suggests making a corresponding change of variables to simplify the Jacobi metric, leading to the result

$$\begin{aligned} ds_J^2 &= X^2(-dT^2+dY^2)+dX^2 \ , \\ X^2 &= e^{-4\sqrt{3}\bar{\beta}^+}, \quad T = 2\sqrt{3}\bar{\beta}^0, \quad Y = 2\sqrt{3}\bar{\beta}^- \ . \end{aligned} \tag{4.35}$$

The solution of the geodesic problem can be obtained in a way very similar to the Bianchi type-I calculation of Appendix A.

Two-dimensional Jacobi power-law-lapse cases

There are four cases in this category: three of type T and one of type N using the same scheme as for the three-dimensional situation. The three type-T cases are the special Bianchi type-II perfect-fluid subsystem discussed earlier, the Bianchi type-V vacuum models and the symmetric Taub-like vacuum Bianchi type-VI$_h$ models. The Taub-like Bianchi type-VI$_0$ vacuum models are instead of type N.

The Jacobi metric for the three T cases has the form

$$ds_J^2 = c^2e^{2mt}(-dt^2+dx^2) \ , \tag{4.36}$$

where c and m are constants which depend on the Bianchi types. By making the change of variables

$$\tilde{t} = \frac{c}{m}e^{mt}\cosh(mx), \quad \tilde{x} = \frac{c}{m}e^{mt}\sinh(mx) \tag{4.37}$$

one obtains

$$ds_J^2 = -d\tilde{t}^2 + d\tilde{x}^2 \ . \tag{4.38}$$

Thus we have reduced the problem to the trivial one of finding timelike geodesics in a flat two-dimensional Minkowski space.

The Jacobi metric in the Bianchi type-VI$_0$ Taub-like case ($\beta^- = 0$) is

$$ds_J^2 = -48e^{4u}du\, dv \ . \tag{4.39}$$

By making the transformation

$$\tilde{t} = 6e^{4u} + \tfrac{1}{2}v, \quad \tilde{x} = 6e^{4u} - \tfrac{1}{2}v \ , \tag{4.40}$$

one obtains the Minkowski metric in the inertial coordinates $(\tilde{t}, \tilde{x})$.

Jacobi metrics admitting a homothetic motion

It follows from the above discussion that the Jacobi metrics for all the nonsemisimple perfect-fluid models admit a homothetic motion which is related to the scale/automorphism symmetry. The vacuum Jacobi geometries, on the other hand, also admit a homothetic motion (given by $\partial/\partial\beta^0$) but which is not related to the scale or automorphism symmetries. For the fluid models, however, the homothetic vector is in general different from $\partial/\partial\beta^0$. The only Jacobi metrics not admitting a homothetic symmetry are those for the perfect-fluid Bianchi type-VIII and -IX models.

Thus for all vacuum models and all nonsemisimple perfect-fluid models the Jacobi Hamiltonian can be written in the form

$$H_J = \tfrac{1}{2} e^{2kq^\delta} F \eta^{\alpha\beta} p_\alpha p_\beta = -\tfrac{1}{2} , \qquad (4.41)$$

where F is a function on the configuration space which is independent of the variable q^δ for a particular index value δ. Then $\dot{p}_\delta = -2kH_J = k$ implying that $p_\delta = -kt$ is a (explicitly time-dependent) constant of the motion. This corresponds to a homothetic motion $\partial/\partial q^\delta$ of the Jacobi metric. The constants of the motion for homothetic symmetries were discussed by Prince and Crampin.[32] Since $p_\delta - kt$ is a constant of the motion it follows that Jacobi time is proportional to p_δ. Thus when $q^\delta = \beta^0$ as is the case for the vacuum models, Jacobi time is proportional to p_0 and can be thought of as being "conjugate" to β^0 time or Ω time.

Jacobi power-law-lapse solutions

As mentioned earlier the Jacobi power-law-lapse cases lead to a set of non-EPL solutions. Table II lists these solutions together with their power-law type (T, S, or N) [see (4.30)] determined by the conformal factor in the Jacobi metric (4.28). Wainwright[22] has previously given these solutions in a form closely related to the one presented here but after making an *ad hoc* assumption about the form. The present analysis explains why one has been able to find these solutions.

V. DISCUSSION

By using a Jacobi time gauge one collects all the dynamical information into the kinetic part of the Hamiltonian which is completely determined by the Jacobi metric. Thus studying the properties of this metric reveals information about the cosmological field equations. In particular the adaptation of the metric variables to the symmetries of the Jacobi metric can yield interesting new parametrizations of the original space-time metric, not only in terms of explaining the various known exact solutions and producing new ones but also in placing them all

TABLE II. This table gives the parameters in the space-time metric (4.29) for the six one-term potential cases. The source is indicated by the abbreviations "PF" (perfect fluid) and "Vac" (vacuum). The generality of the metric is also given. A "G" means that it is the general solution for the given Bianchi type and source while an "S" indicates a special solution. The notation "Taub (G)" is used for the general Taub-like solutions discovered by Ellis and MacCallum[14] ("EM"). The power-law type of the solution according to (4.30) is also given. The p_a are the Kasner parameters satisfying the usual constraints (A15). They are given by $p_a = \bar{p}_a/(\bar{p}_0+1)$ corresponding to the early time Kasner behavior of these models as exemplified in Appendix A. Finally δ is defined by $\delta \equiv \sqrt{(10-3\gamma)(2+\gamma)}$ and k has the canonical value given by $k^{-2} = 1+3a^2$.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)
Bianchi type	I	II	II	VI_0	VI_h	V
Source	PF	Vac	PF	Vac	Vac	Vac
Generality	G	G	S	Taub(G)	Taub(G)	G
Discoverer	Jacobs[33]	Taub[20]	Collins[25]	EM[14]	EM[14]	Joseph[34]
Power-law type	T	S	T	N	T	T
c_1	1	1	$\sqrt{(3\gamma-2)/4(2-\gamma)}$	1	1	1
c_2	1	1	1	1	1	1
c_3	1	1	1	$\sqrt{2}$	$2a$	2
$\bar{p}_0$	$\dfrac{\gamma-1}{2-\gamma}$	$\dfrac{1}{4p_1}-1$	$\dfrac{\gamma-1}{2-\gamma}$	$-\frac{5}{8}$	$-\dfrac{4k+1}{4(k+1)}$	$-\frac{1}{4}$
$\bar{p}_1$	$\dfrac{p_1}{2-\gamma}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}-\dfrac{3ak}{4(1+k)}$	$\frac{1}{4}(1+\sqrt{3})$
$\bar{p}_2$	$\dfrac{p_2}{2-\gamma}$	$\dfrac{p_2}{4p_1}$	$\dfrac{2+\gamma+\delta}{8(2-\gamma)}$	$\frac{1}{4}$	$\frac{1}{4}+\dfrac{3ak}{4(1+k)}$	$\frac{1}{4}(1-\sqrt{3})$
$\bar{p}_3$	$\dfrac{p_3}{2-\gamma}$	$\dfrac{p_3}{4p_1}$	$\dfrac{2+\gamma-\delta}{8(2-\gamma)}$	$-\frac{1}{8}$	$-\dfrac{2k-1}{4(k+1)}$	$\frac{1}{4}$
$\bar{q}_0$	$\bar{p}_0$	$-\dfrac{1}{4p_1}-\dfrac{3}{2}$	$\bar{p}_0$	1	$\dfrac{4k-1}{4(1-k)}$	$-\frac{1}{4}$
$\bar{q}_1$	$\dfrac{\frac{2}{3}-p_1}{2-\gamma}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}+\dfrac{3ak}{4(1-k)}$	$\bar{p}_2$
$\bar{q}_2$	$\dfrac{\frac{2}{3}-p_2}{2-\gamma}$	$-\dfrac{2p_1+p_2}{4p_1}$	$\bar{p}_3$	0	$\frac{1}{4}-\dfrac{3ak}{4(1-k)}$	$\bar{p}_1$
$\bar{q}_3$	$\dfrac{\frac{2}{3}-p_3}{2-\gamma}$	$-\dfrac{2p_1+p_3}{4p_1}$	$\bar{p}_2$	1	$\dfrac{2k+1}{4(1-k)}$	$\frac{1}{4}$

into the context of the more general dynamics, for which one can get a more intuitive picture when seen from the point of view of the new variables and their associated geometry. An example of how the new variables can be exploited is given in Appendix B where we carry out a precise qualitative analysis of the late time dynamics of those nonsemisimple perfect-fluid models for which the Taub time gauge potential has more than one term.

Since all EPL solutions are straight timelike lines in minisuperspace (except for the Kasner solutions which are lightlike), it is always possible to adapt the coordinates to these solutions. In the cases studied here it turns out that such an adaptation is equivalent to adapting to a time-like translational symmetry of the Jacobi metric arising from the coupled scale/automorphism symmetry of the Hamiltonian dynamics, thus explaining the existence of these special solutions. This correspondence suggests that the manipulations of the present discussion motivated by the Jacobi geometry probably have consequences even for the more general Bianchi class-B case and the more general tilted perfect-fluid cases in all types where the system is not completely Hamiltonian.

The cosmological Jacobi metric is conformally flat and has a Lorentz signature. Since the geodesic structure of conformally flat spacetimes has been studied extensively, the methods and ideas developed for this problem can be applied to the cosmological field equations. For example, one can use the compactification scheme for the Minkowski metric described by Hawking and Ellis[35] in order to visualize the global structure. One can also get an intuitive picture of the solution space by using the geodesic deviation equation. By starting from a known geodesic solution curve and computing the curvature of the Jacobi metric along it in a parallel transported orthonormal frame, one can obtain information about how neighboring solutions behave.

A number of generalizations are possible. There are other interesting sources for the Einstein equations which lead to a purely Hamiltonian system for all the variables, among which are an electromagnetic or scalar field. For example, Chmielowski and Page[36] have used a Jacobi time gauge to study inflation in anisotropic Bianchi type-I models with a massive scalar field, while Gurzadyan and Kocharyan[37] have used this technique to study the same problem for the closed Friedmann-Robertson-Walker models. Another possible generalization is the nonorthogonal fluid case via the more complicated Hamiltonian formulation of Bao, Marsden, and Walton[38] in which the combined fluid and Einstein equations are put into Hamiltonian form. Indeed, the tilted symmetric case perfect-fluid models of class A are completely Hamiltonian since the two nontrivial fluid variables are both constants, so the present methods can be directly applied. Finally the ideas of this paper may be easily carried over to the case of higher-dimensional cosmological models, where differential form field sources are often considered with special initial data which lead to potential terms in the Hamiltonian only involving the source through constant parameters.

Perhaps the best conclusion to draw from the results of the present discussion is that the spatially homogeneous Einstein equations are a rich system that should not just be plundered for the special results that can be pulled out by *ad hoc* methods. In fact the problem represents a remarkable intersection of many different facets of classical mechanics, symmetry and differential geometry, brought together in what we think is an elegant synthesis in the present approach. By a more systematic investigation of the properties of the system based on this broader perspective, one can make considerable progress in obtaining a feeling for its true dynamical content.

APPENDIX A: BIANCHI TYPE-I PERFECT-FLUID SOLUTION

To illustrate the way in which the choice of variables motivated by the symmetries of the Jacobi metric can be used to solve the field equations and reveal asymptotic properties of the solutions cleanly and straightforwardly, the Bianchi type-I perfect-fluid case will be discussed in detail. Modulo constants the Jacobi metric and the corresponding Jacobi Hamiltonian are

$$\begin{aligned} ds_J^2 &= -dT^2 + T^2(dX^2+dY^2) , \\ T^2 &= [3(2-\gamma)/2]^{-2} e^{3(2-\gamma)\beta^0} , \\ X &= 3(2-\gamma)\beta^+/2, \quad Y = 3(2-\gamma)\beta^-/2 , \\ H_J &= T_J = \tfrac{1}{2}[-p_T^2 + T^{-2}(p_X^2+p_Y^2)] = -\tfrac{1}{2} . \end{aligned} \tag{A1}$$

Since X and Y are cyclic variables, their conjugate momenta are constants. Let $C^2 = p_X^2 + p_Y^2$. Using a dot for the Jacobi time derivative, the useful Hamiltonian equations are

$$\dot{T} = p_T, \quad \dot{X} = p_X T^{-2}, \quad \dot{Y} = p_Y T^{-2} , \tag{A2}$$

which put the Hamiltonian constraint in the form

$$\dot{T}^2 - C^2 T^{-2} = 1 . \tag{A3}$$

If the constants p_X and p_Y are both zero, one obtains the Friedmann-Robertson-Walker solution. Therefore in what follows at least one of the constants p_X and p_Y will be assumed to be nonzero. By using the expressions for $\dot{X}$ and $\dot{Y}$ and rescaling the variables, one can rewrite the constraint equation in the form

$$\begin{aligned} &\left[\frac{dZ}{d\lambda}\right]^2 - Z^{-2} = 1 , \\ &Z = T/C, \quad t_J = C\lambda, \quad C^2 = p_X^2 + p_Y^2 . \end{aligned} \tag{A4}$$

The solution of this equation is

$$\lambda^2 = Z^2 + 1 . \tag{A5}$$

This solution can be used to find expressions for X and Y whose equations become

$$\begin{aligned} \frac{dX}{d\lambda} &= (p_X/C)(\lambda^2-1)^{-1} , \\ \frac{dY}{d\lambda} &= (p_Y/C)(\lambda^2-1)^{-1} \end{aligned} \tag{A6}$$

and have the solutions

$$(X+X_0, Y+Y_0)=\tfrac{1}{2}\ln[(\lambda-1)/(\lambda+1)](p_X/C, p_Y/C) \,, \tag{A7}$$

where X_0 and Y_0 are constants which are pure gauge. In fact as an alternative method of solution, one can introduce polar coordinates in the X-Y plane and reduce this problem to a flat two-dimensional Jacobi geometry discussed above.

It is convenient to choose a new affine parameter t where $t=\frac{1}{2}(\lambda-1)$, so that the time variable vanishes at the initial singularity, i.e., when $\beta^0\to-\infty$, $Z\to 0$. By using the above expressions one can easily obtain the variables β^A which differ only by additive constants from their actual values before their redefinition used to simplify the Jacobi metric initially. The result is

$$\begin{aligned}\beta^0&=[3(2-\gamma)]^{-1}\ln[t(t+1)]+k_1 \,,\\ (\beta^+,\beta^-)&=[3(2-\gamma)]^{-1}\\ &\quad\times\ln[t/(t+1)](p_X/C+k_2, p_Y/C+k_3) \,,\end{aligned} \tag{A8}$$

where it is convenient to let $k_1, k_2, \ldots$ stand for as many constants as needed. The metric coefficients are given by

$$\begin{aligned}g_{11}&=e^{2(\beta^0+\beta^++\sqrt{3}\beta^-)}=k_4[t^{p_1}(t+1)^{(2/3-p_1)}]^{2/(2-\gamma)} \,,\\ g_{22}&=e^{2(\beta^0+\beta^+-\sqrt{3}\beta^-)}\\ &=k_5[t^{p_2}(t+1)^{(2/3-p_2)}]^{2/(2-\gamma)} \,,\\ g_{33}&=e^{2(\beta^0-2\beta^+)}\\ &=k_6[t^{p_3}(t+1)^{(2/3-p_3)}]^{2/(2-\gamma)} \,,\end{aligned} \tag{A9}$$

where

$$\begin{aligned}p_1&=\tfrac{1}{3}(1+p_X/C+\sqrt{3}p_Y/C) \,,\\ p_2&=\tfrac{1}{3}(1+p_X/C-\sqrt{3}p_Y/C) \,,\\ p_3&=\tfrac{1}{3}(1-2p_X/C) \,.\end{aligned} \tag{A10}$$

Since the lapse function N is proportional to $e^{3(\gamma-1)\beta^0}$ one gets

$$N=k_7[t(t+1)]^{(\gamma-1)/(2-\gamma)} \,. \tag{A11}$$

By using the scale invariance one can scale away the constant k_7. The constants occurring in the metric coefficients can be eliminated by rescaling the variables x,y,z, equivalent to automorphisms of the homogeneity group. This leads to the metric

$$\begin{aligned}ds^2=&-[t(t+1)]^{2(\gamma-1)/(2-\gamma)}dt^2\\ &+[t^{p_1}(t+1)^{(2/3-p_1)}]^{2/(2-\gamma)}dx^2\\ &+[t^{p_2}(t+1)^{(2/3-p_2)}]^{2/(2-\gamma)}dy^2\\ &+[t^{p_3}(t+1)^{(2/3-p_3)}]^{2/(2-\gamma)}dz^2 \,.\end{aligned} \tag{A12}$$

At early times when t is close to zero one has the following relation between t and the synchronous time τ:

$$t=k_8\tau^{2-\gamma} \,. \tag{A13}$$

By again rescaling the variables x,y,z one obtains the following approximate solution at early times:

$$ds^2=-d\tau^2+\tau^{2p_1}dx^2+\tau^{2p_2}dy^2+\tau^{2p_3}dz^2 \,. \tag{A14}$$

It is well known (see, e.g., Figure 1 of Uggla and Rosquist[39]) that at early times the general perfect-fluid Bianchi type-I solution is approximated by the Kasner solution, allowing us to identify the parameters p_a with the Kasner parameters conventionally denoted by these symbols. These parameters satisfy the constraints

$$p_1+p_2+p_3=1=p_1^2+p_2^2+p_3^2 \,. \tag{A15}$$

By using the definition $C^2=p_X^2+p_Y^2$ and inserting the previous expressions for the parameters p_a into the above constraints, one sees that the Kasner constraints are indeed satisfied, thus confirming their identification with the Kasner parameters.

At late times, on the other hand, the approximation $t+1\approx t$ is valid so that we can replace $t+1$ by t in (A12). Then t is related to synchronous time by

$$t=k_9\tau^{(2-\gamma)/\gamma} \,. \tag{A16}$$

This leads to the metric

$$ds^2=-d\tau^2+\tau^{4/3\gamma}(dx^2+dy^2+dz^2) \,, \tag{A17}$$

again after rescaling x,y,z. This is the spatially flat FRW universe thus illustrating the well-known tendency of isotropization of the Bianchi type-I models (we refer again to Figure 1 of Uggla and Rosquist[39]).

APPENDIX B: QUALITATIVE CONSIDERATIONS FOR THE DYNAMICS AT LATE TIMES

In this appendix we discuss the late stage behavior of those nonstiff perfect-fluid models ($\gamma\neq 2$) which have more than one exponential term in the potential in the Taub time gauge and which admit a coupled scale/automorphism timelike symmetry whose $\beta^{\pm}$-dependent part of the potential has a minimum. This occurs for the perfect fluid models of type II, type VI_0, and Taub-like type VI_h. In symmetry adapted variables and in the monotonic-function time gauge these models have a Hamiltonian of the form

$$\begin{aligned}H_{\mathrm{mf}}&=T_{\mathrm{mf}}+V_{\mathrm{mf}}\\ &=\tfrac{1}{2}e^{-k_0\bar\beta^0}\eta^{AB}\bar p_A\bar p_B+V_{\mathrm{mf}}(\bar\beta^+,\bar\beta^-) \,,\end{aligned} \tag{B1}$$

where the indices are restricted to ($A,B=0,+$) in the Bianchi type-VI_h case. In the Bianchi type-II case the graph of the potential has the shape of a trough or gutter over the $\bar\beta^+$-$\bar\beta^-$ plane, while in the Taub-like Bianchi type-VI_h case it is a one-dimensional well and in the Bianchi type-VI_0 case a two-dimensional well.

First consider the non-negative quantity

$$E\equiv\tfrac{1}{2}\bar p_0^2e^{-k_0\bar\beta^0}\geq 0, \quad k_0\equiv 3(2-\gamma)>0 \,. \tag{B2}$$

which has the time derivative

$$dE/dt_{\mathrm{mf}}=\tfrac{1}{2}k_0\bar p_0e^{-2k_0\bar\beta^0}(\bar p_+^2+\bar p_-^2) \,, \tag{B3}$$

where we have used the Hamiltonian equations

$$d\bar{\beta}^0/dt_{\rm mf}=\partial H_{\rm mf}/\partial\bar{p}_0=-\bar{p}_0 e^{-k_0\bar{\beta}^0}\ ,$$
$$d\bar{p}_0/dt_{\rm mf}=-\partial H_{\rm mf}/\partial\bar{\beta}^0=k_0 T_{\rm mf}\ . \tag{B4}$$

It is assumed that both $t_{\rm mf}$ and $\bar{\beta}^0$ increase towards the future, which corresponds to an expanding cosmological model, implying that $\bar{p}_0$ is negative. From these equations note that $|\bar{p}_0|$ and hence $\bar{p}_0^2$ increase with time. We note that as long as we have motion in the $\bar{\beta}^+$-$\bar{\beta}^-$ plane, E will change. Furthermore since $\bar{p}_0$ is negative it follows that E is a monotonically decreasing function, giving rise to the terminology monotonic-function time gauge. This means that $\bar{p}_0^2$ is increasing more slowly than $e^{k_0\bar{\beta}^0}$. This gives severe limits on the evolution of the quantities $\bar{p}_+^2 e^{-k_0\bar{\beta}^0}$, $\bar{p}_-^2 e^{-k_0\bar{\beta}^0}$ and the positive potential $V_{\rm mf}$ since E dominates over these quantities as can be seen by considering the Hamiltonian constraint

$$E=\tfrac{1}{2}e^{-k_0\bar{\beta}^0}(\bar{p}_+^2+\bar{p}_-^2)+V_{\rm mf}$$
$$>\tfrac{1}{2}\bar{p}_+^2 e^{-k_0\bar{\beta}^0},\tfrac{1}{2}\bar{p}_-^2 e^{-k_0\bar{\beta}^0},V_{\rm mf}\ . \tag{B5}$$

Since E is decreasing each of these quantities will have a decreasing upper limit. In particular $V_{\rm mf}$ will have a decreasing upper limit which will force the system towards the minimum value of the potential. In those cases where the potential is a well this means that the system is forced towards the EPL solution which corresponds to the minimum of the well. In the Bianchi type-II non-LRS case where the potential is a trough the situation is slightly more complicated. Here the dynamics is pressed down towards the Collins solution[25] which corresponds to motion along the bottom of the trough, explaining the importance of this solution for the general Bianchi II dynamics.[23] Furthermore since $\bar{p}_-$ is a constant while $|\bar{p}_0|$ is increasing, the motion in the $\bar{\beta}^0$ direction will eventually dominate the dynamics and all solutions will behave like the EPL solution for which the system sits still in the bottom of the trough, corresponding to motion only along the variable $\bar{\beta}^0$.

The above adaptation of the variables to the symmetry is therefore equivalent to adapting them to the EPL solution, whose linear motion in β space is chosen as the direction for $\bar{\beta}^0$. Thus the adaptation to the coupled scale/automorphism direction here is equivalent to adapting the variables to the late stage behavior of the dynamics, so these variables are an ideal tool for investigating the dynamics in this limit.

The qualitative information we have about $\bar{\beta}^0$ and $\bar{p}_0$, combined with the form of the potential in adapted variables, makes the problem very similar to a standard two-dimensional potential problem where E takes the role of a decreasing energy while the factor $e^{-k_0\bar{\beta}^0}$ in the kinetic energy can be interpreted as an increasing mass, $m=e^{k_0\bar{\beta}^0}$. This gives a very intuitive picture of the late stage behavior of these models. By looking at the remaining equations of motion one can give a more detailed analysis than the one above, leading to approximate solutions describing the late stage behavior. However, this is beyond the scope of the present paper and will be dealt with elsewhere.[40]

All the vacuum models are also of the general form (B1) but with $k_0=4$. The only vacuum potential with a minimum, however, is that for Bianchi type IX but then the minimum is not attainable due to the Hamiltonian constraint $H_{\rm mf}=0$. Therefore, although some qualitative conclusions can be drawn also for the vacuum models, the scheme outlined in this appendix is not as powerful for the vacuum models as it is for the perfect-fluid models.

APPENDIX C: HIDDEN SYMMETRIES VIEWED FROM OTHER TIME GAUGES

Hidden symmetries may also appear in time gauges other than the Jacobi time gauge. An example is the Taub time gauge in which the vacuum Bianchi type-VI_0 and -VII_0 Hamiltonian is given by

$$H_T=\tfrac{1}{2}(-4p_u p_v+p_-^2)+6e^{4u}h_-^2\ ,$$
$$(p_u,p_v)=\tfrac{1}{2}(p_0+p_+,p_0-p_+)\ . \tag{C1}$$

In addition to the obvious constant of the motion p_v associated with the cyclic variable v, there is another constant of the motion $u+2p_v t_T$, where t_T is Taub time. It follows that u equals the Taub time modulo an affine transformation. Unlike the homothetic symmetry in the Jacobi metric, its symmetry generator commutes with $\partial/\partial v$. If $p_v\neq 0$, the configuration variable u coincides with Taub time up to an affine transformation and the only remaining coupled equations are those for β^- and p_- which then become time dependent. Since p_v comes from an automorphism symmetry, its value can be set to any convenient (nonzero) value. Combining the equations for β^- and p_- we obtain the following second-order time-dependent equation for β^- (for canonical structure constant values):

$$\ddot{\beta}^-=-48\sqrt{3}e^{4u}\sinh(4\sqrt{3}\beta^-)\ . \tag{C2}$$

Introducing the variables $w=e^{4\sqrt{3}\beta^-}$, $z=e^{4u}$ and setting $p_v=-3/\sqrt{2}$ this equation can be written as

$$w''-w^{-1}w'^2+z^{-1}w'+z^{-1}(w^2-1)=0\ , \tag{C3}$$

where we have used $u=-2p_u t_T$ and a prime signifies differentiation with respect to z. This equation is a special case of Painlevé's third transcendent[41] so it cannot be integrated any further. Wils[42] has recently discussed the appearance of Painlevé solutions in general relativity. Khalatnikov and Pokrovsky[43] have given approximate solutions of (C3) valid for all times using slightly different variables.

The existence of a Killing tensor for the Jacobi metric also leads to other preferred time gauges, namely, the ones in which the Hamiltonian system decouples into separate subsystems, thus leading to an additional constant of the motion.[29] An example of this occurs for the Bianchi type-VIII and -IX LRS cases, where the Taub time is preferred. The Hamiltonian in this time gauge is

$$H_T = \tfrac{1}{2}(-\bar{p}_0^2 + \bar{p}_+^2) - 24 n^{(3)} e^{2\sqrt{3}\bar{\beta}^0} + 6 n^{(3)2} e^{-4\sqrt{3}\bar{\beta}^+}$$
$$= -H(\bar{\beta}^0) + H(\bar{\beta}^+) = 0 \; . \quad \text{(C4)}$$

Each of the individual Hamiltonians is separately conserved but the total energy constant must vanish by the Hamiltonian constraint. This manifest separability of the Hamiltonian occurs since the variables are adapted to the Killing tensor which exists for this class.

*Electronic address: (EARN/BITNET)UGGLA@SESUF51 and KR@SESUF51.

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