Quantum string scattering in a cosmic-string space-time

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We exactly quantize fundamental strings propagating in a straight cosmic-string space-time (conical space-time with deficit angle $8\pi G\mu$, μ being the cosmic-string tension). If the fundamental string collides with the cosmic string the scattering is inelastic since the internal modes of the fundamental string become excited. If there is no collision, the fundamental string only suffers a deflection of $\pm 4\pi G\mu$. If there is collision we find inelastic particle production from the interaction of the string with the (classical) geometry. The string oscillator modes only suffer a change of polarization (rotation) in the elastic case and a Bogoliubov transformation in the inelastic case. As a consequence, for a given initial state, the final particle may be in any state associated with the string oscillators. All transformations are explicitly calculated in closed form. Finally, the quantum scattering amplitude for the lowest scalar (tachyon) is computed exactly. In this calculation the vertex operator in the conical geometry and the oscillator linear transformations we find here are used thoroughly. The peculiar features of the string propagation in this topologically nontrivial space-time are discussed including the question whether string splitting can occur at the classical level.

I. INTRODUCTION

In previous papers,[1-3] the present authors started a program to quantize strings in curved space-times with applications to de Sitter space-time, black holes, and gravitational shock-wave geometries. The string equations of motion and constraints were solved both at the classical and quantum level in an expansion which takes into account strong curvature effects of the classical gravitational field. Physical magnitudes such as the mass spectrum, elastic and inelastic scattering amplitudes, Regge trajectories, and critical dimensions have been computed and a new stringy inelastic scattering of particles was found.[2,3] Thermal and ground-state effects of the string in Rindler-accelerated space and near black holes were also studied.[4,5] Strings in cosmological space-times have been also investigated with these methods.[1,6]

In this paper, we study string quantization in a different type of nontrivial background, a purely topological one. We do not need to apply our expansion method here since the quantization and scattering problem can be exactly solved in a closed explicit form. We study a quantum (fundamental) string in a conical space-time in D dimensions. This geometry describes a straight cosmic string of zero thickness and it is a good approximation for very thin cosmic strings with large curvature radius. The space-time is locally flat but globally it has a nontrivial (multiply connected) topology. There exists a conelike singularity with azimuthal deficit angle

$$\delta\Phi = 2\pi(1-\alpha) = 8\pi G\mu \ . \tag{1.1}$$

$G\mu$ is the dimensionless cosmic-string parameter, G is the Newton constant and μ the cosmic-string tension (mass per unit of length). $G\mu \approx 10^{-6}$ for the standard cosmic strings of grand unified theories.

The string equations of motion are free equations in the Cartesian-type coordinates X^0, X, Y, Z^i ($3 \le i \le D-1$), but with the requirement that

$$0 \le \arctan(Y/X) \le 2\pi\alpha \ . \tag{1.2}$$

We find the solution of the equation of motion and constraints in the light-cone gauge [see Eqs. (2.24)–(2.31)].

The string as a whole is deflected by an angle

$$\Delta = \delta\Phi/2 \ . \tag{1.3}$$

A string passing to the right (left) of the topological defect is deflected by $+(-)\delta\Phi$. This deflection does not depend on the impact parameter, nor on the particle energy due to the fact that the interaction with the space-time is of a purely topological nature. In the description of this interaction we find essentially two different situations. (i) The string does not touch the scatterer body. A deflection $\pm\Delta$ at the origin and a rotation in the polarization of modes takes place. In this case there is no creation or excitation of modes (creation α_n^μ and annihilation operators $\alpha_n^{\mu\dagger}$ are not mixed) and we refer to this situation as elastic scattering. [See Eqs. (2.32)–(2.38)]. (ii) The string collides against the scattering center; then in addition to being deflected, the internal modes of the string become excited. We refer to this situation as inelastic scattering. We describe the evolution of this system and guarantee continuity of the string coordinates and its τ derivatives at the collision times $\tau=\tau_0$ [see Eqs. (2.39)–(2.45)]. (We deal here with open strings.) We find the relations between ingoing ($\tau<\tau_0$) and outgoing

($\tau > \tau_0$) zero modes and oscillators. In this case, in addition to a change in the polarization, there are mode excitations which yield final particle states *different* from the initial one. Particle and antiparticle modes are mixed [see the Bogoliubov transformations Eqs. (2.44)–(2.48)]. Here we have a single (test) string. That is, the initial and final states are *one particle* but *different* states. Notice that here particle states transmute at the classical (tree) level as a consequence of the interaction with the space-time geometry. In the present case this is a topological defect.

We describe the conformal L_n generators and the mass formulas. We explicitly prove that the $L_n^<$ built from the ingoing modes are *identical* to the $L_n^>$ built from the outgoing modes. The mass spectrum is the same as in the standard Minkowski space-time and the critical dimension is the same ($D=26$ for bosonic strings).

We also analyze the question whether the string may split into two pieces as a consequence of the collision with the conical singularity. We find that string splitting only occurs at the quantum level. In Sec. III, we study the scalar particle (lowest string mode) quantum scattering amplitude in the conical space of the cosmic string. This is also calculated exactly and in closed form. We need first to find the solution of the Klein-Gordon equation in conical space-time in D dimensions. We find the solution Ψ_{in} [Eqs. (3.5) and (3.15)] which satisfies the massive free wave equation with the nontrivial requirement to be periodic in the azimuthal angle Φ with period $2\pi\alpha$ [Eq. (3.25)]. This prevents the usual asymptotic behavior for large radial coordinate $R\to\infty$. The full wave function Eq. (3.22) is the sum of two terms. For $D=3$, this solution has been found in Refs. 7 and 8. The incident wave turns out to be a finite superposition of plane waves without distortion. They propagate following wave vectors rotated from the original one by a deflection $\pm\Delta$ and periodically extended with period $2\pi\Delta$. This incident wave although undistorted suffers multiple periodic rotations as a consequence of the multiply connected topology. In addition, the second term describes the scattered wave with scattering amplitude

$$f(\Theta)=\frac{1}{2\pi}\frac{\sin\pi/\alpha}{\cos\pi/\alpha+\cos\Theta/\alpha}\ . \tag{1.4}$$

In the scalar (ground-state) string amplitude Eq. (3.2) we have ingoing $\tau<\tau_0$ and outgoing $\tau>\tau_0$ zero modes and oscillator modes averaged on the ingoing ground state $|0\rangle$. Inserting the wave functions Ψ_{in} and $\Psi_{\text{out}}=\Psi_{\text{in}}^*(-k)$ in this matrix element (A) yields four terms $A=A_{\text{I}}+A_{\text{II}}+A_{\text{III}}+A_{\text{IV}}$, each of these terms splitting into four other ones corresponding to the natural four integration regions of the double τ domain. The detailed computation is reported in Sec. III. For this computation it is convenient to work in the covariant formalism where all string components are quantized on equal footing. The final result is given by Eqs. (3.39)–(3.42). The term A_{I} corresponding to the undistorted waves describes outgoing particles moving in the original incident direction or rotated by the angle $\pm\Delta$ (modulo $2\pi\alpha$). The terms A_{II}, A_{III}, and A_{IV} describe true elastic scattered particles. The effect of the topological defect in space-time on the string scattering amplitudes manifests through the nontrivial vertex operator (which is different from $e^{ik\cdot x}$ and through the fact that ingoing ($\alpha^A_{n<}$) and outgoing ($\alpha^A_{n>}$) mode operators are related by a linear transformation which makes the expectation value on the ingoing ground state $|0_<\rangle$ nontrivial. In the $\alpha=1$ limit (that is, cosmic-string mass $\mu=0$), we recover the Minkowski amplitude. If the oscillator modes $n\neq 0$ are ignored, we recover the point-particle field-theory Klein-Gordon amplitude Eq. (1.4).

We deal here with open strings but the extension to closed strings is straightforward. We find a $\pm\Delta$ rotation for the elastic scattering of closed strings and mode excitation when the string collides with the conical singularity (inelastic case).

Finally, let us notice that strings in conical space-times were considered in Ref. 9 but only for deficit angles $2\pi(1-1/N)$ where the scattering is trivial. (In that case the space becomes an orbifold.) In this paper we have solved the scattering problem for general deficit angles where it is nontrivial. Let us also notice that the condition of conformal invariance (vanishing of the β function) is identically satisfied everywhere in the conical space-time, except eventually at the origin. If such difficulty arises, this space-time will simply *not* be a candidate for a string ground state (vacuum). Anyway, this geometry effectively describes the space-time around a cosmic string.

II. STRING PROPAGATION IN CONICAL SPACE-TIME

We consider a conical space-time in D dimensions defined by the metric

$$(dS)^2=-(dX^0)^2+(dR)^2+R^2(d\Phi)^2+(d\mathbf{Z}^i)^2\ , \tag{2.1}$$

where

$$R=\sqrt{X^2+Y^2}\ ,$$

$$\Phi=\arctan(Y/X)$$

are cylindrical coordinates, but with the range

$$0\le\Phi<2\pi\alpha,\quad \alpha=1-4G\mu \tag{2.2}$$

$(d\mathbf{Z}^i)^2$ is a flat $(D-3)$-dimensional space and $\mathbf{Z}^i$, $3\le i\le D-1$ are Cartesian coordinates. The spatial points (R, Φ, $\mathbf{Z}$) and ($R,\Phi+2\pi\alpha$, $\mathbf{Z}$) are identified. The space-time is locally flat for $R\neq 0$ but has a conelike singularity at $R=0$ with azimuthal deficit angle

$$\delta\Phi=2\pi(1-\alpha)=8\pi G\mu\ . \tag{2.3}$$

This geometry describes a straight cosmic string of zero thickness. It is a good approximation for very thin cosmic strings with large curvature radius. Globally, it has a nontrivial (multiply-connected) topology.

It is also useful to introduce the coordinates

$$\bar{\Phi}=\alpha^{-1}\Phi,\quad \bar{R}=\alpha R \tag{2.4}$$

with the usual range

$$0\le\bar{\Phi}<2\pi \tag{2.5}$$

but in which the metric takes the form

$$dS^2=-(dX^0)^2+\alpha^{-2}(d\bar{R})^2+\bar{R}^2(d\bar{\Phi})^2+(dZ^i)^2 . \tag{2.6}$$

The action of the string in the metric [Eq. (2.1)] is given by

$$S=\frac{1}{2\pi}\int\int d\sigma\, d\tau[-(\partial_\mu X^0)^2+(\partial_\mu R)^2 +R^2(\partial_\mu\Phi)^2+(\partial_\mu Z^i)^2] \tag{2.7}$$

and the string equations of motion are

$$\begin{aligned}&(\partial_\sigma^2-\partial_\tau^2)X^0=0 ,\\ &(\partial_\sigma^2-\partial_\tau^2)R=R[(\partial_\sigma\Phi)^2-(\partial_\tau\Phi)^2] ,\\ &(\partial_\sigma^2-\partial_\tau^2)\Phi=-\frac{1}{2R}(\partial_\sigma R\,\partial_\sigma\Phi-\partial_\tau R\,\partial_\tau\Phi) ,\\ &(\partial_\sigma^2-\partial_\tau^2)Z^i=0 .\end{aligned} \tag{2.8}$$

These are free equations in the coordinates X^0, X, Y, Z^i:

$$\begin{aligned}&\Box^2X^0=0 ,\\ &\Box^2X=0=\Box^2Y=\Box^2Z^i\end{aligned} \tag{2.9}$$

but with the condition

$$0\le\arctan(Y/X)\le 2\pi\alpha .$$

Before solving the string equations (2.8) and (2.9), let us consider the center-of-mass equations

$$\begin{aligned}&\ddot{\rho}=\rho\dot{\varphi}^2 ,\\ &\frac{d}{d\tau}(\rho^2\dot{\varphi})=0,\quad 0<\varphi<2\pi\alpha ,\\ &\ddot{Z}^i=0 .\end{aligned} \tag{2.10}$$

In Cartesian coordinates, they are

$$\begin{aligned}&\ddot{x}^\mu=0,\quad \mu=0,1,\ldots,D-1 ,\\ &0<\arctan(y/x)<2\pi\alpha .\end{aligned} \tag{2.11}$$

The solution is obtained from the free trajectories

$$x^\mu=q^\mu+p^\mu\tau \tag{2.12}$$

and by imposing that the variable $\varphi=\arctan(Y/X)$ be periodic with period $2\pi\alpha$. It is useful to write the trajectories in terms of the coordinates ($\bar{\rho}=\alpha\rho, \bar{\varphi}=\alpha^{-1}\varphi$); that is,

$$\begin{aligned}&\bar{\rho}\cos(\alpha\bar{\varphi})=q_x+p_x\tau ,\\ &\bar{\rho}\sin(\alpha\bar{\varphi})=q_y+p_y\tau .\end{aligned} \tag{2.13}$$

This yields the orbit equation

$$\bar{\rho}\sin(\alpha\bar{\varphi}-\gamma)=-q\sin(\gamma-\beta) , \tag{2.14}$$

where

$$\tan\gamma=p_y/p_x$$

and (2.15)

$$q_x=q\cos\beta,\quad q_y=q\sin\beta .$$

When $\tau\to\pm\infty$, then $\bar{\rho}\to+\infty$ and the angle $\bar{\varphi}\to\bar{\varphi}_\pm$ given by

$$\sin(\alpha\bar{\varphi}_\pm-\gamma)=0 . \tag{2.16}$$

In order to satisfy Eq. (2.13) we choose, for $\bar{\varphi}_\pm$,

$$\bar{\varphi}_+=\frac{\gamma}{\alpha},\quad \bar{\varphi}_-=\frac{\gamma\mp\pi}{\alpha} , \tag{2.17}$$

where the ($\mp$) sign in φ_- depends on whether $\pm\dot{\bar{\varphi}}>0$, that is, on whether the particle passes to the right or to the left of the cosmic string. The scattering angle is defined as usual by

$$\bar{\Delta}=\bar{\varphi}_+-\bar{\varphi}_--\pi . \tag{2.18}$$

This yields, for a particle passing to the right,

$$\bar{\Delta}_R=\pi(\alpha^{-1}-1) \tag{2.19}$$

and for a particle passing to the left we find

$$\bar{\Delta}_L=-\pi(\alpha^{-1}+1)$$

since $\bar{\Delta}_L$ is defined modulo 2π; this is equivalent to

$$\Delta_L=-\pi(\alpha^{-1}-1) ; \tag{2.20}$$

that is,

$$\bar{\Delta}_{\binom{R}{L}}=(\pm)\frac{4\pi G\mu}{1-4G\mu} . \tag{2.21}$$

Finally, in the coordinates (ρ,φ), we have as deflection angles

$$\Delta_{\binom{R}{L}}=(\pm)\pi(1-\alpha)=(\pm)4\pi G\mu . \tag{2.22}$$

Thus, the scattering angle is in absolute value half of the deficit angle:

$$\Delta\equiv|\Delta_{\binom{R}{L}}|=\frac{\delta\Phi}{2} . \tag{2.23}$$

Notice that the deflection does not depend on the impact parameter, nor on the particle energy, as a consequence of the fact that the interaction with the geometry is of purely topological nature.

Let us now consider the solution of the string equations. Let us use the coordinates X^μ that satisfy the free equations of motion [Eq. (2.9)]. We have

$$X^\mu(\sigma,\tau)=q^\mu+p^\mu\tau+i\sum_{n=1}^{\infty}\frac{\cos n\sigma}{n}(\alpha_n^\mu e^{-in\tau}-\alpha_n^{\mu\dagger}e^{in\tau}) \tag{2.24}$$

with the requirement that

$$\tan\Phi=\frac{q_y+p_y\tau+i\sum_{n=1}^{\infty}\frac{\cos n\sigma}{n}(\alpha_n^y e^{-in\tau}-\alpha_n^{y\dagger}e^{in\tau})}{q_x+p_x\tau+i\sum_{n=1}^{\infty}\frac{\cos n\sigma}{n}(\alpha_n^x e^{-in\tau}-\alpha_n^{x\dagger}e^{in\tau})} \tag{2.25}$$

be periodic with period $2\pi\alpha$. This periodicity condition will be enforced by appropriately rotating free string solutions [see Eqs. (2.34), (2.40), and (2.41)]. As a consequence, linear relations between the ingoing and outgoing oscillators associated with the X and Y coordinates appear (see below).

From Eqs. (2.8), and (2.9) we see that we can choose the light-cone gauge

$$U=p_U\tau, \quad U=X^0-Z^3 . \tag{2.26}$$

Then, for $\tau\to\pm\infty$, from Eqs. (2.25) and (2.26) and by expressing the solution in terms of the $\overline{\Phi}$ coordinate, we have

$$\tan(\alpha\overline{\Phi}) \underset{\tau\to\pm\infty}{\equiv} \tan(\alpha\Phi_\pm)=\tan\gamma=\frac{p_y}{p_y} . \tag{2.27}$$

This yields a string scattering angle

$$\overline{\Delta}_{\binom{R}{L}}=(\pm)\pi(\alpha^{-1}-1) \tag{2.28}$$

equal to that of the point particle. That is, it yields

$$\Delta_{\binom{R}{L}}=(\pm)\pi(1-\alpha)=(\pm)4\pi G\mu . \tag{2.29}$$

The string as a whole is deflected by an angle

$$\Delta=|\Delta_{\binom{R}{L}}|=\frac{\delta\Phi}{2} . \tag{2.30}$$

Let us consider now the constraints. The energy-momentum tensor is given by

$$\begin{aligned} T_{\pm\pm}&=\partial_\pm U\partial_\pm V-(\partial_\mu R)^2-R^2(\partial_\mu\Phi)^2-(\partial_\mu \mathbf{Z}^i)^2 \\ &=\pm p_U\partial_\pm V-(\partial_\mu X)^2-(\partial_\mu Y)^2-(\partial_\mu \mathbf{Z}^i)^2\simeq 0 , \end{aligned} \tag{2.31}$$

where $3\le i\le D-1$ and

$$x_\pm=(\sigma\pm\tau), \quad \partial_\pm=\tfrac{1}{2}(\partial_\sigma\pm\partial_\tau) ,$$
$$U=X^0-Z^3, \quad V=X^0+Z^3 .$$

In the light-cone gauge where we are working these constraints completely determine the longitudinal V coordinate of the string in terms of the transverse coordinates, as it should be.

Let us study now the interaction between the fundamental string and the classical cosmic string (or conical space). There appear essentially two different situations.

(i) The string does not touch the scatterer body (here represented by the cosmic string); in this case the string only suffers a deflection at the origin. We refer to this situation as elastic scattering.

(ii) The string collides against the scattering center. In this case, the string gets its normal modes excited in addition to being deflected. This happens each time a point of the string collides with the center. We refer to this process as inelastic scattering.

Let us first consider case (i) (elastic scattering).

For $\tau\to-\infty$, the ingoing solution $(X^A_<)$ is just the free solution without deflection. For $\tau\to+\infty$, the outgoing solution $X^A_>$ is the free solution after deflection. We have

$$X_<=R\cos\Phi, \quad Y_<=R\sin\Phi , \tag{2.32a}$$

and

$$X_>=R\cos(\Phi\pm\Delta), \quad Y_>=R\sin(\Phi\pm\Delta) , \tag{2.32b}$$

where Δ is the deflection angle given by Eq. (2.30) and the $+$ $(-)$sign refers to a string passing to the right (left) of the cosmic string:

$$\begin{aligned} X_>&=X_<\cos\Delta\mp Y_<\sin\Delta , \\ Y_>&=Y_<\cos\Delta\pm X_<\sin\Delta ; \end{aligned} \tag{2.33}$$

that is,

$$\begin{bmatrix} X_> \\ Y_> \end{bmatrix}=\mathcal{R}(\pm\Delta)\begin{bmatrix} X_< \\ Y_< \end{bmatrix} . \tag{2.34}$$

$\mathcal{R}(\Delta)$ being the rotation matrix:

$$\mathcal{R}(\Delta)=\begin{bmatrix} \cos\Delta & -\sin\Delta \\ \sin\Delta & \cos\Delta \end{bmatrix} . \tag{2.35}$$

The $D-3$ $\mathbf{Z}^i$ components are not affected by the scattering.

From Eqs. (2.24) and (2.34) we get for the zero modes and for the oscillation the following transformations between the in and out solutions:

$$\begin{aligned} \begin{bmatrix} q^x_> \\ q^y_> \end{bmatrix}&=\mathcal{R}(\pm\Delta)\begin{bmatrix} q^x_< \\ q^y_< \end{bmatrix} , \\ \begin{bmatrix} p^x_> \\ p^y_> \end{bmatrix}&=\mathcal{R}(\pm\Delta)\begin{bmatrix} p^x_< \\ p^y_< \end{bmatrix} \end{aligned} \tag{2.36}$$

and

$$\begin{bmatrix} \alpha^x_{n>} \\ \alpha^y_{n>} \end{bmatrix}=\mathcal{R}(\pm\Delta)\begin{bmatrix} \alpha^x_{n<} \\ \alpha^y_{n<} \end{bmatrix} , \tag{2.37}$$

$$\begin{bmatrix} \alpha^{x\dagger}_{n>} \\ \alpha^{y\dagger}_{n>} \end{bmatrix}=\mathcal{R}(\pm\Delta)\begin{bmatrix} \alpha^{x\dagger}_{n<} \\ \alpha^{y\dagger}_{n<} \end{bmatrix} . \tag{2.38}$$

We see that there is a change in the polarization of string modes after passing the cosmic string. There is no Bogoliubov transformation relating ingoing and outgoing solutions. That is, there is no creation of excitation of modes after passing the scattering center (α^μ_n and $\alpha^{\mu\dagger}_n$ are not mixed). In this case, the effect of the deficit angle is not like a curved metric or a external potential in which excitation of pairs of modes takes place.[2,3] Pair mode excitations lead to inelastic processes. That is, the final string state (outgoing particle) is different from the initial one (ingoing particle) in the cases of Refs. 2 and 3.

Let us discuss now the inelastic scattering [case (ii)].

The string evolution is described by the free equations except at the collision point (σ_0,τ_0) with the cosmic string. We take the origin at the cosmic string; thus, we have

$$X(\sigma_0,\tau_0)=0 , \quad Y(\sigma_0,\tau_0)=0 . \tag{2.39}$$

The values of σ_0 and τ_0 indeed depend on the string state before the collision, that is, on the dynamical variables $q^x_<, q^y_<, p^x_<, p^y_<, \alpha^x_{n<}$, and $\alpha^y_{n<}$ [see Eq. (2.45)].

Since the deflection angles to the right and to the left of the cosmic string are different, Eqs. (2.36)–(2.38) do not hold. We have now different matching conditions for $0\le\sigma\le\sigma_0$ and for $\sigma_0\le\sigma\le\pi$. That is,

$$X^A_>(\sigma,\tau_0)=\mathcal{R}^A_B(-\Delta)X^B_<(\sigma,\tau_0),\quad 0<\sigma<\sigma_0\ ,\quad A\equiv X,Y\ ,$$
$$\partial_\tau X^A_>(\sigma,\tau_0)=\mathcal{R}^A_B(-\Delta)\partial_\tau X^B_<(\sigma,\tau_0)\ ; \tag{2.40}$$

$$X^A_>(\sigma,\tau_0)=\mathcal{R}^A_B(\Delta)X^B_<(\sigma,\tau_0),\quad \sigma_0<\sigma<\pi\ ,\ A\equiv X,Y\ ,$$
$$\partial_\tau X^A_>(\sigma,\tau_0)=\mathcal{R}^A_B(\Delta)\partial_\tau X^B_<(\sigma,\tau_0)\ , \tag{2.41}$$

where $X^1\equiv X$ and $X^2\equiv Y$. This guarantees continuity of the string coordinates and its derivatives at $\tau=\tau_0$. In addition, the string ends must obey the usual requirements

$$\partial_\sigma X^\mu_<(0,\tau)=\partial_\sigma X^\mu_<(\pi,\tau)=0\ ,$$
$$\tau<\tau_0,\quad 0\le\mu\le D-1\ ,\quad \tau>\tau_0\ .$$
$$\partial_\sigma X^\mu_>(0,\tau)=\partial_\sigma X^\mu_>(\pi,\tau)=0\ . \tag{2.42}$$

The string solutions $X^A_<(\sigma,\tau)$ and $X^A_>(\sigma,\tau)$ admit the usual expansion

$$X^A_<(\sigma,\tau)=q^A_<+p^A_<\tau+i\sum_{n=1}^{\infty}\frac{\cos n\sigma}{n}(\alpha^A_{n<}e^{-in\tau}-\alpha^{A\dagger}_{n<}e^{in\tau})\ ,$$
$$0\le\sigma\le\pi,\tau<\tau_0\ , \tag{2.43}$$

$$X^A_>(\sigma,\tau)=q^A_>+p^A_>\tau+i\sum_{n=1}^{\infty}\frac{\cos n\sigma}{n}(\alpha^A_{n>}e^{-in\tau}-\alpha^{A\dagger}_{n>}e^{in\tau})\ ,$$
$$0\le\sigma\le\pi,\tau>\tau_0\ . \tag{2.44}$$

We recall that σ_0 and τ_0 depend on the initial data of the string through the constraint (2.39) as

$$0=q^A_<+p^A_<\tau_0+i\sum_{\substack{n=-\infty\\ n\neq0}}^{+\infty}\frac{\cos n\sigma_0}{n}\alpha^A_{n<}e^{-in\tau_0}\ . \tag{2.45}$$

Imposing Eqs. (2.40) and (2.41) at $\tau=\tau_0$ yields linear relations between the operators $q_>$ $(p_>)$ and $q_<$ $(p_<)$ appearing in the expansions (2.43)–(2.45). We find, for the zero modes,

$$q^A_>=\mathcal{R}^A_B(\Delta)q^B_<+\frac{\sigma_0}{\pi}[\mathcal{R}^A_B(-\Delta)-\mathcal{R}^A_B(\Delta)]\ ,$$
$$p^A_>=\mathcal{R}^A_B(\Delta)p^B_<+\frac{\sigma_0}{\pi}[\mathcal{R}^A_B(-\Delta)-\mathcal{R}^A_B(\Delta)]\ . \tag{2.46}$$

We get, for the oscillators,

$$\alpha^A_{m>}=\mathcal{R}^A_B(\Delta)\alpha^B_{m<}+\frac{1}{\pi}L^A_B\left\{-ie^{im\tau_0}\sin m\sigma_0\left[q^B_<+p^B_<\left[\tau_0+\frac{i}{m}\right]\right]\right.$$
$$\left.+\frac{1}{2}\sum_{n=-\infty}^{\infty}\left[\frac{\sin(n-m)\sigma_0}{n-m}+\frac{\sin(n+m)\sigma_0}{n+m}\right]\alpha^B_{n<}e^{i(m-n)\tau_0}\left[\frac{m}{n}+1\right]\right\}\ , \tag{2.47}$$

where $L^A_B\equiv\mathcal{R}^A_B(-\Delta)-\mathcal{R}^A_B(\Delta)$. This can be written as

$$\alpha^A_{m>}=\sum_{B=1}^{2}D^A_{B,m}p_{B<}+\sum_{B=1}^{2}\sum_{n=1}^{\infty}(A^{AB}_{mn}\alpha^B_{n<}+B^{AB}_{mn}\alpha^{B\dagger}_{n<})$$
$$=\mathcal{R}^A_B(\Delta)\alpha^B_{m<}+\beta^B_m\ , \tag{2.48}$$

where

$$D^A_{Bm}=\frac{L^A_B}{\pi}\frac{\sin m\sigma_0}{m}\ , \tag{2.49}$$

$$A^{AB}_{mn}=\mathcal{R}^A_B(\Delta)\delta_{mn}+\frac{1}{\pi}L^A_B\frac{\sin(m-n)\sigma_0}{m-n}e^{-in\tau_0}\ , \tag{2.50}$$

$$B^{AB}_{mn}=\frac{1}{\pi}L^A_B\frac{\sin(m+n)\sigma_0}{m+n}e^{in\tau_0}\ ,$$
$$\beta^B_m=\frac{L^A_B}{\pi}\sum_{n\in Z}\frac{\sin(n-m)\sigma_0}{n-m}\alpha^B_{n<}e^{-in\tau_0}\ . \tag{2.51}$$

We used here the constraint (2.44) to simplify the expressions.

The Bogoliubov transformations Eqs. (2.47)–(2.50) are interpreted as follows: if, for $\tau<\tau_0$, the initial string is in a particle mode n polarized in a given direction A $(\alpha^A_{n<}, A=1,2)$, then, for $\tau>\tau_0$, there will be (a) an amplitude $A^{AB}_{nm}(\Delta)$ for a mode m polarized in another direction B $(\alpha^B_{m>})$ and (b) an amplitude $B^{AB}_{nm}(\Delta)$ for an antiparticle mode m polarized in the B direction $[(\alpha^B_{m>})^\dagger]$. The same conclusions hold for the transverse modes α^i_n $(i=3,\ldots,D-2)$, except that they do not suffer polarization changes.

If for $\tau<\tau_0$, the ingoing vacuum $|0_<\rangle$ is defined by

$$\alpha^\mu_{n<}|0_<\rangle=0\quad\text{for all } n>0\ . \tag{2.52}$$

Then, the number of physical modes for $\tau>\tau_0$ at the nth level created from the $\tau<\tau_0$ ground state is

$$N^A_n=(1/n)\langle0_<|(\alpha^A_{n>})^\dagger\alpha^A_{n>}|0_<\rangle\ . \tag{2.53}$$

Since $\alpha^A_{n>}$ contains $(\alpha^B_{n<})^\dagger$ [see Eq. (2.46)] this expectation value is nonzero.

The Bogoliubov coefficients satisfy

$$A^{AB}_{nm}(\Delta)=B^{AB}_{-nm}(\Delta)+\mathcal{R}^A_B(\Delta)\delta_{mn}\ . \tag{2.54}$$

In this inelastic case in which the string collides against the cosmic string, in addition to the change of polarization, the modes become excited. This mode excitation appears also for a string in a curved (static as well as time dependent) space-time or in flat space in the presence of an external potential.[2,3] These mode excitations yield final particle states different from the initial one. With these excited modes one can construct a particle state of higher and well-defined spin and mass. It should be noticed that here we have a single (test) string. That is, both the final and the initial states are one-particle (but different) states. Particle states are created at the classical (tree) level as a consequence of the interaction with the geometry. In the present case, this is a topological defect in the space-time.

It should be also noticed that Eqs. (2.44) and (2.45) reproduce Eqs. (2.36)–(2.38) with signs + or − when $\sigma_0=0$ or $\sigma_0=\pi$, respectively, as it must be.

We have considered the cases when the string does not collide with the conical singularity and when it collides once. It may happen that the string intersects two or more times with the cosmic string. This depends upon the initial shape and velocity of it. The corresponding linear transformation of the string-mode operators $Z_n^A=(\alpha_n^A,\alpha_n^{A\dagger},q^A,p^A)$ can be obtained by repeatedly using Eqs. (2.46) and (2.47). This transformation can be written in matrix form

$$Z_{m>}^A=\sum_{n,B} J_{mn}^{AB}(z_0)Z_{n<}^B\ ,$$
$$Z_n^A\equiv(\alpha_n^A,\alpha_n^{A\dagger},q^A,p^A),\quad z_0=(\sigma_0,\tau_0)\ , \tag{2.55}$$

where

$$J(z_0,\Delta)=\begin{bmatrix} A_{mn}^{AB} & B_{mn}^{AB} & 0 & D_m^{AB}\\ A_{mn}^{AB*} & B_{mn}^{AB*} & 0 & D_m^{AB*}\\ 0 & 0 & \mathcal{L}_B^A & 0\\ 0 & 0 & 0 & \mathcal{L}_B^A \end{bmatrix}. \tag{2.56}$$

Here the coefficients A_{nm}^{AB}, B_{nm}^{AB}, $\mathcal{L}_m^{AB}$, and D_m^{AB} are given by Eqs. (2.49)–(2.51) and $\mathcal{L}_B^A$ is given by

$$\mathcal{L}_B^A=\mathcal{R}_B^A(\Delta)+\frac{\sigma_0}{\pi}[\mathcal{R}_B^A(-\Delta)-\mathcal{R}_B^A(\Delta)]\ .$$

Let us consider, for example, the situation of Fig. 1 where the ingoing string collides with the cosmic string twice: first at $z_0=(\sigma_0,\tau_0)$ and then at $z_1=(\sigma_1,\tau_1)$. We have then

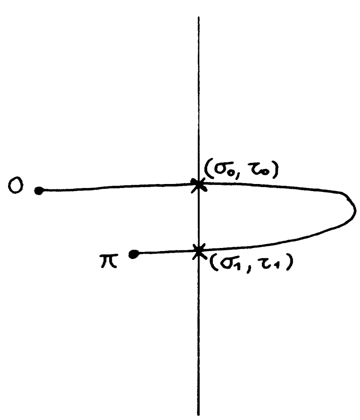

FIG. 1. String colliding twice with the conical singularity.

$$Z_>=J(Z_1,-\Delta)J(Z_0,\Delta)Z_<\ .$$

Let us discuss now the conformal generators

$$L_n=\int_0^\pi d\sigma\, e^{in\sigma}T_{\pm\pm}(\sigma,\tau=0)\ , \tag{2.57}$$

where $T_{\pm\pm}$ has the usual flat-space expression (2.31).

The stress-energy tensor on the world sheet $T_{\mu\nu}(\sigma,\tau)$ is everywhere conserved and traceless.

The L_n operators in its Fourier expansion can be computed in terms of the $\tau<\tau_0$ basis or in terms of the $\tau>\tau_0$ basis:

$$L_n^<=\sum_m \alpha_{m<}^i\alpha_{n-m<}^i,\quad L_n^>=\sum_m \alpha_{m>}^i\alpha_{n-m>}^i\ . \tag{2.58}$$

We find that $L_n^<=L_n^>$ in both the elastic and inelastic cases. In the elastic case the equality $L_n^<=L_n^>$ easily follows from Eqs. (2.36)–(2.38) and (2.58). In the inelastic case, by inserting Eq. (2.49) in Eq. (2.58) we find

$$L_{n>}-L_{n<}=2(\cos 2\Delta-1)\times\left[\sum_m(\beta_{n-m}^A\alpha_{m<}^A-\beta_m^A\beta_{n-m}^A)\right]\ , \tag{2.59}$$

where we have used the properties

$$RL^T=\cos(2\Delta)-1+\epsilon\sin 2\Delta,\quad \epsilon=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},$$
$$L^TL=2(1-\cos 2\Delta)\ .$$

Using now the series

$$\frac{1}{\pi}\sum_{n\in z}\frac{(\sin n\sigma_0)\sin(n-m)\sigma_0}{n(n-m)}=\frac{\sin m\sigma_0}{m}$$

we see that both terms in the right-hand side (RHS) of Eq. (2.59) just cancel. Thus,

$$L_n^<=L_n^>\ . \tag{2.60}$$

The mass formula follows from the $L_0\approx 0$ constraint as usual. We have both in the elastic and inelastic cases

$$M_<^2=\frac{1}{\alpha'}\sum_{n=1}^\infty \alpha_{n<}^{i\dagger}\alpha_{n<}^i\ ,$$
$$M_>^2=\frac{1}{\alpha'}\sum_{n=1}^\infty \alpha_{n>}^{i\dagger}\alpha_{n>}^i\ . \tag{2.61}$$

These two operators have identical spectra which are simply the flat-space-time mass spectrum. However, they do not have common particle eigenvectors. As discussed above, if the string collides with the conical singularity, a particle state in one of the regions (< or >) is an infinite superposition of particle states associated with the other region.

The critical dimension for bosonic strings propagating in this conical space-time turns out to be the same as in flat space-time ($D=26$).

Can the string split? When the string collides against the conical singularity, since the deflection angles to the right and the left of the scattering center are different, one could think that the splitting of the string into two pieces will be favored by the motion. Such splitting solution exists and is consistent. However, its classical action is larger than the one without splitting. Therefore, this splitting may take place only quantum mechanically. In fact, such a possibility of string splitting always exists and already in the simplest case for strings *freely* propagating in *flat* space-time. The free equations of motion of strings in flat space-time admit consistent solutions which describe splitting but once more their action is larger than the one without splitting.

When the string propagates in curved space-time, the interaction with the geometry modifies the action. In particular, the possibility arises that the action for the splitting solution becomes smaller than the one without splitting. Therefore, string splitting will occur classically.

III. THE TWO-BODY AMPLITUDE FROM STRING THEORY IN CONICAL SPACE-TIME

We study now the scattering of the scalar particle corresponding to the open-string ground state in the conical space of the cosmic string. Here the scalar vertex operator satisfies the free D-dimensional Klein-Gordon equation

$$(\Box^2 - m^2)T(X) = 0 \tag{3.1a}$$

with the condition that $T(X^\mu)$ be periodic function of

$$\Phi = \arctan(Y/X) \tag{3.1b}$$

with period $2\pi\alpha$.

The scattering amplitude is given by

$$A(\mathbf{k}_1,\mathbf{k}_2) = \int\int d^2z_1 d^2z_2 \langle 0_< | :\Psi^*_{\text{out}}(\mathbf{k}_2, X(Z_2)) : \times :\Psi_{\text{in}}(\mathbf{k}_1, X(z_1)) : | 0_< \rangle \,, \tag{3.2}$$

where $(\mathbf{k}_1,\mathbf{k}_2)$ are the on-shell ingoing and outgoing momenta of the scalar particle. $X(z)$ is the string coordinate operator and the normal ordering : : is taken with respect to the ingoing ground state $|0_<\rangle$:

$$\alpha^i_{n<}|0_<\rangle = 0, \quad n>0 \,, \qquad p_<|0_<\rangle = 0 \,. \tag{3.3}$$

z_1, z_2 stand for the world-sheet coordinates (σ,τ). $\Psi_{\text{in}}(\mathbf{k},X)$ and $\Psi_{\text{out}}(\mathbf{k},X)$ are solutions of Eqs. (3.1) with plane-wave boundary conditions for $X_0 \to -\infty$ and $X_0 \to +\infty$, respectively. In our problem, the solutions Ψ_{out} and Ψ_{in} satisfy the relation

$$\Psi^*_{\text{out}}(\mathbf{k},X^\mu) = \Psi_{\text{in}}(-\mathcal{R}(\Delta)\mathbf{k}, X^\mu) \,, \tag{3.4}$$

where the rotation matrix $\mathcal{R}(\Delta)$ is defined by Eq. (2.35).

The Klein-Gordon equation in the D-dimensional metric of a cosmic string can be solved in closed form. In $D=4$, this was found in Refs. 7 and 8. The nontrivial dependence is in the two Cartesian string coordinates X and Y (or R and Φ):

$$\Psi_{\text{in}}(\mathbf{k},X^\mu) = e^{i(\mathbf{k}\cdot\mathbf{Z} - EX^0)} F(k_1,k_2,X,Y) \,. \tag{3.5}$$

Here $F(X,Y)$ satisfies

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial y^2} + \kappa^2\right] F = 0 \,, \tag{3.6}$$

where

$$\kappa^2 \equiv k_1^2 + k_2^2 \,, \qquad E^2 = \mu^2 + \mathbf{k}^2 + \kappa^2 \,, \tag{3.7}$$

and the periodicity requirement reads

$$F(R, \Phi + 2\pi\alpha, \kappa) = F(R,\Phi,\kappa) \,. \tag{3.8}$$

That is, F obeys the two-dimensional free Klein-Gordon equation (3.6) with the unusual angular periodicity of Eq. (3.8). This prevents the usual asymptotic behavior of the scattering solution for $R \to \infty$,

$$e^{i\kappa R\cos\Phi} + \frac{f(\Phi)}{\sqrt{2\pi i\kappa R}} e^{i\kappa R} \,, \tag{3.9}$$

since the ingoing wave in Eq. (3.9) *does not* satisfy the condition (3.8). The simplest way to impose Eq. (3.8) is to expand F in the functions

$$\exp\left[\frac{il\Phi}{\alpha}\right], \quad l \in \mathbb{Z} \tag{3.10}$$

which is a complete set satisfying Eq. (3.8). One finds, for the radial solution[7]

$$\mu_l(\kappa R) = (-1)^{(l-|l|)/2} J_{|l/\alpha|}(\kappa R) \,, \tag{3.11}$$

where the factor in front is chosen for simplicity of notation. Their asymptotic behavior reads

$$\mu_l(\kappa R) \underset{R\to\infty}{\sim} \left[\frac{2}{\pi\kappa R}\right]^{1/2} \times \cos\left[\kappa R - \left[\frac{|l|}{\alpha} + \frac{1}{2}\right]\frac{\pi}{2} + (|l| - l)\frac{\pi}{2}\right] . \tag{3.12a}$$

In the absence of cosmic string, then $\alpha = 1$ and one finds

$$\mu_l(\kappa R)_{\alpha=1} = \left[\frac{2}{\pi\kappa R}\right]^{1/2} \cos\left[\kappa R - (l + \tfrac{1}{2})\frac{\pi}{2}\right] . \tag{3.12b}$$

Therefore, the phase shift reads here

$$\delta_l = \frac{\pi|l|}{2}\left[1 - \frac{1}{\alpha}\right] . \tag{3.13}$$

The scattering solution can be defined as the one whose ingoing wave does not suffer of any phase shift. That is

$$F(x,y,\kappa_1,\kappa_2) = \frac{\kappa}{2\pi\alpha} \sum_{l\in Z} e^{i\delta_l} i^l \mu_l(\kappa R) e^{il\Phi/\alpha} \,, \tag{3.14}$$

where the factor in front ensures a correct normalization (see below). In the $\alpha = 1$ limit this yields

$$\lim_{\alpha\to 1} F(X,Y,\kappa_1,\kappa_2) = e^{i\kappa R\cos\Phi}$$

as one could expect. The asymptotic behavior of Eq. (3.13) is easily computed with the result

$$F(R,\Phi,\kappa) \underset{R\to\infty}{\sim} \left[\frac{i}{2\pi R}\right]^{1/2} e^{-i\kappa R}\delta_{2\pi\alpha}(\Phi+\pi\alpha) + \frac{1}{\sqrt{2\pi i R}} e^{i\kappa R}\left[\frac{f(\Phi)}{2\pi\alpha}+\delta_{2\pi\alpha}(\Phi)\right] . \tag{3.15}$$

Here $\delta_{2\pi\alpha}(x)$ stands for the periodic δ function with period $2\pi\alpha$:

$$\delta_{2\pi\alpha}(x)=\sum_{n\in Z}\frac{e^{i(2\pi n/2\pi\alpha)x}}{2\pi\alpha}$$

and f stands for the scattering amplitude

$$f(\Phi)=\sum_{l\in \mathbf{Z}}(e^{2i\delta_l}-1)e^{i(l/\alpha)\Phi} = \frac{i\sin\pi/\alpha}{\cos\Phi/\alpha+\cos\pi/\alpha} . \tag{3.16}$$

The wave functions $F(R,\Phi,\kappa)$ obey the orthogonality relation

$$\int d^2x\, F^*(R,\Phi-\Theta,\kappa')F(R,\Phi,\kappa)=\delta(\kappa-\kappa')\delta_{2\pi\alpha}(\Theta) \tag{3.17}$$

and

$$\int d^2x\, F^*(R,\Phi-\Theta+\pi\alpha,\kappa')F(R,\Phi,\kappa) =\delta(\kappa-\kappa')\left[\delta_{2\pi\alpha}(\Theta)+\frac{f(\Theta)}{2\pi\alpha}\right] . \tag{3.18}$$

It should be noticed that Θ has here the natural interpretation of the scattering angle that is the angle between the ingoing ($\mathbf{k}_1$) and outgoing ($\mathbf{k}_2$) momenta [see Eq. (3.5)].

An integral representation for $F(R,\Phi,\kappa)$ follows by resumming the series (3.14) (Ref. 10) with the help of the Schafli representation of the Bessel functions. One finds

$$F(R,\Phi,\kappa)=\int_{-\pi}^{\pi}dx\,\rho(x)e^{-i\kappa R\cos(x-\Phi)} +\int_{-\infty}^{\infty}dy\,\tilde\rho(y)e^{i\kappa R\cosh(y+i\Phi)} , \tag{3.19}$$

where

$$\rho(x)=\alpha\delta_{2\pi\alpha}(x-\pi\alpha) ,\qquad \tilde\rho(y)=\frac{1}{2\pi}\,\frac{\sin\pi/\alpha}{\cos\frac{\pi}{\alpha}+\cosh\frac{y}{\alpha}} . \tag{3.20}$$

By contour deformation, Eq. (3.19) can be alternatively written as

$$F(R,\Phi,\kappa)=\int_{\Phi-\pi}^{\Phi+\pi}dx\,\rho(x)e^{-i\kappa R\cos(x-\Phi)} +\int_{-\infty}^{\infty}dy\,\tilde\rho(y+i\Phi)e^{i\kappa R\cosh y} . \tag{3.21}$$

In summary, the scattering wave function Eq. (3.5) reads

$$\Psi_{\rm in}(\mathbf{k},X^\mu)=e^{i(\mathbf{k}\cdot\mathbf{Z}-EX^0)}\left[\int_{-\pi}^{\pi}dx\,\rho(x)e^{-i\kappa(X\cos x+Y\sin x)}+\int_{-\infty}^{\infty}dy\,\tilde\rho(y)e^{i\kappa(X\cosh y+iY\sinh y)}\right] . \tag{3.22}$$

Let us analyze the first term

$$F_0=\alpha\int_{-\pi}^{\pi}dx\,\delta_{2\pi\alpha}(x-\pi\alpha)e^{-i\kappa R\cos(x-\Phi)} =\alpha\sum_l e^{i\kappa R\cos[\Phi+(2l-1)\Delta]} , \tag{3.23}$$

where Δ is the deflection angle [Eq. (2.30)] and

$$-\tfrac{1}{2}\left[\frac{1}{\alpha}+1\right]<l<\tfrac{1}{2}\left[\frac{1}{\alpha}-1\right] .$$

The incident wave is an infinite superposition of plane waves without distortion. They propagate following wave vectors rotated from the original one by the deflection angle $\pm\Delta$ and they are periodically continued with period 2Δ. In Eq. (3.23) the original wave vector points in the x direction.

In this conical space-time, the incident ingoing wave although undistorted, suffers multiple periodic rotations as a consequence of the multiply connected topology. In addition, we have the emergence of a scattered wave

$$\hat F=\int_{-\infty}^{\infty}dy\,\tilde\rho(y)e^{i\kappa(X\cosh y+iY\sinh y)} . \tag{3.24}$$

Let us compute now the amplitude Eq. (3.2). Notice that the string coordinates (X^A) dependence of $\Psi_{\rm in}(k,X)$ is always through exponential functions in Eq. (3.22). This makes representation Eq. (3.22) especially suitable to compute the amplitude (3.2). In this matrix element we have zero modes and oscillator modes averaged on the $|0_<\rangle$ ground state.

Since we are interested in having one scalar particle in the ingoing and outgoing state, only elastic string evolution contributes. We use here for $(X(\sigma,\tau),Y(\sigma,\tau))$ the solutions $(X_<,Y_<)$ and $(X_>,Y_>)$ given by Eqs. (2.31)–(2.35) for $\tau<\tau_0$ and $\tau>\tau_0$, respectively. In order to apply the operator $X_>,Y_>$ on the ground state $|0_<\rangle$, it is convenient to express them in terms of the operators $X_<,Y_<$, by using Eq. (2.34).

The appropriate wave functions $\Psi_{\rm in}$ and $\Psi_{\rm out}$ read here

$$\Psi_{\rm in}(\mathbf{k}_1,X^\mu)=e^{i(\mathbf{k}_1\cdot\mathbf{Z}-E_1X^0)}[F_0(\mathbf{k}_1,X,Y)+\hat F(\mathbf{k}_1,X,Y)] \tag{3.25}$$

where

$$F_0(\mathbf{k}_1,X,Y)=\int_{-\pi}^{\pi}dx\,\rho(x)e^{-ik_1[X\cos(x+\Theta_1)+Y\sin(x+\Theta_1)]}\,, \tag{3.25a}$$

$$\hat{F}(\mathbf{k}_1,X,Y)=\int_{-\infty}^{\infty}dy\,\tilde{\rho}(y)e^{ik_1[X\cosh(y-i\Theta_1)+iY\sinh(y-i\Theta_1)]} \tag{3.25b}$$

and

$$\Psi^*_{\rm out}(\mathbf{k}_2,X^\mu)=\Psi_{\rm in}(-k_2,k_2,\Theta_2+\Delta-\pi,X_\mu)=e^{-i(\mathbf{k}_2\cdot\mathbf{Z}+E_2X^0)}[F_0(R\mathbf{k}_2,X,Y)+\hat{F}(Rk_2,X,Y)]\,,$$

where

$$F_0(R\mathbf{k}_2,X,Y)=\int_{-\pi}^{\pi}dx\,\rho(x)e^{ik_2[X\cos(x+\Theta_2+\Delta)+Y\sin(x+\Theta_2+\Delta)]}\,, \tag{3.26a}$$

$$\hat{F}(R\mathbf{k}_2,X,Y)=\int_{-\infty}^{\infty}dy\,\tilde{\rho}(y)e^{-ik_2[X\cosh/y-i\Theta_2-i\Delta)+iY\sinh(y-i\Theta_2-i\Delta)]}\,. \tag{3.26b}$$

Here

$$k_{1x}=k_1\cos\Theta_1,\quad k_{2x}=k_2\cos\Theta_2,\quad k_{1y}=k_1\sin\Theta_1,\quad k_{2y}=k_2\sin\Theta_2\,.$$

Inserting Eqs. (3.25) and (3.26) in the amplitude Eq. (3.2) yields four terms $A_{\rm I}$, $A_{\rm II}$, $A_{\rm III}$, $A_{\rm IV}$:

$$A=A_{\rm I}+A_{\rm II}+A_{\rm III}+A_{\rm IV}\,, \tag{3.27}$$

where

$$A_{\rm I}=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}dx\,dx'\rho(x)\rho(x')\int_0^{\pi}\int d\sigma_1 d\sigma_2\int_{-\infty}^{\infty}\int d\tau_1 d\tau_2\langle 0_<|{:}e^{-i(\mathbf{k}_2\cdot\mathbf{Z}+E_2X^0)}e^{i(\mathbf{k}_1\cdot\mathbf{Z}-E_1X^0)}$$
$$\times F_0(R\mathbf{k}_2,X,Y)F_0(\mathbf{k}_1,X,Y){:}|0_<\rangle\,, \tag{3.28}$$

$$A_{\rm II}=\int_{-\pi}^{\pi}dx\,\rho(x)\int_{-\infty}^{\infty}dy\,\tilde{\rho}(y)\int_0^{\pi}\int d\sigma_1 d\sigma_2\int_{-\infty}^{\infty}\int d\tau_1 d\tau_2\langle 0_<|{:}e^{-i(\mathbf{k}_2\cdot\mathbf{Z}+E_2X^0)}e^{i(\mathbf{k}_1\cdot\mathbf{Z}-E_1X^0)}$$
$$\times \hat{F}(R\mathbf{k}_2,X,Y)F_0(\mathbf{k}_1,X,Y){:}|0_<\rangle\,, \tag{3.29}$$

$$A_{\rm III}=\int_{-\pi}^{\pi}dx\,\rho(x)\int_{-\infty}^{\infty}dy\,\tilde{\rho}(y)\int_0^{\pi}\int d\sigma_1 d\sigma_2\int_{-\infty}^{\infty}\int d\tau_1 d\tau_2\langle 0_<|{:}e^{-i(\mathbf{k}_2\cdot\mathbf{Z}+E_2X^0)}e^{i(\mathbf{k}_1\cdot\mathbf{Z}-E_1X^0)}F_0(R\mathbf{k}_2,X,Y)$$
$$\times F(\mathbf{k}_1,X,Y){:}|0_<\rangle\,, \tag{3.30}$$

$$A_{\rm IV}=\int_{-\pi}^{\pi}dx\,\rho(x)\int_{-\infty}^{\infty}dy\,\tilde{\rho}(y)\int_0^{\pi}\int d\sigma_1 d\sigma_2\int_{-\infty}^{\infty}\int d\tau_1 d\tau_2\langle 0_<|{:}e^{-i(\mathbf{k}_2\cdot\mathbf{Z}+E_2X^0)}e^{i(\mathbf{k}_1\cdot\mathbf{Z}-E_1X^0)}\hat{F}(R\mathbf{k}_2,X,Y)$$
$$\times \hat{F}(\mathbf{k}_1,X,Y){:}|0_<\rangle\,. \tag{3.31}$$

The integration domain in each term naturally separates into four regions:

$$(\tau_1<0,\tau_2<0),\quad(\tau_1<0,\tau_2>0)\,,$$
$$(\tau_1>0,\tau_2<0)\ \text{and}\ (\tau_1>0,\tau_2>0)\,.$$

Depending on the sign of $\tau-\tau_0$, $X^A(\tau)$ is given by $X^A_<$ or $X^A_>$ (see Sec. II).

Now, we start by computing the expectation value on the oscillatory modes. The computation is involved but since all the dependence on the string coordinates is exponential, we can performed it in closed form. For this amplitude, it is convenient to work in the covariant formalism where all components of the string coordinates are quantized on equal footing and the gauge conditions

$$L_n|\Psi\rangle=0,\quad n=1,2,\ldots \tag{3.32}$$

are imposed on the physical states.

The ground-state expectation value on the oscillatory modes yields for all the terms in Eqs. (3.28), a contribution of the form

$$e^{-[\mathbf{k}_1\cdot\mathbf{k}_2-E_1E_2+k_1k_2\cos\chi]\sum_1^\infty\frac{\cos n\sigma_1\cos n\sigma_2}{n}e^{in(\tau_2-\tau_1)}}$$
$$=g(\sigma_1+\sigma_2,\tau_1-\tau_2)q(\sigma_1-\sigma_2,\tau_1-\tau_2)e^{2i\nu(\tau_1-\tau_2)}4^\nu\,,$$

where

$$g(\sigma,\tau)=(\cos\tau-\cos\sigma)^\nu\,,$$
$$\nu=\tfrac{1}{4}(\mathbf{k}_1\cdot\mathbf{k}_2-E_1E_2+k_1k_2\cos\chi)\,.$$

χ depends on the terms chosen (I, . . . , IV) and on the τ-integration region considered. We find

$$\begin{aligned}\chi_{\rm I}&=x-x'-\Theta-\Delta H(-\tau_1\tau_2)\,,\\ \chi_{\rm II}&=x-iy-\Theta-\Delta H(\tau_1\tau_2)\,,\\ \chi_{\rm III}&=x-iy+\Theta-\Delta H(\tau_1\tau_2)\,,\\ \chi_{\rm IV}&=i(y-y')+\Theta-\Delta H(\tau_1\tau_2)\,,\end{aligned} \tag{3.33}$$

where $\Theta=\Theta_2-\Theta_1$ and H is the Heaviside step function.

For all the terms (I,II,III,IV) of the amplitude, we find that the contribution of the zero modes has the form

$$e^{-(i/2)\mu^2(z_1-z_2)}\delta(E_1-E_2)\delta(\mathbf{k}_1-\mathbf{k}_2)\int\frac{d^2q}{(2\pi)^2}e^{iq\cdot v(\gamma_1,\gamma_2)},$$

where

$$v_1=-k_1\cos\gamma_1+k_2\cos\gamma_2,\qquad v_2=-k_1\sin\gamma_1+k_2\sin\gamma_2 \tag{3.34}$$

and

$$\begin{aligned}\gamma_{1\mathrm{I}}&=\gamma_{2\mathrm{II}}=x+\Theta_1+\Delta H(-\tau_1),\\ \gamma_{1\mathrm{III}}&=\gamma_{1\mathrm{IV}}=iy+\Theta_1+\pi+\Delta H(-\tau_1),\\ \gamma_{2\mathrm{I}}&=\gamma_{2\mathrm{III}}=x'+\Theta_2+\Delta H(-\tau_2),\\ \gamma_{2\mathrm{II}}&=\gamma_{2\mathrm{IV}}=iy'+\Theta_2+\pi+\Delta H(\tau_2).\end{aligned}\tag{3.35}$$

The integrals over the world-sheet coordinates σ_1,σ_2 can be computed as follows:

$$\int_0^\pi d\sigma_1\int_0^\pi d\sigma_2 g(\sigma_1+\sigma_2,\tau_1-\tau_2)g(\sigma_1-\sigma_2,\tau_1-\tau_2)=\left[\int_0^\pi d\mu\, g(u,\tau_1-\tau_2)\right]^2,$$

where we used $g(\sigma+2\pi,\tau)=g(\sigma,\tau)$. Moreover,

$$J(\beta)\equiv\int_0^\pi du\, g(u;i\beta)=\pi\sinh^\nu\beta P_\nu(\coth\beta), \tag{3.36}$$

where $P_\nu(z)$ is a Legendre function.

All the integrals over (τ_1,τ_2) are conveniently expressed as an integral on a single imaginary time variable:

$$L(M)=2^{2\nu}\int_0^\infty d\lambda\,\lambda e^{-M\lambda}[J(\lambda)]^2, \tag{3.37}$$

where

$$M(x)=\frac{\mu^2}{2}-2\nu=\tfrac{1}{2}(\mu^2+\mathbf{k}_1\cdot\mathbf{k}_2+E_1E_2-k_1k_2\cos\chi) \tag{3.38}$$

and μ^2 is the scalar particle mass:

$$\mu^2=-k_1^{\mu^2}=-k_2^{\mu^2}=-4.$$

Finally, we arrive at the result

$$A_{\mathrm{I}}=L(\mu^2)\alpha^2[2\delta_{2\pi\alpha}(\Theta)-\delta_{2\pi\alpha}(\Theta+\Delta)-\delta_{2\pi\alpha}(\Theta-\Delta)]\delta(k_1-k_2)\delta(E_1-E_2)\delta(\mathbf{k}_1-\mathbf{k}_2), \tag{3.39}$$

$$\begin{aligned}A_{\mathrm{II}}=\int_{-\infty}^{\infty}\frac{d^2q}{(2\pi)^2}\int_{-\pi}^{\pi}dx\int_{-\infty}^{\infty}dy\,\rho(x)\tilde\rho(y)\{&-L[M(x-iy+\Theta)](e^{iq\cdot v(x+\Theta_1,iy+\Theta_2+\Delta+\pi)}+e^{iq\cdot v(x+\Theta_1,iy+\Theta_2+\pi)})\\ &+L[M(x-iy+\Theta+\Delta)](e^{iq\cdot v(x+\Theta_1,iy+\Theta_2+\pi)}+e^{iq\cdot v(x+\Theta_1+\Delta,iy+\Theta_2+\pi)})\}\\ &\times\delta(\mathbf{k}_1-\mathbf{k}_2)\delta(E_1-E_2),\end{aligned}\tag{3.40}$$

$$\begin{aligned}A_{\mathrm{III}}=\int\frac{d^2q}{(2\pi)^2}\int_{-\pi}^{\pi}dx\int_{-\infty}^{\infty}dy\,\rho(x)\tilde\rho(y)\{&L[M(x+\Theta+\Delta-iy)](e^{iq\cdot v(\Theta_1+iy+\pi,x+\Theta_2)}+e^{iq\cdot v(\Theta_1+iy+\Delta+\pi,x+\Theta_2+\Delta)})\\ &-L[M(x-\Theta-iy)](e^{iq\cdot v(\Theta_1+iy+\pi,x+\Theta_2+\Delta)}+e^{iq\cdot v(\Theta_1+iy+\Delta+\pi,x+\Theta_2)})\},\end{aligned}\tag{3.41}$$

$$\begin{aligned}A_{\mathrm{IV}}=\int\frac{d^2q}{(2\pi)^2}\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dy'\tilde\rho(y)\tilde\rho(y')\{&L[M(\Theta-\Delta+iy-iy')](e^{iq\cdot v(iy+\Theta_1,iy'+\Theta_2+\Delta)}+e^{iq\cdot v(iy+\Theta_1+\Delta,iy'+\Theta_2)})\\ &+L[M(\Theta+iy-iy')](e^{iq\cdot v(iy+\Theta_1,iy'+\Theta_2)}+e^{iq\cdot v(iy+\Theta_1+\Delta,iy'+\Theta_2+\Delta)})\}.\end{aligned}\tag{3.42}$$

In conclusion, we have reduced the two-body scalar particle scattering amplitudes in cosmic-string space-time to quadratures. [Eqs. (3.39)–(3.42)]. It must be noted that in Minkowski space-time the lowest nontrivial amplitude is the four-legs amplitude whereas here the two-body S-matrix is nontrivial. It describes the elastic scattering of scalar particles by the cosmic string. In the $\alpha=1$ limit, the amplitude vanishes as it must be. The detailed analysis of its properties is beyond the scope of this paper. It can be done by analyzing the integrals in Eqs. (3.39)–(3.42). Let us briefly comment on Eqs. (3.39)–(3.42). The term A_{I} [Eq. (3.39)] corresponds to the undistorted waves. These terms describe outgoing particles moving in the original ingoing direction or rotated by the deflection $\pm\Delta$ (modulo $2\pi\alpha$). The terms A_{II}, A_{III}, A_{IV} describe true elastic scattered particles. One sees that the effect of nontrivial space-time on the string scattering amplitudes manifests in two coherently combined ways: (i) the vertex operator [Eq. (3.26)] is different than the one (e^{ikx}) in Minkowski space-time; (ii) the outgoing harmonic-oscillator operators $\alpha_n{}^{A}{}_{>}$ are related to the ingoing operators $\alpha_n{}^{A}{}_{<}$ by a linear transformation and therefore their expectation values in $|0_<\rangle$ are different than their Minkowski values. If one just ignores the harmonic oscillators $n\neq0$ in the string coordinates expansion [Eq. (2.24)], the two-body string amplitude Eq. (3.2) becomes the point-particle field-theory Klein-Gordon amplitude Eq. (3.16).

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