Neutrino oscillations in inhomogeneous matter

Raymond F. Sawyer
Department of Physics, University of California, Santa Barbara, California 93106
(Received 30 July 1990)

An equation is derived for the evolution of the density matrix for neutrinos propagating, and mixing, in inhomogeneous matter. A measure of the decrease in the efficiency of the Mikheyev-Smirnov-Wolfenstein (MSW) transformation of ν_e to ν_x is defined, and is proved to increase steadily as the neutrino traverses a lumpy medium. It is estimated that, at the least, 1% density variations in the solar core on a scale of 1000 km would be required in order to disrupt the MSW predictions for the solar-neutrino problem. In the case of the supernova core, 0.01% variations on the scale of the c.m. system would be sufficient to alter drastically the results of scenarios which have been proposed.

I. INTRODUCTION

The assumption of vacuum neutrino oscillations combined with the standard model for the interaction of neutrinos with the components of matter has led to the prediction of a number of interesting phenomena. The most important is the possibility of drastic changes in the solar-neutrino flux and spectrum, for some ranges of the vacuum parameters, resulting from the Mikheyev-Smirnov-Wolfenstein (MSW) effect.[1-3] Another possible MSW effect, demanding another range of vacuum parameters, may affect the dynamics of supernovae and the nature of the neutrino pulse from the supernova.[4,5] Finally, MSW effects could have some effects in the early Universe, just prior to nucleosynthesis,[6] especially in variants with large lepton number.[7]

All of the works of the MSW effect deal with inhomogeneous matter in the sense that it is the change in the eigenstates of the medium as the neutrino moves, say, from high density to low density, that transforms the flavor of the neutrino beam. There are also a number of papers[8-10] that address the effects of smaller scale irregularities along the way. The authors of Ref. 8, for example, calculate the effects of a sinusoidal modulation of the electron density function; they conclude that, for modulation wavelengths of the order of the oscillation length (in the matter) and a 0.01 amplitude of modulation, there could be appreciable effects on the conversion probability for a neutrino of a particular energy, but that the effect itself could have either sign, depending on the neutrino energy, and would probably be obliterated in an average over energies. Along the same lines, the authors of Ref. 9 consider specifically the case of resonance conversion when the modulation wavelength is tuned exactly to the oscillation length.

The basis of the treatment of neutrino propagation presented in all of the papers mentioned above is the following: the neutrino follows a classical trajectory $\mathbf{x}(t)$ through the matter; it has a flavor-space wave function, with as many components as there are neutrino flavors, each component a function of time only, the wave function obeying a Schrödinger equation with a matrix potential which depends on time through evaluation of a position-dependent potential at the point $\mathbf{x}(t)$.

In the present paper we, too, consider the effects of a lumpy environment on the MSW effect. However, we follow a different approach, one in which, up to a certain stage of the calculation, the momentum (or space) coordinate of the neutrino is retained. We obtain results that are superficially very different from those of Refs. 8 and 9, both from the standpoint of theoretical principle and from the standpoint of the results. On the theoretical side the differences in the answers are traceable to the existence of nonvanishing interference terms, effectively between scattered and unscattered waves, in the treatment with the flavor coordinates only, while the corresponding terms in the treatment with an extended wave function do not interfere. On the practical side, our treatment leads to a steady erosion of the $\nu_e \rightarrow \nu_\mu$ conversion efficiency, as more irregularities are introduced along the path, with a rate which is not a sensitive function of the neutrino energy. This is in contrast with the results of Ref. 8, in which the effect was found to be of either sign, and oscillating as a function of neutrino energy.

We shall argue that the classical ray treatment could lead to the same results as our method if an appropriate average over trajectories were taken. Nevertheless, our approach leads more directly to what we believe is the essential result: an equation for a quantity which we call the depolarization, which provides a measure of the importance of the degradation through scattering of the MSW conversion process, and which increases steadily as the neutrino moves through a lumpy medium.

We use our approach to estimate the possible effects in the Sun and in the supernova, with the conclusions that (a) the large density fluctuations, of the order of 1% over distances in the 1000-km range, would be required for the scattering effect to degrade appreciably the proposed MSW transformation in the Sun, (b) much smaller relative fluctuations, e.g., at the 0.01% level over c.m. scales, would severely alter suggested MSW effects in supernovae.

Since we have not found in the literature the equations for discussing the evolution of the complete density matrix, in combined momentum and flavor space, for a neutrino traversing a nonuniform medium, we begin with a development, *ab initio,* of such an equation.

II. AN EQUATION FOR THE PROPAGATION OF A NEUTRINO IN INHOMOGENEOUS MATTER THAT IS UNIFORM ON THE AVERAGE

We consider the scattering of a neutrino from a medium of N particles in a volume (Vol), the particles individually positioned at the points $\mathbf{x}_\alpha$. In an application to the solar-neutrino problem, the scatterers would be electrons; in the application to core collapse, scattering from neutrinos may be significant as well. For illustration, we consider only the scattering from one species, and take the effective interaction potential in which the neutrino moves to be that appropriate to a cloud of electrons, as described in the standard model

$$\begin{aligned} H_0 &= \sum_{i,j} \int d^3x\, \psi^\dagger_{\nu_i}(\boldsymbol{\alpha}\cdot\mathbf{p}\delta_{ij} + \beta M_{ij})\psi_{\nu_j} \,, \\ H_I &= \sqrt{2}G_F \int d^3x\, \psi^\dagger_{\nu_e}\psi_{\nu_e} n_e(\mathbf{x}) \,, \end{aligned} \tag{1}$$

where

$$n_e(\mathbf{x}) = \sum_\alpha \delta(\mathbf{x}-\mathbf{x}_\alpha) \tag{2}$$

is the electron density. We use units $\hbar = c = 1$.

Here H_0 describes neutrinos in a flavor representation, with a nondiagonal mass matrix. H_I gives the interaction of the electron neutrinos with the electrons in the medium, through the charged-current couplings, retaining only the portion which will contribute to coherent, small momentum transfer, scattering off of the inhomogeneities. The neutral-current couplings, if they are universal for all neutrino flavors, will not contribute to the transformations among neutrino colors. The electrons' role is to transfer momentum to the neutrino, as well as to change the neutrino type, but the momentum transfers which will enter our treatment will be so small that the electrons can be described as fixed sources at positions $\mathbf{x}_\alpha$, as above.

In this case the evolution of the system can be described by a multicomponent single neutrino wave function ψ_i. Henceforth we take the representation in the internal space to be that in which the (mass)^2 matrix is diagonal. The interaction potential arising from H_I will therefore not be diagonal:

$$V_{ik}(\mathbf{x}) = w_{ik}\sqrt{2}G_F n_e(\mathbf{x}) \,, \tag{3}$$

where

$$w_{ik} = U_{ie}U_{ke} \,. \tag{4}$$

Here the real orthogonal matrix U_{ie} is the transformation from the flavor to the diagonal-mass representation, $(U^{\mathrm{tr}}MMU)_{ij} = \delta_{ij}m_i^2$. The equation for the neutrino wave function in this representation is

$$\sum_{m,k} \left[i\frac{\partial}{\partial t}\delta_{im} - V_{im} \right] \left[i\frac{\partial}{\partial t}\delta_{mk} - V_{mk} \right] \psi_k = (-\nabla^2 + m_i^2)\psi_i - i\sum_k \boldsymbol{\alpha}\cdot\nabla V_{ik}\psi_k \,. \tag{5}$$

Now we consider a neutrino propagating with momentum $\mathbf{p}_0$ and approximate energy $E_0 = |\mathbf{p}_0|$. The magnitude of the momentum will be changed only by small amounts during the time evolution. In a more or less standard way, we define a new neutrino wave function ϕ by removing the dominant time dependence:

$$\psi_i = e^{-iE_0 t}\phi_i \,. \tag{6}$$

The new neutrino wave function obeys the equation

$$i\frac{\partial\phi_i(\mathbf{x},t)}{\partial t} = \sum_k \left[\frac{-\nabla^2 + m_i^2 - E_0^2}{2E_0}\delta_{ik} + V_{ik}(\mathbf{x}) - \sum_m \frac{V_{im}(\mathbf{x})V_{mk}(\mathbf{x})}{2E_0} + \frac{\partial^2/\partial t^2}{2E_0}\delta_{ik} - \frac{iV_{ik}(\partial/\partial t)}{E_0} - \frac{i\boldsymbol{\alpha}\cdot\nabla V_{ik}}{2E_0} \right] \phi_k(\mathbf{x},t) \,. \tag{7}$$

We shall neglect the last four terms on the right-hand side (RHS) of (7). It would be straightforward to see *a posteriori* that the first three neglected terms produce results of higher order in the small quantities G_w and m_i^2/E_0 than do the retained terms. The final term, containing the Dirac matrix α, will be negligible as long as the potential is slowly varying, on the scale of the neutrino wavelength, a condition which will be satisfied by many orders of magnitude in our application. Thus, the Dirac spinor space disconnects completely from our problem. The potential function in (5) can be separated into an average potential, over the entire volume, and a part which represents the inhomogeneities:

$$\begin{aligned} V_{ik}(\mathbf{x}) &= w_{ik}\sqrt{2}G_F[n_e^{(0)} + \delta n_e(\mathbf{x})] \\ &= V_{ik}^{(0)} + \delta V_{ik}(\mathbf{x}) \,. \end{aligned} \tag{8}$$

The average term in the potential, $V_{ik}^{(0)}$, times a factor of $2E_0$ can be considered part of a (mass)^2 matrix, which is then to be rediagonalized. The states $|p,i\rangle$ are now taken to represent the eigenstates in a uniform medium with density equal to the average density. In this representation the Schrödinger equation for the neutrino is

$$i\frac{\partial\phi_i(\mathbf{x},t)}{\partial t} = \sum_k \left[\frac{-\nabla^2 + \hat{m}_i^2 - E_0^2}{2E_0}\delta_{ik} + \delta V_{ik}(\mathbf{x}) \right] \phi_k(\mathbf{x},t) \,, \tag{9}$$

where $\hat{m}_i^2$ are the (mass)^2 eigenvalues after rediagonalization. We define a function $\rho_{ij}(\mathbf{p},t)$ describing the density of neutrinos in momentum-flavor space,

$$\rho_{ij}(\mathbf{p},t)=\phi_i^*(\mathbf{p},t)\phi_j(\mathbf{p},t)\ , \tag{10}$$

where we have introduced Fourier components through

$$\phi(\mathbf{p},t)=(\mathrm{Vol})^{-1/2}\int_{(\mathrm{Vol})} e^{i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{x},t)d^3p\ . \tag{11}$$

Defining

$$\langle \mathbf{p}|\delta V_{ij}|\mathbf{q}\rangle = w_{ij}G_F\int_{(\mathrm{Vol})}\delta n_e(\mathbf{x})e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}d^3x$$
$$\equiv w_{ij}G_F\delta n_e(\mathbf{p}-\mathbf{q})\ , \tag{12}$$

we now obtain, from (9),

$$i\frac{\partial}{\partial t}\rho_{ij}(\mathbf{p},t)=(\omega_p^j-\omega_p^i)\rho_{ij}(\mathbf{p},t)+\phi_i^*(\mathbf{p},t)\sum_{\mathbf{q},k}\langle\mathbf{p}|\delta V_{jk}|\mathbf{q}\rangle\phi_k(\mathbf{q},t)(\mathrm{Vol})^{-1}-\sum_{\mathbf{q},k}\phi_k^*(\mathbf{q},t)\langle\mathbf{q}|\delta V_{ki}|\mathbf{p}\rangle\phi_j(\mathbf{p},t)(\mathrm{Vol})^{-1}\ , \tag{13}$$

where

$$\omega_p^i=|\mathbf{p}|+\frac{\hat{m}_i^2}{2E_0}-E_0+\frac{(|\mathbf{p}|-E_0)^2}{2E_0}\ , \tag{14}$$

the last term being negligible for our case in which $|\mathbf{p}|\approx E_0$.

In order to calculate, from (13), the effects which are second order in δV, we use the wave function to first order in δV:

$$\phi_i(\mathbf{q},t)=\varepsilon_i\delta_{r,q}e^{-i\omega_r^i t}-\sum_m\frac{e^{-i\omega_q^i t}-e^{-i\omega_r^m t}}{(\mathrm{Vol})(\omega_r^m-\omega_q^i)}\langle\mathbf{q}|\delta V_{im}|\mathbf{r}\rangle\varepsilon_m\ . \tag{15}$$

Here we have taken the neutrino state at $t=0$ to be characterized by momentum $\mathbf{r}$, and polarization (in the flavor space) ε_i. We define a density matrix $\hat{\rho}$ in flavor space only, by summing over momenta $\mathbf{p}$ in (10):

$$\hat{\rho}_{ij}(t)=\sum_{\mathbf{p}}\rho_{ij}(\mathbf{p},t)\ . \tag{16}$$

Using (11) for the wave functions in (10), summing over modes, converting sums to integrals, and introducing (12), we obtain

$$i\frac{d}{dt}\hat{\rho}_{ij}=-\Delta_{ij}\hat{\rho}_{ij}+\frac{2G_W^2}{(\mathrm{Vol})}\int\frac{d^3\mathbf{q}}{(2\pi)^3}|\delta n_e(\mathbf{r}-\mathbf{q})|^2$$
$$\times\sum_{m,k}\left[-w_{kj}w_{mk}\hat{\rho}_{im}^{(0)}\frac{e^{i(\omega_r^m-\omega_q^k)t}-1}{\omega_r^m-\omega_q^k}+w_{ik}w_{km}\hat{\rho}_{mj}^{(0)}\frac{e^{-i(\omega_r^m-\omega_q^k)t}-1}{\omega_r^m-\omega_q^k}\right.$$
$$\left.-w_{im}w_{kj}\hat{\rho}_{mk}^{(0)}\frac{e^{-i(\omega_r^m-\omega_q^i)t}-1}{\omega_r^m-\omega_q^i}+w_{mj}w_{ik}\hat{\rho}_{km}^{(0)}\frac{e^{i(\omega_r^m-\omega_q^j)t}-1}{\omega_r^m-\omega_q^j}\right]\ , \tag{17}$$

where

$$\Delta_{ij}=\omega_p^i-\omega_p^j\approx\frac{\hat{m}_i^2-\hat{m}_j^2}{2E_0}\ ,$$
$$\omega_r^i-\omega_q^j\approx\Delta_{ij}+|\mathbf{r}|-|\mathbf{q}|\ . \tag{18}$$

In (17) we have introduced the notation

$$\hat{\rho}_{mk}^{(0)}=\varepsilon_m^*\varepsilon_k\exp(i\Delta_{mk}t) \tag{19}$$

for the time-dependent flavor-density matrix in the absence of the perturbing potential, corresponding to the initial conditions embodied in (15).

Equation (17) is the basic formal result of our development. The physical picture with respect to which we interpret (17) is the following: Prior to $t=0$, a plane wave of momentum $\mathbf{r}$ and polarization ε_i fills our box. At $T=0$ the interaction δV is turned on. For a localized δV, the rate at which the system's average polarization changes is proportional to the inverse volume, because the amount of probability in the region $\delta V\neq 0$ is proportional to $(\mathrm{Vol})^{-1}$.

We are interested in the steady effects of the interaction, not the turn-on transients, and therefore consider the limit of large time in (17). Under the integrals we take $t_{\to\infty}$ limit of the imaginary part of the time-dependent factors, using

$$\lim_{t\to\infty}\frac{e^{\pm i\eta t}-1}{\eta}=\pm\pi i\delta(\eta)+\lim_{t\to\infty}\frac{\cos\eta t-1}{\eta}\ . \tag{20}$$

The physical upper limit on the size of t is set by how far the neutrino can travel without encountering a significantly altered *average* density. On the other hand, the limit for the imaginary part of (20) cannot be used for too short times either; the requirement is that multiplying factors in the integrands in (17) not vary appreciably over the range of a q defined by

$$|\omega_q^\beta - \omega_r^\alpha| < t^{-1} \ . \tag{21}$$

For this to be true t must be large compared to the time it takes the neutrino to traverse the inhomogeneity in question. Thus, reasonably enough, the two limitations on time, for our considerations to apply, demand that the inhomogeneities of small scale compared to the distance scale for continuous change of the medium.

Substituting (20) in (17), we obtain

$$\begin{aligned}\frac{d}{dt}\hat\rho_{ij} = i\Delta_{ij}\hat\rho_{ij} + \sum_{\alpha\beta} B_{ij\alpha\beta}\hat\rho^{(0)}_{\alpha\beta} + \frac{G_F^2}{(\mathrm{Vol})}\int \frac{d^3\mathbf{q}}{2\pi^2}|\delta n_e(\mathbf{r}-\mathbf{q})|^2 \\ \times \sum_{m,k}[\delta(|\mathbf{r}|-|\mathbf{q}|-\Delta_{km})(-w_{jk}w_{km}\hat\rho^{(0)}_{im} - w_{ij}w_{km}\hat\rho^{(0)}_{mj}) \\ +\delta(|\mathbf{r}|-|\mathbf{q}|-\Delta_{im})w_{im}w_{kj}\hat\rho^{(0)}_{mk} + \delta(|\mathbf{r}|-|\mathbf{q}|-\Delta_{jm})w_{mj}w_{ik}\hat\rho^{(0)}_{km}] \ ,\end{aligned} \tag{22}$$

where the B term stands for the sum of all the contributions from the real term of the RHS of (20), and the imaginary terms are written out explicitly. We shall see below that the B term does not enter our principal results; however, we pause for a moment to consider the significance of the two kinds of terms. The δ-function terms are the ones describing real transitions, the arguments of the delta functions expressing conservation of energy in the process of neutrino scattering off of the lumps in the medium. The B term, the contribution to which would vanish in the equation for the density function in a one-channel problem, represents the effects of local changes in the real part of a mass matrix, induced by the additional potential δV.

We assume that the dominant wave numbers in the Fourier transform of the density fluctuations are of the order of the inverse size of the fluctuation, so that there is an effective upper cutoff in the q integration in (22) at $(L_{\mathrm{inhom}})^{-1}$. [The important exception to this condition is the case in which the lump not only has internal structure, but also has periodicity that is tuned to the oscillation length. This is the case that would lead to resonant enhancement, as in Ref. 9. We shall not consider this possibility, for the following reasons: (a) it requires fine-tuning of the neutrino energy; (b) in any reasonable scenario which produces the periodicity, such as radial pulsation of the solar core, the periodicity would not be perfectly sinusoidal, nor would a neutrino be able to take advantage if it were, unless the neutrino were on a radial trajectory.]

The δ-function terms, as explicitly exhibited in (22), then have a significant characteristic: The scale of the density fluctuation L_{inhom} sets an upper limit on the dominant momentum differences which enter the integral: $|\mathbf{r}|-|\mathbf{q}| < (L_{\mathrm{inhom}})^{-1}$. Therefore, terms in which a neutrino oscillation length (Δ_{km}) is *smaller* than L_{inhom} will be suppressed in the integral. We shall see in Sec. IV that the suppression of these terms gives a flavor index structure in (22) that leads to no modification of the MSW efficiencies. Thus, there is a kind of cutoff, at the oscillation length, in the size of structures which are most interesting for our purposes.

The results simplify considerably in the opposite limit, that in which the inhomogeneities are small compared to oscillation lengths. In this case we can discard the Δ terms inside the δ functions in (22), and make similar simplifications in the B term, obtaining

$$\begin{aligned}\frac{d\hat\rho_{ij}}{dt} = i\Delta_{ij}\hat\rho_{ij} + \sum_{\alpha\beta} B_{ij\alpha\beta}\rho^{(0)}_{\alpha\beta} \\ + \frac{G_F^2}{2\pi^2(\mathrm{Vol})} \\ \times \sum_{\alpha\beta} R_{ij\alpha\beta}\hat\rho^{(0)}_{\alpha\beta}\int d^3\mathbf{k}|\delta n_e(\mathbf{k})|^2\delta(|\mathbf{r}-\mathbf{k}|-|\mathbf{r}|) \ ,\end{aligned} \tag{23}$$

where

$$\begin{aligned}R_{ij\alpha\beta} &= 2w_{i\alpha}w_{\beta j} - \delta_{i\alpha}\sum_k w_{\beta k}w_{kj} - \delta_{\beta j}\sum_k w_{ik}w_{k\alpha} \\ &= 2U_{ie}U_{\alpha e}U_{\beta e}U_{je} - \delta_{i\alpha}U_{\beta e}U_{je} - \delta_{\beta j}U_{ie}U_{\alpha e}\end{aligned} \tag{24}$$

and

$$B_{ij\alpha\beta} = \hat B(t)\sum_k(\delta_{i\alpha}w_{kj}w_{\beta k} - \delta_{j\beta}w_{ik}w_{k\alpha}) \ , \tag{25}$$

the flavor dependence being all that we shall need to know about the B term. To get the last line in (24) we substituted (4) and used the orthogonality of the U's.

It is easy to see directly from (23) that

$$\left[\frac{d}{dt}\right]\sum_i \hat\rho_{ii} = 0 \ , \tag{26}$$

that is, the total probability is conserved. We define the depolarization function $D(t)$ by

$$D(t) = 1 - \sum_{ij}\hat\rho_{ij}\hat\rho_{ji} \ . \tag{27}$$

For a normalized state which factors into a flavor vector and a coordinate function we have $D=0$. If the initial flavor density matrix is of this form, i.e., $\epsilon_i^*\epsilon_j$, then the initial time derivative of D must be positive, since

$$D(t)=1-\sum_{ij}\int d^3\mathbf{p}\int d^3\mathbf{q}\varphi_i^*(\mathbf{p})\varphi_j(\mathbf{p})\varphi_j^*(\mathbf{q})\varphi_i(\mathbf{q})$$
$$=\tfrac{1}{2}\sum_{ij}\int d^3\mathbf{p}\int d^3\mathbf{q}|\varphi_i(\mathbf{p})\varphi_j(\mathbf{q})-\varphi_j(\mathbf{p})\varphi_i(\mathbf{q})|^2\geq 0\ . \tag{28}$$

The rate of depolarization in the case under consideration can be calculated to second order in δV (the first nonvanishing order):

$$\frac{dD}{dt}=-\sum_{ij}\left[\hat{\rho}_{ji}^{(0)}\frac{d\hat{\rho}_{ij}}{dt}+\frac{d\hat{\rho}_{ji}}{dt}\hat{\rho}_{ij}^{(0)}\right], \tag{29}$$

where $(d\hat{\rho}_{ij})/(dt)$ is given by (23). Inserting (23) into (29), and using (25) it is now easy to see that the B term on the RHS of (29) makes no contribution to dD/dt. Henceforth we shall drop this term entirely.

III. EXAMPLES

A. Incoherent scattering from electrons

First we recapture the results for the effects of incoherent scattering of the neutrino off of the separate electrons in the medium. From (1) we have

$$n_e(\mathbf{k})=\sum_\alpha e^{i\mathbf{k}\cdot\mathbf{x}_\alpha}\ . \tag{30}$$

If we average over positions for each electron, so as to eliminate the interference terms, we obtain $n_e(k)n_e(-k)\to N$, where N is the number of electrons, giving

$$\frac{d\hat{\rho}_{ij}}{dt}=\frac{G_F^2E_0^2}{\pi}n_e^{(0)}\sum_{\alpha\beta}R_{ij\alpha\beta}\hat{\rho}_{\alpha\beta}^{(0)}\ , \tag{31}$$

where $N/(\mathrm{Vol})$ has been replaced by the average electron density, $n_e^{(0)}$. The kinematical factors in (31) are correct only for the case of neutrino momentum small compared to the electron rest mass, and (31) omits the terms which depend on the electron spin as well, since in (1) we kept only the coherent (Coulomb-like) term in the interaction, and we neglected electron recoil throughout. The flavor dependence in (31) would be the same for the complete calculation, and is equivalent to the flavor dependence implicit in the results of Ref. 11, for the case of a two-flavor system. It is easy to see in any case that the depolarization from the incoherent single-particle scattering effect will be negligible, even at neutron star densities.

B. Scattering from slabs

Next we consider the effects of scattering off of a single macroscopic density variation, of smaller scale than the mixing lengths, $(\Delta_{12})^{-1}$. We take a density perturbation which has a constant magnitude A within a slab of thickness L oriented perpendicular to the z axis

$$\delta n_e(\mathbf{x})=A\,\theta(L^2-4z^2)\ , \tag{32}$$

$$|\delta n_e(\mathbf{k})|^2=16\pi^2A^2\frac{\sin^2(k_zL/2)}{k_z^2}\delta(k_x)\delta(k_y)(\mathrm{area})\ , \tag{33}$$

where (area) is the cross section of the box. We take $\hat{\mathbf{n}}$ as the direction of initial propagation, $\mathbf{r}\approx\hat{\mathbf{n}}E_0$. For very small momentum transfer $\mathbf{k}$, we can approximate the argument of the energy δ function in (23) as $k\cos(\alpha)$, where α is the angle between the incident momentum $\mathbf{r}$ and $\mathbf{k}$. Note that the transverse δ functions in (33) set $\alpha=\theta$, where θ is the angle between the incident neutrino momentum and the normal to the slab. We obtain the result

$$\frac{d\hat{\rho}_{ij}}{dt}=i\Delta_{ij}\hat{\rho}_{ij}+\frac{G_F^2A^2L^2(\mathrm{area})}{(\cos\theta)(\mathrm{Vol})}\sum_{\alpha\beta}R_{ij\alpha\beta}\hat{\rho}_{\alpha\beta}^{(0)}\ . \tag{34}$$

We calculate the total change in the depolarization, ΔD, that takes place when a particle passes through the slab. This will be given by (26), using the rate (34), and dividing by the probability per unit time that the neutrino enters the slab, given by

$$[(\cos\theta)(\text{area of box})/(\text{volume of box})]\ .$$

We obtain

$$\Delta D=2G_F^2\langle S\rangle A^2L_{\mathrm{eff}}^2\ , \tag{35}$$

where

$$\langle S\rangle=-\sum_{ij\alpha\beta}\hat{\rho}_{ji}^{(0)}R_{ij\alpha\beta}\hat{\rho}_{\alpha\beta}^{(0)}\ , \tag{36}$$

and $L_{\mathrm{eff}}=L/(\cos\theta)$, the distance through the slab traversed by the neutrino.

C. Scattering from spheres

As a third example, we consider a sphere of radius R, inside of which there is a density change of magnitude A. In this example there will be finite momentum transfer (of order R^{-1}) to the medium in the $t\to\infty$ limit, as there was not in the case of a parallel-faced slab. We obtain

$$\delta n_e(\mathbf{k})=4\pi Ak^{-3}[\sin(kR)-(kR)\cos(kR)]\ . \tag{37}$$

Taking the momentum of the incident neutrino to be in the $\hat{\mathbf{z}}$ direction, we obtain

$$\frac{dP}{dt}=2^{-1}G_F^2\langle S\rangle\pi^{-2}(\mathrm{Vol})^{-1}\int d^3(k)\delta(k_z)|\delta n_e(k)|^2$$
$$=4G_F^2\langle S\rangle W^2\pi R^4(\mathrm{Vol})^{-1}\ . \tag{38}$$

In this case we obtain the (volume-independent) physical quantity of interest by taking a randomly placed group of $N=n_S(\mathrm{Vol})$ such spheres in our box, and letting the system evolve for a time T, obtaining

$$\Delta D=4n_SG_F^2\langle S\rangle A^2\pi R^2T\ . \tag{39}$$

We can compare the results (35) and (39), for the two kinds of geometry, in the following way: for the sphere case, take the density of scatterers to be the maximum allowed, i.e., $n_s\approx R^{-3}/8$; for the slabs, let the system traverse N slabs in time T, where $N \approx T/L$, again the maximum number of scatterers. We obtain

$$\text{spheres: } (\Delta D)_{\max} \approx 2^{-1}\pi G_F^2 A^2 \langle S \rangle RT \;, \qquad \text{slabs: } (\Delta D)_{\max} \approx 2 G_F^2 A^2 \langle S \rangle LT \;, \tag{40}$$

essentially the same results in the two cases.

IV. FLAVOR DEPENDENCE

In our discussion we shall need several properties of the flavor-dependent quantities $\hat{\rho}_{ij}$, $R_{ij\alpha\beta}$, S, and D.

(a) We write $\langle S \rangle$, defined in (36) in terms of the flavor representation, in which the matrix w_{ij} is diagonal, with eigenvalues w_i ($w_i \neq 0$ for $i=e$), obtaining

$$\langle S \rangle = \sum_{ij} \hat{\rho}_{ij}\hat{\rho}_{ji}(w_i - w_j)^2 > 0 \;. \tag{41}$$

Thus the system heads steadily to a state of more depolarization, for any initial density matrix.

(b) Specializing to the two-flavor case, let us denote the diagonal elements of ρ_{ij} defined in (23), as γ and $1-\gamma$. Then we obtain an upper bound on the depolarization function, $D(t)$:

$$D(t) < 2\gamma - 2\gamma^2 < \tfrac{1}{2} \;. \tag{42}$$

The case $D = \frac{1}{2}$ corresponds to an incoherent 50-50 mixture of ν_e and ν_μ.

(c) Now we restrict to a pure initial state in flavor space, $P(t=0)=1$, or $\hat{\rho}_{ij} = \varepsilon_i^* \varepsilon_j$. From (36) and (28) we then have

$$\langle S \rangle = 2\left[\sum_k \varepsilon_k U_{ke}\right]^2 - 2\left[\sum_k \varepsilon_k U_{ke}\right]^4 \;. \tag{43}$$

If the initial state is one of definite flavor, $\varepsilon_i = U_{if}$, where f is the flavor in question, then there is no depolarization, no matter what are the eigenstates of the medium, $dD/dt = 0$. This follows from noting that, because of the orthogonality of the U matrices, the quantity in large parentheses in (43) is unity or zero, respectively, for the case of $f=e$ or $f=$(another flavor).

(c) In our application we shall be most interested in the results for the case in which the initial state is an eigenstate of the locally averaged system; although our neutrino is produced in a coherent mixture of the two eigenstates, the two components become out of phase by an amount depending on the distance traveled, and averaging over the position of production removes interference terms. (One caveat to be applied to the above remark, however, is that the slowly changing background density will coherently mix the eigenstates; these are "nonadiabatic" corrections calculated by several authors.[12-14] To the extent that this happens before the neutrino encounters the inhomogeneity, we would have to deal with noneigenstates.) For the case of two flavors, we can express $\langle S \rangle$, calculated, e.g., for eigenstate No. 1, in terms of mixing angle θ_M, in the matter,

$$|1\rangle = \cos\theta_M |\nu_e\rangle - \sin\theta_M |\nu_\mu\rangle \;. \tag{44}$$

We obtain

$$\langle S \rangle = \tfrac{1}{2}\sin^2(2\theta_M) \;. \tag{45}$$

We note that the rate of depolarization will be fastest when $\sin^2(2\theta_M)=1$; that is, at the resonance value of the density.

(e) Finally, we turn to the limit in which the size of the inhomogeneity is large compared to the oscillation distance. In accord with the remarks preceding Eq. (23) we can factor out the flavor dependence of those δ-function terms on the RHS of (22) which do not contain a Δ term within the δ function, obtaining an evolution equation in which the tensor R, of (24), is replaced by R^{eff}, where

$$R^{\text{eff}}_{ij\alpha\beta} = -w_{\beta j}w_{\beta\beta}\delta_{i\alpha} - w_{i\alpha}w_{\alpha\alpha}\delta_{\beta j} + w_{ii}w_{\beta j}\delta_{i\alpha} + w_{jj}w_{i\alpha}\delta_{j\beta} \quad (\text{for } L_{\text{inhom}} \gg \Delta^{-1}) \;. \tag{46}$$

From (46) it is easy to see that $\langle S \rangle$ vanishes in the case that the initial state is an eigenstate of the medium.

V. VARYING BACKGROUND DENSITY

We have presented a perturbation theoretic calculation of the effects of scattering from the density fluctuations in the medium on the "flavor-density" matrix of a neutrino. The calculation demanded a background density which is constant across the box, and a fluctuation size small compared to the size of the box. We neglected recoil of the medium, since we were concerned with the coherent scattering from large-scale structures only. For the case over the scale of the maximum sized fluctuation to be considered, this assumption is as weak as, or weaker than, the assumption that the density does not change rapidly over the oscillation length in the medium ("adiabaticity") since the oscillation length provides the cutoff on coherence in scattering from the fluctuations.

At the semiclassical level the evolution of flavor during the neutrino's passage through matter of changing density, $n_e(\mathbf{x})$, is conventionally described by an equation involving time only:

$$i\frac{d}{dt}\psi_j(t) = \sum_k \omega_{jk}(t)\psi_k(t) \;, \tag{47}$$

where $\omega_{ij}(t)$ is the effective Hamiltonian matrix in the flavor representation. The equation for the "flavor-density" matrix is

$$\frac{d}{dt}\hat{\rho}_{ij} = i\sum_m [\omega_{mi}(t)\hat{\rho}_{mj} - \omega_{jm}(t)\hat{\rho}_{im}] \;. \tag{48}$$

Equation (48), in the absence of the fluctuations, is perfectly superfluous, since it merely combines the equations for the individual flavor wave functions in the absence of scattering from the fluctuations. To include the effects of the fluctuations, we add the last term on the RHS of (23), as calculated for constant background density. [The first term on the RHS of (23) becomes exactly the first term on the RHS of (47) when reexpressed in the flavor-diagonal representation; the second term is the energy-shift term discussed earlier, which does not contribute to dD/dt]:

$$\frac{d\hat{\rho}_{ij}}{dt}=i\sum_{m}[\omega_{mi}\hat{\rho}_{mj}-\omega_{jm}\hat{\rho}_{im}]+\frac{G_F^2}{2\pi^2(\mathrm{Vol})}\sum_{\alpha\beta}R_{ij\alpha\beta}\hat{\rho}_{\alpha\beta}\int d^3k\,|\delta n_e(\mathbf{k})|^2\delta(|\mathbf{r}-\mathbf{k}|-|\mathbf{r}|)\ . \tag{49}$$

In the added term on the right-hand side, we have replaced the density matrix in the absence of scattering, $\hat{\rho}^{(0)}$, by the corrected density matrix $\hat{\rho}$.

We justify this step by the following argument: Equation (23) is to be applied in situations in which the neutrino passes through numbers of inhomogeneities, each of small scale compared to the oscillation length, each resulting in a small alteration of the density matrix from the initial value. At some coarse-graining distance (or time) we can restart the clock, that is, deal with the subsequent evolution of the density matrix by inserting, as the new initial state under the scattering integral, the density matrix calculated to that time. In this way, the density matrix can evolve into one with large cumulative corrections due to the scattering, even though the rates were calculated in perturbation theory. The procedure is exactly analogous to the way one calculates exponential decay in perturbation theory, giving an answer that includes the leading term, to every order, in the product G_F^2T, where T is the elapsed time, and missing the higher-order terms in the coupling which carry lower powers of T. As in the case of exponential decay, some terms, which are specific to the way the system is produced in the beginning, or measured at the end, are not included (and should be totally inconsequential). Unlike the case of single-particle propagation, the present case does not lead to an exponential solution, because of the 2×2 matrix structure in the equations of development.

The use of the words "coherent" and "incoherent" in somewhat varying contexts in the course of this paper may cause confusion to the reader. The scattering from the whole of a single fluctuation is coherent; the coherence over the scale of the inhomogeneity is what will give us a big effect. But the part of the wave scattered from this inhomogeneity, with momentum transfer $\delta p \approx R_{\mathrm{imhom}}$, is not coherent with waves scattered from other inhomogeneities many times R_{inhom} away, nor with the unscattered wave. The latter circumstance is of critical significance in distinguishing our results from those of Refs. 8 and 9, as we see below.

Why is it necessary to splice together results in the manner of (49)? It is because in the real problem in which a neutrino needs to get out of the entire star, we cannot use perturbation theory for the wave function over distances over which the flavor wave function changes drastically. But we are not prepared to address nonperturbatively a problem in which momentum and flavor are entangled in the evolution of the wave function, as the neutrino traverses a disordered medium. Thus we need the perturbation theory in order to derive the effects of the inhomogeneities; but we need an equation such as (48) to determine the large changes in the neutrino's state while traversing the whole star.

VI. COMPARISON WITH OTHER APPROACHES

In most or all of the literature on the MSW effect, it is assumed that the neutrino follows a classical trajectory in space and time, $\mathbf{x}(t)$, and the Hamiltonian for the system is taken to operate in flavor space only, with a time dependence which arises from the changing properties of the medium as seen by the moving neutrino. In Refs. 8 and 9, density inhomogeneities have been incorporated by letting the effective Hamiltonian in (47), $\omega_{jk}(t)$, respond to the local density changes $\delta n_e[\mathbf{x}(t)]$, as the semiclassical particle traverses them at time t. Thus the problem is formulated in terms of the usual coupled differential equations in the time variable; the momentum variable of our approach does not enter. We call this the purely semiclassical approach. There is one obvious necessary condition for this approach to be correct that is indeed satisfied by many orders of magnitude, namely, that the wavelength of the neutrino be much less than the size of the inhomogeneities.

But this approach misses an important physical effect: that as the neutrino propagates over the whole MSW transition region, the amplitude scattered from the smaller scale inhomogeneities does not interfere with the unscattered amplitude. The scattered amplitudes have momenta which are distributed in a range $\delta p=(L_{\mathrm{inhom}})^{-1}$ around the incident momentum, these momenta having been deposited in the regions of inhomogeneity. The purely semiclassical approach, however, has the interference built in, as a consequence of suppressing the momentum (or space) coordinate. A closely related difference, in the two approaches, is that in the purely semiclassical approach the flavor-depolarization function $D(t)$ remains fixed at zero.

Looking at the results of Ref. 8, we see persuasive evidence of two kinds that these interference terms are dominating the results of the numerical calculations: (a) it is stated that the changes in the *probability* of the ν_e to ν_μ transition are, to a good approximation, proportional to the amplitude of the density perturbation; (b) the effects of the perturbation are as often, or almost as often, in the direction of *increasing* the efficiency of neutrino transformation as in the direction of decreasing the efficiency. The sign of the effect oscillates as the neutrino energy is varied.

These features can be contrasted with the qualitative features of our results: (a) the effects of the perturbation are quadratic in the amplitude of the density perturbation (and G_F) (of course, as we shall see when we do numerical estimates, the fact that the quadratic terms are of higher order in weak coupling does not make them automatically negligible; factors of G_F which get multiplied by macroscopic numbers can be of order unity, as in the MSW calculation itself); (b) in our approach the sign of dD/dt is positive under all circumstances. If we apply the analysis to a case in which nearly all the ν_e's are transformed to ν_μ's in the absence of inhomogeneities, then the progressive depolarization coming from the scattering must necessarily lead to a decrease in the transformation efficiency. This is true because an all-ν_μ state necessarily has $D=0$; a reduction in D can only be achieved if there is (incoherent) mixing with ν_e states.

Our treatment begins with plane waves coming in. Yet, on the macroscopic scale of the medium, we usually, and with good reason, think of neutrinos as rays. In a treatment in which the neutrino travels as a classical ray, along a particular path, it naturally does not sample a macroscopic volume, as does our plane wave. But, as one can easily see, if one, e.g., considered an expectation value of a translationally invariant operator (flavor matrices, in the case at hand) for an ensemble of raylike wavepackets, identical except for transverse displacements which are weighted equally in the ensemble, then all interference terms between components of different transverse momenta vanish in the ensemble average. Thus the plane-wave calculation which we have made gives the average results for rays traversing the medium.

Above, we have emphasized the substantial differences between our results and those of Ref. 8, or the completely semiclassical approach, differences that stem from the difference in the way which the terms in G_F are (implicitly) treated in the two approaches. It is the case, however, that if we carried out the perturbation expansion of the solution to the completely semiclassical equations, the *quadratic* terms would be of exactly the same order of magnitude as the terms which we calculate. If in addition we perform an average over paths, at least in a problem that has been defined with a perturbation that averages to zero over the whole volume, the linear terms in G_F should cancel, and the quadratic terms, we would assume, become the same as those presented in Sec. II.

Thus one's intuition that the waves could be treated as particles when the wavelength is very small compared to the relevant scale of the inhomogeneities would be correct in a sense; and the approach would be a matter of taste. We would still advocate (49) as the correct transport equation, so long as the inhomogeneities are no larger than the oscillation length, and smaller than the scale for the steady density change. It is both more practical to carry out the analysis of (49) and more physically transparent than to carry out an average over sets of numerical solutions for particular paths. We also advocate using the depolarization parameter to estimate the importance of the effects. Finally, we observe that (49) is the most concise way of combining the standard form for the scattering of a wave in a medium with the standard form for the time evolution of an integral degree of freedom that responds to the environment.

With appropriate changes to include relativistic electron recoil, (23) is also the equation which describes the evolution of the neutrino color-density matrix under the influence of submicroscopic density fluctuations; as an example, we took uncorrelated electrons in (31). The evolution in this latter environment has been studied by Stodolsky,[11] and others, in somewhat different fashions, with results in agreement with the ones which our formalism would give. However, in an interacting medium, collective effects, e.g., plasma waves, may be important; and the appropriate generalization of (23) is the correct method for incorporating them, with the square of the Fourier transform of the density function in (23) being replaced by the Fourier transform of the appropriate correlation function (a time-dependent correlation function, since energy may be transferred).

We emphasize, however, that all of these microscopic correlations are insignificant at solar densities; only the macroscopic ones could change the solar-neutrino signal.

VII. ESTIMATES OF THE EFFECTS OF DEPOLARIZATION ON THE MSW EFFECT

A. MSW effect in the Sun

For illustration, we choose the conditions under which the depolarization effect could decrease the transformation efficiency by an interesting amount. In Sec. IV it was shown that the depolarization effect will be most potent when the state is a 50-50 mixture of the two flavors, i.e., at or near resonance. The depolarization will be greater, for a given fractional amplitude of variation, $\delta n_e/n_e$, in a region of high density than it will in a region of low density. Since in (22) a natural cutoff in the inverse coherence length for scattering from a single inhomogeneity is given by $2\pi\Delta_{12}$, we have the *possibility* of larger scattering when Δ_{12} is small, i.e., when the oscillation length at resonance is large.

For illustration we choose the following values of parameters, following the notation of Bahcall;[3] in our system of $\hbar=c=1$, we choose to measure most dimensional quantities in MeV (1 MeV$\approx 0.508\times10^{11}$ cm^{-1}). We choose the background electron density in the resonance region to be that at a distance of $0.1R_{\rm Sun}$ from the center, $n_e^{(0)}=65N_A$ cm$^{-3}=3.02\times10^{-7}$ MeV^3. Taking $\langle S\rangle=\frac{1}{2}$, at resonance, we estimate the depolarization which would result from the passage through a slab of thickness L, and density contrast $\eta=\delta n_e/n_e$, obtaining, from (34),

$$\Delta P=2G_F^2\langle S\rangle(n_e^{(0)})^2L^2\eta^2=301\eta^2\left[\frac{L}{1000\text{ km}}\right]^2 . \tag{50}$$

For example, for $\eta=0.01$ we could obtain important depolarization only for inhomogeneity of scale $L\geq1000$ km. The oscillation length in matter, at resonance, is given by

$$L_M=\frac{L_\upsilon}{\sin2\theta_\upsilon}=\frac{4\pi E^{(0)}}{\Delta m^2\sin2\theta_\upsilon} . \tag{51}$$

We consider two of the adiabatic possibilities discussed by Bahcall:[3] (a) $\Delta m^2\approx10^{-4}$ (eV)^2 and and $\sin2\theta_\upsilon$, ranging from as low as 0.02 to as high as 0.85; the range of L_M for the 10-MeV neutrino then being from 300 to 10 000 km; (b) $\sin2\theta_\upsilon$ approximately unity, and a range of Δm^2 from 10^{-4} to 10^{-6}, the range of L_M then being approximately as in (a).

We now estimate the maximum effect of scattering from a single inhomogeneity by taking L equal to L_M. For example, if we take a one-percent density change, $\eta=0.01$, over a distance of 2000 km, we would get a ΔD of about 0.1. Since this could be repeated several times over while the neutrino is in the resonance region, it is possible for D to increase to values approaching 0.5. From Sec. IV we know that in a two-state system, the value of D must be less than 0.5, a value which signifies an equal and incoherent mixture of the two flavors. From (42) it is also easy to demonstrate that any value of D greater than zero sets a maximum conversion efficiency to a particular kind of neutrino, say ν_μ,

$$(\text{Prob } \nu_\mu)_{\max}=\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-2D}\ . \tag{52}$$

In principle we would solve (49) to find the exact effects of a particular configuration on the flavor-density matrix; for the moment there is not enough information on the neutrino parameters, or a well-enough determined hypothesis for the internal fluctuations to make this useful. However, we see that in the parameter ranges which are interesting for the possibility of the MSW effect in the Sun, there are significant regions in which the scattering effect would result in a large decrease in the conversion efficiency *if* there were density fluctuations at the 1% level on the scale of a few thousand km. As was noted in Ref. 8, there is no strong reason to believe that the solar core should be inhomogeneous at the $\eta=0.01$ level. Nevertheless, if the analysis of the solar-neutrino signal advances to the point at which the MSW mechanism is confirmed, and yields improved limits on the vacuum masses and mixing angles, then these observations, coupled with a more sophisticated version of the present analysis, should put a useful limit on, for example, the amplitude of g modes in the core of the Sun.

It is possible that coherent scattering from density inhomogeneities located well outside the MSW resonance region also could make significant modifications in the transformation efficiency and the spectrum. Away from resonance, the depolarization from a given inhomogeneity will be less than that under resonance conditions, since, by (45), the driving term in the depolarization is proportional to $\sin^2(2\theta_M)$. And the cutoff on size dictated by the oscillation length will be smaller, away from resonance, other parameters being equal. However, the off-resonance depolarization effects could gain from operating over a larger region, or a region in which the fluctuations were more pronounced.

B. MSW effect in supernovae

For this case we carry out the same estimate, this time choosing parameters from Ref. 6: $n_e^{(0)}=1.8\times10^{36}\ \text{cm}^{-3}$ (at $\rho_{\text{mass}}=10^{13}\ \text{g cm}^{-3}$, $X_e=0.3$), $L_M=2\text{cm}$, $\Delta m^2=25\ \text{keV}^2$, $E^{(0)}=40$ MeV, $\theta=10$ rad. For a fluctuation of size L we obtain

$$\Delta P=6.2\times10^7\eta^2\left[\frac{L}{1\ \text{cm}}\right]^2 . \tag{53}$$

We get large depolarization for the case of a single-density variation of one part in 10^4 over the oscillation distance. Bearing in mind that under adiabatic conditions the neutrino traverses many times the oscillation region while in or near resonance, we see that it requires very great homogeneity on the scale of centimeters to avoid complete depolarization, that is, ending in a state of incoherent, equal mixtures of all species involved in resonance. In view of the large accelerations of the material in the supernova case, this degree of homogeneity would appear to be highly unlikely.

VIII. DISCUSSION

We have presented our estimates in terms of D, the depolarization, instead of directly calculating the survival probability of an electron. The advantage of discussing D is its property of steady increase. The property, $D\geq0$, clearly distinguishes our calculation, in which the momentum variable is included, from previous calculations, which lack the momentum variable. In the latter, if we start with a pure state, we end with a pure state, $D=0$. In our formulation, the flavor-only density matrix is no longer pure; D measures the impurity. The depolarization is not a quantity which could be directly measured, even in idealized measurements of the neutrino signal, because averaging over the depths of production will destroy the phase relation between ν_1 and ν_2 in any case. But, as we saw in (42) for the two-flavor case, depolarization limits the probability for a particular flavor, and vice versa. For example, in a model in which 0.9 of the ν_e's are converted to ν_μ's, we must have $D<0.18$. If our estimate of the accumulation of ΔD's, over the course of the propagation, exceeds this number, then the scatterings must cause a significant decrease in the ν_e to ν_μ conversion efficiency resulting from the scatterings. In this way we can avoid much of the model-dependent detail implicit in (49), in order to estimate the importance of our effect (perhaps for cases of greater conversion efficiency than the present data demand, but, we argue, of great value as test cases).

The principal difference in our results and those of Refs. 8 and 9 is that we predict a steady erosion of the conversion efficiency from the scattering, rather than a contribution the sign of which depends on details of the perturbation and the energy of the neutrino. We attribute this difference to the fact that in our case the linear terms in a perturbation development do not interfere with the unscattered term. Note that the depolarization (or impurification of the color-density matrix) which we have calculated follows directly from the Schrödinger equation, with no introduction of phase averaging. The complete density matrix remains that for a pure state; it is the summation over the momentum coordinate to get the flavor-density matrix which introduces the impurity in the flavor space.

As to the implications for the solar-neutrino problem, we believe that the present approach, suitably extended and refined, can put limits on the amplitude of core oscillations with wavelengths of the order of thousands of kilometers, given definitive values of the vacuum neutrino parameters.

In the case of the supernova process, where reliable predictions of the texture of the exploding matter on the scale of a few centimeters will be even harder to come by, the demands for homogeneity, to avoid complete depolarization, are much more stringent. It seems likely that complete depolarization, that is, equal mixtures of different flavors, would prevail, if neutrino parameters are in the range in which there could be an MSW effect.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation under Grant No. PHY86-14185.

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