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Electroweak radiative corrections to the semihadronic decay rate of the τ lepton

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The one-loop electroweak radiative correction to the decay rate of the τ lepton into a neutrino plus hadrons is calculated. The correction includes a large logarithm of M_Z/M_τ whose coefficient was calculated previously. The constant under the logarithm gives an additional electroweak correction of 0.1%. It is argued that modifications of this electroweak correction due to nonperturbative QCD effects are suppressed by powers of M_τ.

The τ lepton is unique in that it is the only charged lepton that can decay semihadronically, i.e., into a neutrino plus hadrons. The electron is stable and the muon is too light to decay into hadrons. Precision measurements of the total and partial decay rates of the τ could be used to sharpen the determination of the parameters of the standard model of particle physics and to search for deviations from its predictions. The partial decay rates into the leptonic modes $\nu_\tau e^- \bar{\nu}_e$ and $\nu_\tau \mu^- \bar{\nu}_\mu$ and the semihadronic mode $\nu_\tau \pi^-$ can all be calculated accurately. The lifetime of the τ and its branching fractions into the leptonic decay modes and many of its semihadronic modes can be measured accurately. Combining these experimental measurements with strong theoretical constraints from isospin symmetry, one can determine whether the known partial decay rates account for the total decay rate. Attempts to carry out this accounting using the world averages for the measured lifetime and branching fractions have revealed discrepancies of several standard deviations.[1] This puzzle has opened up an opportunity for theoretical predictions of the total decay rate of the τ to point the way toward the resolution of the discrepancy.[2] An accurate theoretical prediction for this decay rate will also allow a precise determination of the QCD coupling constant from measurements of the lifetime of the τ.[3,4]

The decay of the τ into semihadronic final states necessarily involves nonperturbative aspects of QCD. Nevertheless, the total semihadronic decay rate can be calculated theoretically. It is convenient to define a ratio R by normalizing the semihadronic decay rate to the electronic decay rate:

$$R \equiv \frac{\Gamma(\tau^- \to \nu_\tau \text{hadrons}(\gamma))}{\Gamma(\tau^- \to \nu_\tau e^- \bar{\nu}_e(\gamma))} . \qquad (1)$$

The decay rates in (1) are for inclusive final states which may contain additional photons or lepton pairs. A naive estimate $R \simeq 3$ is obtained by comparing the decay rate into the quark pairs $d\bar{u}$ and $s\bar{u}$ with the decay rate into $e^- \bar{\nu}_e$.[5] The QCD corrections to this naive prediction can be calculated systematically. The operator-product expansion can be used to organize the nonperturbative QCD corrections into an expansion in powers of $1/M_\tau^2$.[2,4,6] The coefficients involve matrix elements of local operators for which phenomenological estimates are available. These estimates indicate that the nonperturbative corrections to R are less than 1%. The purely perturbative QCD corrections are much larger. They can be expressed as an expansion in the QCD running coupling constant $\alpha_s(M_\tau)$ evaluated at the mass of the τ.[7] The corrections of order α_s and α_s^2 to the naive prediction $R=3$ are approximately 10% and 5%, respectively. In a previous calculation of the order α_s^3 correction, it was found to have an enormous coefficient so that it gave a correction of about 10%, casting doubt on the validity of the entire perturbation expansion. However, this calculation was based on a calculation of the order-α_s^3 correction to the ratio R of e^+e^- annihilation.[8] An error has been found in this calculation. The order-α_s^3 correction is being recalculated and is expected to be significantly smaller.[9] Thus there is reason to expect that the uncertainty in the QCD corrections can be reduced to the level of 1% of R, a truly astonishing accuracy.

If the uncertainty in the QCD corrections can be reduced to the 1% level, then the electroweak radiative corrections become important.[3] They are on the order of 2% of R, which is much larger than one would naively expect. The reason they are so large is that the purely QED one-loop correction to the decay $\tau^- \to \nu_\tau d\bar{u}$ is ultraviolet divergent. In the standard electroweak model, the divergence is cut off at the Z mass, but it leaves a large logarithm $\ln(M_Z/M_\tau)$. The coefficient of the logarithm has been calculated[10] and this correction has been taken into account in previous predictions. The purpose of this paper is to present the calculation of the constant under the logarithm, thus reducing the uncertainty in R from electroweak corrections to much less than 1%.

At the tree level, the decay $\tau^- \to \nu_\tau f_2 \bar{f}_3$ proceeds through an intermediate virtual W^- via the diagram in Fig. 1. We neglect corrections suppressed by M_τ^2/M_W^2 and m_f^2/M_τ^2, where m_f is the mass of a final-state fermion. The tree-level result for the decay into $\nu_\tau e^- \bar{\nu}_e$ is

$$\Gamma_0 = \frac{\alpha^2}{384\pi s_W^4} \frac{M_\tau^5}{M_W^4} , \qquad (2)$$

where $\alpha \simeq \frac{1}{137}$ is the electromagnetic coupling constant, $s_W = \sin\theta_W$, and θ_W is the Weinberg angle. For the decay into $\nu_\tau d\bar{u}$, (2) must be multiplied by $3|V_{ud}|^2$, where V_{ud} is an element of the Kobayashi-Maskawa matrix.

We calculate the electroweak radiative corrections to first order in α. The natural choice for the electromagnetic coupling constant is the running coupling constant $\alpha(M_\tau)$ evaluated at the mass of the τ, whose numerical

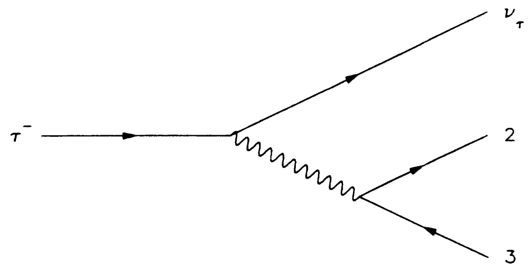

FIG. 1. Lowest-order diagram for the decay of the τ^- into ν_τ plus a fermion and an antifermion.

value[3] is $\alpha(M_\tau) \simeq \frac{1}{133.3}$. We neglect corrections suppressed by m_f^2/M_τ^2, where m_f is the mass of a final-state fermion, so the masses of all final-state fermions are set to zero throughout the calculation. We also neglect corrections proportional to M_τ^2/M_W^2. We work in the on-shell renormalization scheme[11] in which the parameters of the theory are chosen to be the masses of all the particles (W, Z, Higgs bosons, and fermions), together with α. The Weinberg angle is then defined to all orders by $\cos\theta_W \equiv M_W/M_Z$. We separate the radiative corrections into three classes: QED corrections due to the radiation of real photons, QED corrections due to the exchange of virtual photons between the external fermion lines, and the remaining weak corrections which arise from the embedding of QED in the standard model. We calculate the QED radiative corrections in the Feynman gauge and we use the 't Hooft–Feynman gauge for the propagators of virtual W and Z bosons. Ultraviolet divergences are regularized using dimensional regularization in $4-2\epsilon$ dimensions. It is verified that there are no ambiguities associated with the definition of γ_5. Infrared divergences from soft photons and from photons collinear to the massless fermions are regularized by giving the photon a small mass λ.

The electroweak corrections to the denominator in (1) are identical to the corrections for muon decay and have been calculated previously.[11] We therefore organize our calculation so that this can be used as a check on our result. We calculate the correction for the decay of a lepton with charge $Q_0=-1$ into a neutrino, a fermion of charge Q_2, and an antifermion of charge Q_3, with $Q_2+Q_3=-1$. In the case of the electronic decay, the charges are $Q_2=-1$, $Q_3=0$. For the semihadronic decay, the charges are those of the d quark and the $\bar{u}$ antiquark: $Q_2=-\frac{2}{3}, Q_3=-\frac{1}{3}$.

The matrix element for the QED real-photon corrections is the sum of the three Feynman diagrams in which a real photon is radiated from the charged-fermion lines in Fig. 1. The matrix element is squared and integrated over the phase space of the four final-state particles. The result is

$$\delta\Gamma^{\rm QED}_{\rm real} = \Gamma_0 \frac{\alpha}{2\pi}\left[Q_0^2\left(2\ln\frac{\lambda}{M_\tau}+\frac{137}{24}\right) + (Q_2^2+Q_3^2)\left(-\ln\frac{\lambda}{M_\tau}-\frac{49}{24}\right) + Q_0Q_2\left(2\ln^2\frac{\lambda}{M_\tau}+\frac{17}{3}\ln\frac{\lambda}{M_\tau}+\frac{119}{24}-\zeta_2\right) \right.$$
$$\left. + Q_0Q_3\left(2\ln^2\frac{\lambda}{M_\tau}+\frac{19}{3}\ln\frac{\lambda}{M_\tau}+\frac{427}{72}-\zeta_2\right) + Q_2Q_3\left(-4\ln^2\frac{\lambda}{M_\tau}-\frac{43}{3}\ln\frac{\lambda}{M_\tau}-\frac{1261}{72}+2\zeta_2\right)\right], \qquad (3)$$

where $\zeta_2=\pi^2/6$. The diagrammatic origin of each term is evident from the electric charges in the coefficients multiplying it. Infrared divergences from soft and collinear photons appear as logarithms of the photon mass λ.

The QED virtual-photon corrections come from the interference between the lowest-order matrix element and the order-α corrections to that matrix element. The correction is the sum of the three Feynman diagrams in which a virtual photon is exchanged between two of the charged fermion lines in Fig. 1, plus the three diagrams with self-energy corrections on the charged lines (multiplied by one-half to account for wave-function renormalization). The interference term is integrated over the phase space of the three final-state fermions. The result is

$$\delta\Gamma^{\rm QED}_{\rm virtual} = \Gamma_0 \frac{\alpha}{2\pi}\left(\frac{4\pi\mu^2}{e^\gamma M_\tau^2}\right)^\epsilon \left[Q_0^2\left(-\frac{1}{2\epsilon}-2\ln\frac{\lambda}{M_\tau}-2\right) + (Q_2^2+Q_3^2)\left(-\frac{1}{2\epsilon}+\ln\frac{\lambda}{M_\tau}+\frac{1}{4}\right)\right.$$
$$+ Q_0Q_2\left(2\ln\frac{M_W}{M_\tau}-2\ln^2\frac{\lambda}{M_\tau}-\frac{17}{3}\ln\frac{\lambda}{M_\tau}-\frac{1}{8}-5\zeta_2\right)$$
$$+ Q_0Q_3\left(8\ln\frac{M_W}{M_\tau}-2\ln^2\frac{\lambda}{M_\tau}-\frac{19}{3}\ln\frac{\lambda}{M_\tau}+\frac{47}{72}-5\zeta_2\right)$$
$$\left. + Q_2Q_3\left(-\frac{1}{\epsilon}+4\ln^2\frac{\lambda}{M_\tau}+\frac{43}{3}\ln\frac{\lambda}{M_\tau}+\frac{895}{72}-2\zeta_2\right)\right]. \qquad (4)$$

In the expression raised to the power ϵ, γ is Euler's constant and μ is the arbitrary mass scale introduced by dimensional continuation. They will cancel out of the final answer. The diagrammatic origin of each term is again evident from the electric charges in the coefficient multiplying it. The infrared divergences in (4) cancel against those of (3). The ultraviolet divergences in (4) appear as poles in ϵ and the overall coefficient of the pole is proportional to $Q_0^2+(Q_2+Q_3)^2$. The terms proportional to Q_0Q_2 and Q_0Q_3 are ultraviolet finite because the W propagator provides a convergence factor at large momentum. An ultraviolet divergence is recovered in the limit $M_W\to\infty$.

A momentum cutoff Λ could be used to cut off both the

$1/\epsilon$ divergence in (4) and the divergence as $M_W \rightarrow \infty$. The net effect is the replacements $1/\epsilon \rightarrow 2\ln(\Lambda/M_\tau)+\frac{1}{2}$ and $\ln(M_W/M_\tau) \rightarrow \ln(\Lambda/M_\tau)$. The divergence is then proportional to $Q_0^2-2Q_0(Q_2+4Q_3)+(Q_2+Q_3)^2 = 3(-1-Q_2+Q_3)$. In the case of the electronic decay ($Q_2=-1$, $Q_3=0$), the dependence on the cutoff Λ cancels and the correction factor in the sum of (3) and (4) reduces to $(\alpha/2\pi)(25/4-\pi^2)$. This is the classic result for the QED correction to the muon decay rate in the Fermi model for weak interactions[12] and provides a valuable check on our calculation. For the semihadronic decay ($Q_2=-\frac{1}{3}, Q_3=-\frac{2}{3}$), the QED correction in the Fermi model is divergent. In the standard electroweak model, the divergence is replaced by a large logarithm[10] $\ln(M_Z/M_\tau)$.

The remaining radiative corrections are weak corrections that arise from the embedding of electromagnetism in the standard model. They are all short-distance corrections that reduce to a multiplicative renormalization of the tree-level matrix element. The diagrams consist of four box diagrams in which a virtual Z is exchanged between the fermion lines in Fig. 1, six vertex corrections involving a virtual Z, four vertex corrections in which a virtual photon attaches to the W line, plus propagator corrections on the W line and the external fermion lines. There is also a counterterm associated with the renormalization of the W vertices in Fig. 1.[11] The sum of all these corrections is

$$\delta\Gamma_{\text{virtual}}^{\text{weak}} = \Gamma_0\left\{-2\frac{\Pi_W(0)}{M_W^2}+\frac{\alpha}{2\pi}\left[\frac{4\pi\mu^2}{e^\gamma M_\tau^2}\right]^\epsilon\left[\frac{1}{\epsilon}-2\ln\frac{M_W}{M_\tau}+\frac{6}{s_W^2}-\frac{1}{2}+\left[\frac{7}{s_W^4}-\frac{4}{s_W^2}-3+3Q_0(Q_2-Q_3)\right]\ln c_W\right]\right\}, \tag{5}$$

where $s_W=\sin\theta_W$ and $c_W=\cos\theta_W$. In the first term in (5), $\Pi_W(0)$ is the W propagator correction renormalized on the mass shell and evaluated at zero momentum. It cancels between numerator and denominator in the ratio R defined in (1), so its explicit form is not needed here.

Adding (3)–(5), we get the complete electroweak correction to the decay rate:

$$\delta\Gamma^{\text{electroweak}} = \Gamma_0\left\{-2\frac{\Pi_W(0)}{M_W^2}+\frac{\alpha}{2\pi}\left[3[1-Q_0(Q_2-Q_3)]\ln\frac{M_Z}{M_\tau}+\left[\frac{7}{s_W^4}-\frac{4}{s_W^2}\right]\ln c_W+\frac{6}{s_W^2}-\frac{1}{2}+\left[\frac{89}{24}-\pi^2\right]Q_0^2 -\frac{43}{24}(Q_2^2+Q_3^2)+\frac{29}{6}Q_0Q_2+\frac{237}{36}Q_0Q_3-\frac{61}{12}Q_2Q_3\right]\right\}. \tag{6}$$

Setting $Q_0=Q_2=-1$ and $Q_3=0$, we get the complete order-α correction to the electronic decay rate of the τ in the standard model. The correction is identical to the correction for muon decay in the standard model,[11] thus providing another check on our calculation. The electroweak correction to the semihadronic decay rate proceeding through the $d\bar{u}$ current or the $s\bar{u}$ current is obtained by setting $Q_0=-1$, $Q_2=-\frac{1}{3}$, and $Q_3=-\frac{2}{3}$. Taking the ratio R defined in (1), we find that the result is

$$R=3(|V_{ud}|^2+|V_{us}|^2)\left[1+\frac{\alpha}{2\pi}\left[4\ln\frac{M_Z}{M_\tau}+\frac{5}{6}\right]+\text{QCD corrections}\right]. \tag{7}$$

The logarithm in (7) represents a short-distance correction due to virtual particles with energies ranging from M_τ to M_Z. It should therefore be pulled out as an overall factor $S(M_\tau)=1+2(\alpha/\pi)\ln(M_Z/M_\tau)$ multiplying both the electroweak and the QCD corrections. Its numerical value is 1.018. If the renormalization group is used to sum up all the leading logarithms of M_Z/M_τ, it increases slightly to $S(M_\tau)=1.019$.[3] Our final result for the ratio R, including the leading electroweak correction together with the known perturbative QCD corrections[7] is

$$R=3(|V_{ud}|^2+|V_{us}|^2)S(M_\tau)\left[1+\frac{5}{12}\frac{\alpha(M_\tau)}{\pi}+\frac{\alpha_s(M_\tau)}{\pi}+5.20\left[\frac{\alpha_s(M_\tau)}{\pi}\right]^2+\cdots\right]. \tag{8}$$

In addition to higher-order perturbative corrections to (8), there are also nonperturbative corrections. One might expect the nonperturbative QCD corrections to be large, since the invariant-mass distribution of hadrons differs dramatically from the invariant-mass distribution for quarks and gluons computed in perturbation theory. However the total decay rate is given by the integrated distribution and this is much less sensitive to nonperturbative effects than the shape of the distribution. The operator-product expansion can be used to systematically organize the nonperturbative corrections into an expansion in powers of $1/M_\tau^2$. The coefficients involve vacuum matrix elements of local operators such as the quark condensate $\langle\bar{\psi}\psi\rangle$ and the gluon condensate $\langle G_{\mu\nu}^a G^{a\mu\nu}\rangle$.[13] The values of these matrix elements can be estimated phenomenologically. The best available estimate is that nonperturbative QCD effects should give a negative correction between 0.3% and 0.6%.[4]

The remaining nonperturbative corrections involve the influence of nonperturbative QCD effects on the electroweak correction. The spectrum of photons emitted in semihadronic decays of the τ differs significantly from the spectrum obtained by the perturbative calculation of the decay into quarks plus a photon. However it is likely that the integrated spectrum is much less sensitive to nonperturbative effects. It is possible that these nonperturbative corrections can also be systematically organized into an expansion in powers of $1/M_\tau^2$. The coefficients would involve vacuum matrix elements of local operators evaluated in the presence of a weak background electromagnetic field,[14] such as the magnetic susceptibility of the quark condensate. A careful treatment of these effects analogous to the operator-product-expansion treatment of the nonperturbative effects of pure QCD has not been carried out. If it is indeed possible, then these corrections would be suppressed relative to the perturbative electroweak corrections by the ratio of some hadronic matrix element to the appropriate power of M_τ and would be much smaller than 1%.

We have computed the order-α electroweak correction to the ratio R for the decay of the τ. The correction contains a large logarithm of M_Z/M_τ which gives a short distance enhancement of +1.9%. The constant under the logarithm increases the correction further by 0.1%. A precise theoretical prediction of the τ lifetime, or a precise determination of $\alpha_s(M_\tau)$ from the measured lifetime, must await the recalculation of the order-α_s^3 perturbative QCD correction.

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