Brief Reports

Brief Reports are accounts of completed research which do not warrant regular articles or the priority handling given to Rapid Communications; however, the same standards of scientific quality apply. (Addenda are included in Brief Reports.) A Brief Report may be no longer than four printed pages and must be accompanied by an abstract.

Effective low-energy, large-N Lagrangian calculation of the $K \to \pi\pi\pi\gamma$ decay modes

S. Fajfer and K. Suruliz
Institut za Fiziku, University of Sarajevo, Sarajevo, Yugoslavia

Robert J. Oakes
Department of Physics and Astronomy, Northwestern University, Evanston, Illinois 60208
(Received 7 May 1990)

Branching ratios for the decays $K^+ \to \pi^+\pi^+\pi^-\gamma$, $K^+ \to \pi^+\pi^0\pi^0\gamma$, and $K_L^0 \to \pi^+\pi^-\pi^0\gamma$ are calculated in the large-N limit using an effective low-energy Lagrangian which gives a good description of the nonradiative $K \to \pi\pi\pi$ decays. In addition to the bremsstrahlung contributions to the amplitude, the direct-emission terms are systematically included and found to be significant. The results are in reasonable agreement with the existing data.

INTRODUCTION

Effective chiral Lagrangians can provide a useful low-energy description of hadronic interactions, particularly the pseudoscalar mesons.[1] They are also of considerable interest as the large-N limit of quantum chromodynamics.[2] Calculations based on this approach have been rather successful in describing the nonleptonic kaon decays $K \to 2\pi$ (Refs. 3–5) and $K \to 3\pi$.[6–8] Including the Wess-Zumino[9,10] terms in the effective chiral Lagrangian, the amplitude for the radiative weak decay $K \to \pi\pi\gamma$ has also been calculated.[11–13] Here we extend this work to the processes $K \to \pi\pi\pi\gamma$. Our results agree reasonably well with the existing data on the decay modes $K^+ \to \pi^+\pi^+\pi^-\gamma$ (Ref. 14) and $K^+ \to \pi^+\pi^0\pi^0\gamma$.[15] We also present results for the decay mode $K_L^0 \to \pi^+\pi^-\pi^0\gamma$, which has not yet been observed.

The bremsstrahlung contributions to $K^+ \to \pi^+\pi^+\pi^-\gamma$ were considered some time ago by Dalitz.[16] Our calculations show the amplitude for direct emission of the γ is a significant contribution, particularly since it interferes with the amplitude for bremsstrahlung. The effective-Lagrangian approach now enables one to systematically include these direct-emission contributions.

In the next section we briefly review the effective low-energy Lagrangian and then proceed to use it to calculate the partial widths for the $K \to \pi\pi\pi\gamma$ decays. Analytic expressions for the distributions in the energy of the odd pion and energy of the photon were calculated in the center of mass of the three pions. The integrations over the two remaining energies were finally done numerically to obtain the partial decay widths.

EFFECTIVE LAGRANGIAN

In the large-N limit, the effective low-energy Lagrangian describing the strong interactions of pseudoscalar mesons contains only the terms with single traces over color indices:[3–6]

$$\mathcal{L}_{\text{strong}} = \frac{f^2}{4}\mathrm{Tr}\left[\partial_\mu U\,\partial_\mu U^\dagger + rM(U+U^\dagger) - \frac{r}{\Lambda_0^2}M(\partial^2 U + \partial^2 U^\dagger) + \frac{1}{\Lambda_1^2}(\partial_\mu U^\dagger \partial_\mu U\,\partial_\nu U^\dagger \partial_\nu U) - \frac{1}{\Lambda_2^2}(\partial_\mu U^\dagger \partial_\nu U\,\partial_\mu U^\dagger \partial_\nu U)\right]. \quad (1)$$

Here $U = \exp(i\Pi/f)$, where $\Pi = \sum \lambda_a \pi^a$ is the usual 3×3 matrix of pseudoscalar meson fields and $M = \mathrm{diag}(m_u, m_d, m_s)$ is the mass matrix. The coupling of constant f is related to the pion decay constant $F_\pi = 93$ MeV by

$$F_\pi = f(1 + m_\pi^2/\Lambda_0^2)\,, \quad (2)$$

to lowest order in the pion mass. Taking into account flavor-SU(3) breaking, the kaon decay constant $F_K = 1.22F_\pi$ is given by

$$F_K = F_\pi[1 + (m_K^2 - m_\pi^2)/\Lambda_0^2]\,. \quad (3)$$

From Eqs. (2) and (3), one finds $f = 91.3$ MeV and

$\Lambda_0=1.02$ GeV.[3,4] The parameter r is determined from the π and K masses and the values of the current quark masses:

$$r=2m_\pi^2/(m_u+m_d)=2m_K^2/(m_u+m_s)\ . \tag{4}$$

In our numerical calculations only the combination rM was needed and its value was determined by the meson masses m_π^2 and m_K^2. We have also used the values $\Lambda_1=0.93$ GeV and $\Lambda_2=1.26$ GeV, obtained from the best fit to the $K\to 3\pi$ data.[6]

The effective weak interaction is determined in the large-N limit through the procedure of "bosonization" of the weak current. The bosonized weak quark current is[6]

$$\begin{aligned} J_{ij}^\mu &= \bar{q}_i\gamma^\mu(1-\gamma_5)q_j \\ &= if^2\left[\partial^\mu U\,U^\dagger-\frac{r}{2\Lambda_0^2}(M\,\partial^\mu U^\dagger-\partial^\mu U\,M)-\frac{1}{\Lambda_1^2}(U\,\partial^\nu U^\dagger\partial^\nu U\,\partial^\mu U^\dagger-\partial^\mu U\,\partial^\nu U^\dagger\partial^\nu U\,U^\dagger)+\frac{2}{\Lambda_2^2}(U\,\partial^\nu U^\dagger\partial^\mu U\,\partial^\nu U^\dagger)\right]_{ji}\ . \end{aligned} \tag{5}$$

Neglecting CP-nonconserving contributions, the effective weak interaction is then the sum of $\Delta I=\frac{1}{2}$ and $\frac{3}{2}$ parts, which can be written in terms of the quark operators Q_1 and Q_2:

$$\mathcal{L}_{\text{weak}}=-g_0[(Q_2-Q_1)+(\omega/\sqrt{2})(Q_2+2Q_1)]\ . \tag{6}$$

The operators Q_1 and Q_2 are given in terms of quark operators by

$$Q_1=\bar{s}\gamma_\mu(1-\gamma_5)u\;\bar{u}\gamma_\mu(1-\gamma_2)d\ , \tag{7}$$

and

$$Q_2=\bar{s}\gamma_\mu(1-\gamma_5)d\;\bar{u}\gamma_\mu(1-\gamma_5)u\ , \tag{8}$$

where it is understood that the corresponding bosonized weak currents are given by Eq. (5). The values of g_0 and ω in Eq. (6) have previously been determined empirically and are in quite good agreement with the theoretical values found in explicit calculations of the next-to-leading order in N.[4] We shall use the values[6] $g_0=8.8\times10^{-6}$ GeV^2 and $\omega=\frac{1}{22}$ in our numerical calculations.

Finally, the electromagnetic interactions are introduced by gauging the effective strong-interaction Lagrangian [Eq. (1)] and the effective weak interaction [Eq. (6)]. That is, the derivatives $\partial_\mu U$ are replaced by covariant derivatives

$$D_\mu U=\partial_\mu U+ieA_\mu[Q,U]\ , \tag{9}$$

where $Q=\text{diag}(\frac{2}{3},-\frac{1}{3},-\frac{1}{3})$ is the usual 3×3 charge matrix. The lowest-order terms in the interaction which contribute to the processes $K\to\pi\pi\pi\gamma$ are

$$\mathcal{L}_{\text{strong}}^{\text{em}}=ie\frac{f^2}{4}A_\mu\text{Tr}(\partial_\mu U[Q,U^\dagger]+[Q,U]\partial_\mu U^\dagger)\ , \tag{10}$$

and

$$\mathcal{L}_{\text{weak}}^{\text{em}}=ieg_0f^4A_\mu(\partial_\mu U[Q,U^\dagger]+[Q,U]\partial_\mu U^\dagger)_{ds}^{\Delta I=1/2}\ . \tag{11}$$

The $\Delta I=\frac{3}{2}$ contribution is suppressed by a factor $\omega=\frac{1}{22}$ and therefore is negligible.

DECAY AMPLITUDES

The bremsstrahlung contributions A_B to the decay amplitude A are given by the diagrams in Fig. 1. Explicitly, for the decay $K^+\to\pi^+\pi^+\pi^-\gamma$, one finds

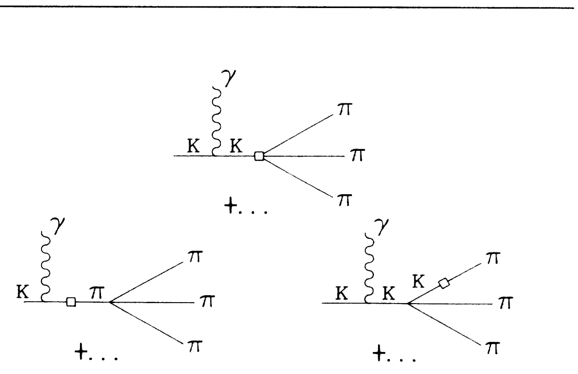

FIG. 1. Bremsstrahlung diagrams for the processes $K(k)\to\pi(p_1)+\pi(p_2)+\pi(p_3)+\gamma(q)$. The weak interaction is indicated by a square box. The ellipses indicate the photon can be emitted from any of the charged mesons.

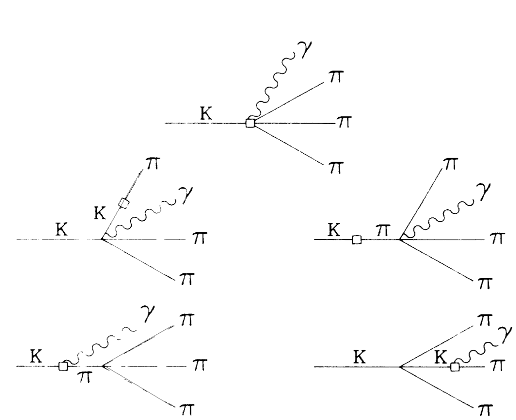

FIG. 2. Direct-emission diagrams for the processes $K(k)\to\pi(p_1)+\pi(p_2)+\pi(p_3)+\gamma(q)$. The weak interaction is indicated by a square box.

$$A_B(K^+\to\pi^+\pi^+\pi^-\gamma)=\tfrac{2}{3}eg_0m_K^2\left[\frac{\epsilon\cdot p_1}{q\cdot p_1}+\frac{\epsilon\cdot p_2}{q\cdot p_2}-\frac{\epsilon\cdot p_3}{q\cdot p_3}-\frac{\epsilon\cdot k}{q\cdot k}\right]\left[1+6\frac{m_\pi^2}{\Lambda_0^2}+\tfrac{4}{3}(m_K^2-3m_\pi^2)\left[\frac{1}{\Lambda_1^2}-\frac{1}{\Lambda_2^2}\right]\right]. \tag{12}$$

In Eq. (12) we have used the value of the $K^+\to\pi^+\pi^+\pi^-$ amplitude in the center of the Dalitz plot and have retained only the leading terms in the photon momentum q.

The direct-emission amplitude $A_{\rm DE}$, which can now be treated systematically in the low-energy effective-Lagrangian framework, is given by the contributions of the diagrams in Fig. 2. The direct-emission amplitude for the process $K^+\to\pi^+\pi^+\pi^-\gamma$ is

$$A_{\rm DE}(K^+\to\pi^+\pi^+\pi^-\gamma)=\tfrac{8}{3}eg_0[\epsilon\cdot(p_1+p_2)-\tfrac{1}{2}\epsilon\cdot p_3]\ , \tag{13}$$

where higher-order terms in the photon momentum q and terms of order m_π^2 and m_K^2 have been neglected. We have also neglected the direct-emission terms corresponding to diagrams in which there are internal η or η' meson poles since these violate strong isospin and are suppressed by the factor (m_d-m_u).

For the process $K^+\to\pi^+\pi^0\pi^0\gamma$, the bremsstrahlung amplitude shown in Fig. 1 is

$$A_B(K^+\to\pi^+\pi^0\pi^0\gamma)=\tfrac{1}{3}eg_0m_K^2\left[\frac{\epsilon\cdot p_3}{q\cdot p_3}-\frac{\epsilon\cdot k}{q\cdot k}\right]\times\left[1+6\frac{m_\pi^2}{\Lambda_0^2}+\tfrac{4}{3}(m_K^2-3m_\pi^2)\left[\frac{1}{\Lambda_1^2}-\frac{1}{\Lambda_2^2}\right]\right]. \tag{14}$$

For this process the leading contributions to the direct-emission amplitude shown in Fig. 2 actually cancel, and the next-to-leading contributions are entirely negligible compared to the bremsstrahlung amplitude of Eq. (14).

For the process $K_L^0\to\pi^+\pi^-\pi^0\gamma$, the bremsstrahlung amplitude shown in Fig. 1 is

$$A_B(K_L^0\to\pi^+\pi^-\pi^0)=-\tfrac{1}{6}eg_0m_K^2\left[\frac{\epsilon\cdot p_1}{q\cdot p_1}-\frac{\epsilon\cdot p_2}{q\cdot p_2}\right]\times\left[1+6\frac{m_\pi^2}{\Lambda_0^2}+\tfrac{4}{3}(m_K^2-3m_\pi^2)\left[\frac{1}{\Lambda_1^2}-\frac{1}{\Lambda_2^2}\right]\right]. \tag{15}$$

The direct-emission amplitude from Fig. 2 corresponding to the process $K_L^0\to\pi^+\pi^-\pi^0\gamma$ is

$$A_{\rm DE}(K_L^0\to\pi^+\pi^-\pi^0\gamma)=\tfrac{5}{3}eg_0\epsilon\cdot(p_1-p_2)\ . \tag{16}$$

NUMERICAL RESULTS

The bremsstrahlung and direct-emission amplitudes must be added coherently to obtain the total amplitude

$$A=A_B+A_{\rm DE}\ , \tag{17}$$

in contrast with the processes $K\to\pi\pi\gamma$, where these two contributions lead to different final states and do not interfere.[11–13] The partial decay width is then obtained by integrating over phase space. This was carried out analytically to obtain the distributions in the energies of the odd pion and photon in the center of mass of the three pions. The resulting formulas are rather lengthy. The integrals over the odd-pion energy and photon energy were then carried out numerically for various values of the minimum-photon-energy cut. The branching ratios relative to the nonradiative $K\to\pi\pi\pi$ partial width, calculated using the same effective Lagrangian, are given in Table I. The measured values[14]

$$\Gamma(K^+\to\pi^+\pi^+\pi^-\gamma)/\Gamma(K^+\to\pi^+\pi^+\pi^-)=(1.79\pm0.72)\times10^{-3}\ ,$$

corresponding to a photon-energy cut $E_{\min}=11$ MeV, and[15]

TABLE I. Branching factions for the $K\to\pi\pi\pi\gamma$ decays assuming various values of the minimum-photon-energy cut $E_{\min}$.

$E_{\min}$ (MeV)	$\frac{\Gamma(K^+\to\pi^+\pi^+\pi^-\gamma)}{\Gamma(K^+\to\pi^+\pi^+\pi^-)}$	$\frac{\Gamma(K^+\to\pi^+\pi^0\pi^0\gamma)}{\Gamma(K^+\to\pi^+\pi^0\pi^0)}$	$\frac{\Gamma(K_L^0\to\pi^+\pi^-\pi^0\gamma)}{\Gamma(K_L^0\to\pi^+\pi^-\pi^0)}$
5	2.43×10^{-3}	2.47×10^{-4}	8.53×10^{-4}
10	1.17×10^{-3}	1.30×10^{-4}	4.76×10^{-4}
11	1.03×10^{-3}		
15	6.19×10^{-4}	7.61×10^{-5}	2.90×10^{-4}
20	3.36×10^{-4}	4.57×10^{-5}	1.80×10^{-4}
25	1.74×10^{-4}	2.75×10^{-5}	1.13×10^{-4}
30	5.71×10^{-5}	1.63×10^{-5}	6.85×10^{-5}

$$\Gamma(K^+\to\pi^+\pi^0\pi^0\gamma)/\Gamma(K^+\to\pi^+\pi^0\pi^0) = (4.28^{+3.18}_{-1.67})\times 10^{-4}\ ,$$

corresponding to a photon-energy cut $E_{\min}=10$ MeV, are in reasonably good agreement with the calculated values.

SUMMARY

The $K\to\pi\pi\pi\gamma$ widths were calculated by introducing gauge-invariant electromagnetic interactions into an effective Lagrangian which describes $K\to\pi\pi\pi$ decays quite well. The direct-emission contributions were systematically included coherently with the bremsstrahlung contributions. The numerical results, which are summarized in Table I, agree reasonably well with the rather imprecise data presently available. Since the direct-emission amplitude interferes coherently with the dominant bremsstrahlung amplitude, unlike the $K\to\pi\pi\gamma$ case, the direct-emission contributions are significant, typically amounting to about 10%. Unfortunately, the present data are not sufficiently precise to exhibit these effects.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation, Division of International Programs, Grant No. JF829, and the Department of Energy. One of us (S.F.) is grateful to Northwestern University for the kind hospitality during their visit.

[1]See, for example, J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985), and references therein.

[2]G.'t Hooft, Nucl. Phys. **B72**, 461 (1974).

[3]W. A. Bardeen, A. J. Buras, and J.-M. Gerard, Phys. Lett. B **192**, 138 (1987).

[4]W. A. Bardeen, A. J. Buras, and J.-M. Gerard, Nucl. Phys. **B293**, 787 (1987).

[5]A. J. Buras and J.-M. Gerard, Nucl. Phys. **B264**, 371 (1986).

[6]S. Fajfer and J.-M. Gerard, Z. Phys. C **42**, 425 (1989).

[7]H. Y. Cheng, C. Y. Cheung, and W. B. Yeung, Z. Phys. C **43**, 391 (1989).

[8]J. F. Donoghue, E. Golowich, and B. R. Holstein, Phys. Rev. D **30**, 587 (1984).

[9]J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971).

[10]E. Witten, Nucl. Phys. **B223**, 422 (1983).

[11]S. Fajfer, Z. Phys. C **45**, 293 (1989).

[12]H. Y. Cheng, S. C. Lee, and H. L. Yu, Z. Phys. C **41**, 223 (1988).

[13]Y. C. R. Lin and G. Valencia, Phys. Rev. D **36**, 1423 (1987).

[14]P. Stamer *et al.*, Phys. Rev. **138**, B440 (1965).

[15]V. N. Bolotov *et al.*, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 390 (1985) [JETP Lett. **42**, 481 (1985)].

[16]R. H. Dalitz, Phys. Rev. **99**, 915 (1955).