Decay and evolution of the neutral kaon

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(Received 16 April 1990)

The time evolution of the neutral-kaon complex is investigated, making use of the constraints of *CPT* invariance, a non-negative energy spectrum, and the corrections to the Lee-Oehme-Yang phenomenological theory. Some numerical estimates are made which are compared with the work of Khalfin, which stimulated the present work.

I. INTRODUCTION

The study of the decay of a metastable quantum system began with Gamow's theory of the α decay[1] of atomic nuclei and Dirac's theory of spontaneous emission of radiation by excited atoms.[2] A general treatment of decaying systems was given by Weisskopf and Wigner,[3] which led to the familiar exponential decay. That the decay could not be strictly exponential for a quantum system with an energy spectrum bounded from below was pointed out by Khalfin three decades ago.[4] The exact solution to a decaying system of the kind that Dirac considered was investigated by Friedrichs[5] and in an elegant form by Lee,[6] followed by Glaser and Källen.[7] (See also the papers by Höhler, Williams, and Fleming.[8])

The short-term behavior of decaying system and the quantum Zeno effect were investigated by Misra and Sudarshan[9] and Chiu, Misra, and Sudarshan.[10] (See also the papers by Ghirardi *et al.*, Peres, Fleming,[11] and Valanju.[12]) The quantum Zeno effect has been verified by Itano, Heinzen, Bollinger, and Wineland[13] using metastable atoms "interrogated" by microwaves.

In view of these, it would be desirable to consider the decay of a more complex system consisting of two (or more) communicating metastable states, particularly in view of the recent refinements in atomic physics techniques including ion traps. We have carried out such an investigation in great generality, which we will present elsewhere.

In this paper we have reexamined the important special case of the decay of the neutral-kaon system. Three and half decades ago, Gell-Mann and Pais[14] pointed out that K^0 and $\bar{K}^0$ communicated via the decay channels and therefore the decay contained two superpositions K_1 and K_2, which were the orthonormal combinations of K^0 and $\bar{K}^0$, which were, respectively, even and odd under charge conjugation. With the discovery of parity violation and *CP* conservation, the terms K_1 and K_2 were redefined to correspond to, respectively, *CP*-even and -odd superpositions. With the discovery of the small *CP* violation, qualitatively new phenomena were obtained with nonorthonormal short- and long-lived neutral kaons K_S and K_L. Lee, Oehme, and Yang[15] formulated the necessary generalization of the Weisskopf-Wigner formalism, which has been used in the discussion of the empirical data.[16,17] This phenomenological theory has the same kind of shortcoming as the Weisskopf-Wigner theory as discussed earlier.

Khalfin has pointed out[18,19] some of these theoretical deficiencies and gave some estimates of the departure from the Lee-Oehme-Yang (LOY) theory to be expected in the neutral-kaon system as well as in the $D^0\bar{D}^0$ and $B^0\bar{B}^0$ systems. He asserts that there are possibly measurable "new *CP*-violation effects." We have reexamined this question in detail, formulated a generic solvable model, and studied the exact solution. While bearing out the need to upgrade the LOY formalism to be in accordance with the boundedness from below of the total Hamiltonian, our estimates of the corrections are more modest than Khalfin's. We review Khalfin's work to pose the problem and establish notation.

In the LOY formalism, the short- and long-lived particles are linear combinations of K^0 and $\bar{K}^0$:

$$\begin{bmatrix} |K_S\rangle \\ |K_L\rangle \end{bmatrix} = U \begin{bmatrix} |K^0\rangle \\ |\bar{K}^0\rangle \end{bmatrix}, \quad U = \begin{bmatrix} p & -q \\ p' & q' \end{bmatrix}, \tag{1.1}$$

with $|p|^2+|q|^2=1$ and $|p'|^2+|q'|^2=1$. The parameters p,q,p',q' are complex; their phases may be altered by redefining the phases of $|K_S\rangle$ and $|K_L\rangle$. Generally, the states are not orthogonal, but linearly independent:

$$\langle K_L|K_S\rangle = p'^*p - q'^*q \neq 0 . \tag{1.2}$$

Let i denote $K^0,\bar{K}^0$, and α denote K_S,K_L. Equation (1.1) can be rewritten as

$$|\alpha\rangle = \sum_i |i\rangle\langle i|\alpha\rangle \equiv \sum_i |i\rangle R_{i\alpha} , \tag{1.3}$$

where $R = U^T$. For a right eigenstate $|\alpha\rangle$, let the corresponding left eigenstate be $\langle\tilde{\alpha}|$. Then in terms of the oblique bases,

$$|i\rangle = \sum_\alpha |\alpha\rangle\langle\tilde{\alpha}|i\rangle = \sum_\alpha |\alpha\rangle R^{-1}_{\alpha i} . \tag{1.4}$$

Let the "time-evolution matrix " of K^0 and $\bar{K}^0$ states be defined by

$$\begin{bmatrix} |K^0(t)\rangle \\ |\bar{K}^0(t)\rangle \end{bmatrix} = A(t) \begin{bmatrix} |K^0\rangle \\ |\bar{K}^0\rangle \end{bmatrix} , \tag{1.5}$$

with $A_{ij}(t)=\langle i|e^{-iHt}|j\rangle$, and the corresponding matrix in the K_S and K_L bases by

$$\begin{bmatrix} |K_S(t)\rangle \\ |K_L(t)\rangle \end{bmatrix} = B(t) \begin{bmatrix} |K_S\rangle \\ |K_L\rangle \end{bmatrix} , \tag{1.6}$$

with $B_{\alpha\beta}(t)=\langle\tilde{\alpha}|e^{-iHt}|\beta\rangle$. The matrices A and B can be related in the following way:

$$A_{ij}=\sum_{\alpha,\beta}\langle i|\alpha\rangle\langle\tilde{\alpha}|e^{-iHt}|\beta\rangle\langle\tilde{\beta}|j\rangle = (RBR^{-1})_{ij} . \tag{1.7}$$

As in the LOY theory, for the time being, if we were to assume that K_L and K_S do not regenerate into each other,

$$B(t)=\begin{bmatrix} S(t) & 0 \\ 0 & L(t) \end{bmatrix} . \tag{1.8}$$

Then

$$A(t)=RB(t)R^{-1} = \frac{1}{pq'+p'q}\begin{bmatrix} pq'S+qp'L & -pp'(S-L) \\ -qq'(S-L) & qp'S+pq'L \end{bmatrix} . \tag{1.9}$$

At this point let us invoke CPT invariance, which implies $A_{11}=A_{22}$ or $pq'(S-L)=qp'(S-L)$. Since K_L and K_S are states with distinct masses and lifetimes, $S-L\neq 0$. In turn, $p/q=p'/q'$. The states $|K_S\rangle$ and $|K_L\rangle$ are defined only to within phases of our choice; we may therefore set $p'=p$ and $q'=q$. At this point we shall relax the normalization condition on p and q and write $|p|^2+|q|^2=\zeta^2$. The transformation matrix and its inverse are now given by

$$R=\frac{1}{\zeta}\begin{bmatrix} p & p \\ -q & q \end{bmatrix}, \quad R^{-1}=\frac{\zeta}{2pq}\begin{bmatrix} q & -p \\ q & p \end{bmatrix} . \tag{1.10}$$

We shall adhere to this convention in the rest of this paper. Equation (1.9) also implies that the ratio of the off-diagonal elements, that is, the ratio of the transition amplitude of $\bar{K}^0$ to K^0 to that of K^0 to $\bar{K}^0$, is given by

$$r(t)=\frac{A_{12}(t)}{A_{21}(t)}=\frac{p^2}{q^2}=\text{const} . \tag{1.11}$$

To sum up, the assumptions that (i) K_S and K_L are superpositions of K^0 and $\bar{K}^0$ states, (ii) there is no regeneration between K_S and K_L, and (iii) CPT invariance holds, imply the constancy of $r(t)$. Khalfin's theorem states that if the ratio $r(t)$ of (1.11) is constant, then the magnitude of this ratio must be unity. His proof goes as follows.

The matrix elements $A_{ij}(t)$ are given by the Fourier transform of the corresponding energy spectra, i.e.,

$$A_{ij}(t)=\int_0^\infty d\lambda\, e^{-i\lambda t}C_{ij}(\lambda) , \tag{1.12}$$

where

$$C_{ij}(\lambda)=\sum_n \langle i|\lambda n\rangle\langle\lambda n|j\rangle . \tag{1.13}$$

The summation is over different channels. λ is the energy variable. To be precise, it is the difference between the relevant energy and the threshold value. So $\lambda=0$ is the lower bound of the spectrum. Equation (1.13) implies that

$$C_{ij}(\lambda)=C^*_{ji}(\lambda) . \tag{1.14}$$

Assume there exists some constant r, such that, for $t\geq 0$,

$$D(t)=A_{12}(t)-A_{21}(t) = \int_0^\infty d\lambda\, e^{-i\lambda t}[C_{12}(\lambda)-rC_{21}(\lambda)] = 0 . \tag{1.15}$$

Positivity of λ within the range of integration implies that $D(t)$ is an analytic function of t in the lower half t plane. So $D(t)$ vanishes in the entire lower half t plane. By the Paley-Wiener theorem, it implies the vanishing of $D(t)$ along its boundary, or

$$D(t)=0 \quad \text{for } -\infty<t<\infty . \tag{1.16}$$

Inverse Fourier transform of $D(t)$ implies

$$C_{12}(\lambda)-rC_{21}(\lambda)=C_{12}(\lambda)-rC^*_{12}(\lambda)=0 , \tag{1.17a}$$

or

$$|r|=1 . \tag{1.17b}$$

This conclusion contradicts the expectation of the LOY theory. In particular, when there is CP violation, it is expected that

$$|r|=\left|\frac{p}{q}\right|^2=\text{const}\neq 1 . \tag{1.18}$$

We have investigated the situation in the framework of the Friedrichs-Lee model in the lowest section with the particle V_1 and its antiparticle V_2. They are coupled to an arbitrary number of continuum $N\theta$ channels. We express the time-evolution matrix in terms of pole contributions plus a background contribution. We show that because of the form-factor effect, both the correction to the pole contribution and the background contribution give rise to a tiny regeneration between K_L and K_S. This invalidates one of the original assumptions needed to conclude the constancy of the ratio $r(t)$. Therefore, in the generic Friedrichs-Lee model, the constancy of this ratio does not obtain.

Khalfin[18,19] also predicted an appreciable "new CP-nonconservation effect" in the ratio $r(t)$. Our investigation does *not* confirm his prediction. For instance, for the neutral-K system our result gives an effect which is 12 orders of magnitude smaller than that predicted by Khalfin. On the other hand, our investigation implies that for the type of quantum system we are studying, there is a certain parameter region, for which the background contribution, including a new CP-nonconserving effect, can be appreciable.

The outline of the discussions below is as follows. Section II sets up the dynamical system and derives the expression of the time-evolution matrix. Section III evaluates the pole and background contributions to the time-evolution matrix. In Sec. IV we investigate the implications of our solution.

II. TWO-LEVEL MULTICHANNEL SYSTEM AND TIME-EVOLUTION MATRIX

A. Eigenvalue problem

In the generalized Friedrichs-Lee model,[5,6] the Hamiltonian is given by

$$H=\sum_{i,j} m_{ij}V_i^\dagger V_j+\sum_{n=1}^{N}\mu_n N_n^\dagger N_n+\int_0^\infty d\omega\,\omega\phi^*(\omega)\phi(\omega)+\int_0^\infty d\omega\sum_{i,n} g_{in}(\omega)V_i N_n^\dagger\phi^*(\omega)+\int_0^\infty d\omega\sum_{i,n} g_{in}^*(\omega)V_i^\dagger N_n\phi(\omega)\ . \quad (2.1)$$

Here the bare particles are V_1, V_2, N_n ($1\le n\le N$), and θ particles. The following number operators commute with the Hamiltonian:

$$Q_1=\sum_i V_i^\dagger V_i+\sum_n N_n^\dagger N_n\ ,\qquad Q_2=\sum_n N_n^\dagger N_n-\int d\omega\,\phi^*(\omega)\phi(\omega)\ . \quad (2.2)$$

Denote the corresponding eigenvalues by q_1 and q_2. The Hilbert space of the Hamiltonian is divided into sectors, each with a different assignment of q_1 and q_2 values. We will only consider the eigenstates of the lowest nontrivial sector, where $q_1=1$ and $q_2=0$. Here the bare states are labeled by $|V_1\rangle$, $|V_2\rangle$, and $|n,\omega\rangle$, with $n=1,2,\ldots,N$. Since there are N independent continuum states, for each eigenvalue λ, there are N independent eigenstates which can be written as

$$|\lambda,n\rangle=\sum_i |V_i\rangle[a_\lambda]_{in}+\int_0^\infty d\omega\sum_m |m,\omega\rangle[b_\lambda(\omega)]_{mn}\ , \quad (2.3)$$

where

$$[a_\lambda]_{in}=\langle V_i|\lambda,n\rangle,\quad [b_\lambda(\omega)]_{mn}=\langle m,\omega|\lambda,n\rangle\ . \quad (2.4)$$

In (2.3) the integration variable of the $|m,\omega\rangle$ state, ω, begins from 0. So it now stands for the difference between the energy of the state and the threshold energy.

Using the Einstein product convention, the corresponding eigenvalue equation is given by

$$\begin{bmatrix} m_{ij} & g_{il}(\omega') \\ g_{mj}^\dagger(\omega) & \omega\delta(\omega-\omega')\delta_{ml}\end{bmatrix}\begin{bmatrix}[a_\lambda]_{jn}\\ [b_\lambda(\omega')]_{ln}\end{bmatrix}=\lambda\begin{bmatrix}[a_\lambda]_{in}\\ [b_\lambda(\omega)]_{mn}\end{bmatrix}\ . \quad (2.5)$$

For brevity, hereafter we will suppress the matrix indices. Equation (2.5) leads to

$$(\lambda I-m)a_\lambda=\langle g(\omega')b_\lambda(\omega')\rangle\ , \quad (2.6)$$

$$(\lambda-\omega)b_\lambda(\omega)=g^\dagger(\omega)a_\lambda\ , \quad (2.7)$$

where

$$\langle\cdots\rangle\equiv\int_0^\infty d\omega\cdots\ .$$

We choose the boundary condition such that, in the uncoupled limit, b_λ is given by

$$[b_\lambda(\omega)]_{mn}=\delta(\lambda-\omega)\delta_{mn}\ . \quad (2.8)$$

Such a solution is given by

$$b_\lambda(\omega)=\delta(\lambda-\omega)I+\frac{g^\dagger(\omega)a_\lambda}{\lambda-\omega+i\epsilon}\ . \quad (2.9)$$

Substituting (2.9) into (2.6) leads to

$$(\lambda I-m)a_\lambda=g(\lambda)+\left\langle\frac{g(\omega')g^\dagger(\omega')}{\lambda-\omega'+i\epsilon}\right\rangle a_\lambda\ , \quad (2.10)$$

or

$$a_\lambda=K^{-1}g\ , \quad (2.11)$$

where

$$K=\lambda I-m-G(\lambda) \quad (2.12)$$
$$=\begin{bmatrix}\lambda-m_{11}-G_{11} & -m_{12}-G_{12}\\ m_{21}-G_{21} & \lambda-m_{22}-G_{22}\end{bmatrix}\ ,$$

with

$$G(\lambda+i\epsilon)=\left\langle\frac{g(\omega)g^\dagger(\omega)}{\lambda-\omega+i\epsilon}\right\rangle=\int_0^\infty d\omega\frac{g(\omega)g^\dagger(\omega)}{\lambda-\omega+i\epsilon}\ . \quad (2.13)$$

B. Time-evolution matrix

It follows from (2.13), that, for λ real,

$$[G(\lambda+i\epsilon)]^\dagger=\int_0^\infty d\omega\frac{g(\omega)g^\dagger(\omega)}{\lambda-\omega-i\epsilon}=G(\lambda-i\epsilon)\ . \quad (2.14)$$

This in turn implies the identity that, for real λ,

$$G(\lambda+i\epsilon)-G^\dagger(\lambda+i\epsilon)=-2\pi i g(\lambda)g^\dagger(\lambda)=K^\dagger(\lambda+i\epsilon)-K(\lambda+i\epsilon)\ . \quad (2.15)$$

The time-evolution matrix is easily evaluated:

$$A_{ij}(t)=\langle i|e^{-iHt}|j\rangle=\int_0^\infty d\lambda\, e^{-i\lambda t}\sum_n\langle i|\lambda n\rangle\langle\lambda n|j\rangle=\int_0^\infty d\lambda\, e^{-i\lambda t}[a(\lambda)a^\dagger(\lambda)]_{ij}\ . \quad (2.16)$$

From (2.11) and (2.15),

$$aa^\dagger=K^{-1}gg^\dagger(K^{-1})^\dagger=K^{-1}\left[\frac{K^\dagger-K}{-2\pi i}\right](K^{-1})^\dagger=\frac{i}{2\pi}[K^{-1}-(K^{-1})^\dagger]\ . \quad (2.17)$$

Substituting (2.17) into (2.16), we get

$$A_{ij}(t)=\frac{i}{2\pi}\int_0^\infty d\lambda\, e^{-i\lambda t}\{K^{-1}(\lambda+i\epsilon) -[K^{-1}(\lambda+i\epsilon)]^\dagger\} . \quad (2.18)$$

But

$$[K^{-1}(\lambda+i\epsilon)]^\dagger=\{[\lambda-m-G(\lambda+i\epsilon)]^\dagger\}^{-1} =[K(\lambda-i\epsilon)]^{-1} . \quad (2.19)$$

Based on (2.19), (2.18) can be written in a contour integral representation (see Fig. 1):

$$A_{ij}(t)=\frac{i}{2\pi}\int_C d\lambda\, e^{-i\lambda t}[K^{-1}(\lambda)]_{ij} =\frac{i}{2\pi}\int_C d\lambda\, e^{-i\lambda t}\frac{N_{ij}(\lambda)}{\Delta(\lambda)} , \quad (2.20)$$

where

$$\Delta=\det K , \quad (2.21a)$$

and

$$N(\lambda)=\mathrm{Cof}K=\begin{bmatrix}\lambda-m_{22}-G_{22} & m_{12}+G_{12}\\ m_{21}+G_{21} & \lambda-m_{11}-G_{11}\end{bmatrix} . \quad (2.21b)$$

Since $G(\lambda)$ is defined through the dispersion integral (2.13), the λ dependence of G, in turn, the integrand of (2.20) may be extended to the entire cut plane of λ.

C. Completeness relation

At $t=0$, from (2.16) and (2.20),

$$A_{ij}(0)=\int_0^\infty d\lambda \sum_n \langle i|\lambda n\rangle\langle \lambda n|j\rangle =\frac{i}{2\pi}\int_C d\lambda \frac{N_{ij}(\lambda)}{\Delta(\lambda)} . \quad (2.22)$$

From (2.21) the asymptotic behaviors are

$$N_{ij}(\lambda)\to\lambda \quad \text{for } i=j ,$$
$$N_{ij}(\lambda)\to\lambda^0, \quad \text{for } i\neq j , \quad (2.23)$$
$$\Delta(\lambda)\to\lambda^2 .$$

Deform the contour as depicted in Fig. 1. Using (2.23), we arrive at

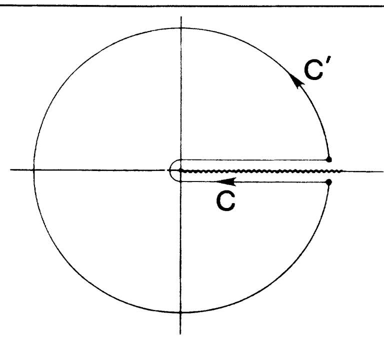

FIG. 1. The contours C and C' in the complex λ plane.

$$A_{ij}(0)=-\frac{i}{2\pi}\int_{C'} d\lambda \frac{N_{ij}(\lambda)}{\Delta(\lambda)}=\delta_{ij} , \quad (2.24)$$

or

$$\int_0^\infty d\lambda \sum_n \langle i|\lambda n\rangle\langle\lambda n|j\rangle=\delta_{ij} , \quad (2.25)$$

which is the completeness relation.

III. POLES AND BACKGROUND INTEGRAL OF $A_{12}(t)$

So far our treatment has been general. Now we want to specialize in the neutral-K system. We identity K^0 and $\bar K^0$ as V_1 and its antiparticle V_2, respectively, and proceed to evaluate the time-evolution matrix specified by (2.20) and (2.21). The 12 element of this matrix is given by

$$A_{12}(t)=\langle K^0|e^{-iHt}|\bar K^0\rangle =\frac{i}{2\pi}\int_C d\lambda\, e^{-i\lambda t}\frac{N_{12}}{\Delta(\lambda)} , \quad (3.1)$$

where $N_{12}=m_{12}+G_{12}$. For λ real and $\lambda>0$,

$$\frac{G_{12}(\lambda+i\epsilon)-G^*_{12}(\lambda+i\epsilon)}{2i} =-\pi[g(\lambda)g^\dagger(\lambda)]_{12} =-\pi\sum_n\langle K^0|H|\lambda n\rangle\langle\lambda n|H|\bar K^0\rangle . \quad (3.2)$$

In the Weisskopf-Wigner approximation,

$$G_{12}(\lambda)=-i\Gamma_{12}=\text{const} .$$

In general, there is λ dependence in G. With CP violation, there are also complex phases in the off-diagonal elements of $[g(\lambda)g^\dagger(\lambda)]$. We assume that these phases are global phases; i.e., they are independent of λ. We may write

$$[g(\lambda)g^\dagger(\lambda)]_{ij}=\Gamma_{ij}f_{ij}(\lambda) , \quad (3.3)$$

where $f_{ij}(\lambda)=f^*_{ij}(\lambda)$, Γ_{ij} is independent of λ, and it contains global phases. By construction f_{ij} is real. Introduce the form-factor matrix F such that

$$G_{ij}(\lambda)=-i\Gamma_{ij}F_{ij}(\lambda) , \quad (3.4)$$

where

$$F_{ij}(\lambda)=i\int\frac{f_{ij}(z)}{\lambda-z+i\epsilon}dz .$$

CPT invariance and Hermiticity of the Hamiltonian imply, respectively,

$$\Gamma_{11}=\Gamma_{22},\quad F_{11}=F_{22} , \quad (3.5a)$$

and

$$\Gamma_{12}=\Gamma^*_{21},\quad F_{12}=F_{21} . \quad (3.5b)$$

To arrive at the last two equalities, the following consideration was made. From (3.3),

$$[g(\lambda)g^{\dagger}(\lambda)]_{21}=[g(\lambda)g^{\dagger}(\lambda)]_{12}^{*}\ ,$$

or

$$\Gamma_{21}f_{21}=\Gamma_{12}^{*}f_{12}^{*}=\Gamma_{12}^{*}f_{12}\ .$$

This leads to the identities

$$\Gamma_{21}=\Gamma_{12}^{*}\ \text{ and } f_{21}=f_{12}\ .$$

The last equality together with the definition of F in (3.4) implies $F_{21}=F_{12}$. It turns out that our conclusions below are not sensitive to the details of the form factors assumed. For simplicity, hereafter we will work with one common form factor and rewrite

$$G_{ij}=-i\Gamma_{ij}F(\lambda)\ , \tag{3.6}$$

with $F(m_{11})=1$. Note that the normalization point of F is chosen to be at $\lambda=m_{11}$.

From (3.6) we obtain the subtracted dispersion relation

$$F(\lambda)=1-i(\lambda-m_{11})\times\int_0^{\infty}\frac{dzf(z)}{(\lambda-z+i\epsilon)(m_{11}-z+i\epsilon)}\ . \tag{3.7}$$

For the weight function, we choose the simple form

$$f(z)=\text{const}\times\frac{\sqrt{z}}{z+\omega_0}\ , \tag{3.8}$$

which has the expected square-root threshold behavior[10] and provides enough convergence in the dispersion integral. Substituting (3.8) into (3.7) after some algebra we get

$$\begin{aligned}F(\lambda)&=n\left[\frac{\sqrt{\lambda}+i\sqrt{\omega_0}}{\lambda+\omega_0}+C\right]\\&=n\left[\frac{1}{\sqrt{\lambda}+i\sqrt{\omega_0}}+C\right]\ ,\end{aligned} \tag{3.9}$$

where the normalization factor and the subtraction term are, respectively,

$$n=\frac{m_{11}+\omega_0}{\sqrt{m_{11}}}\ \text{ and }\ C=\frac{i\sqrt{\omega_0}}{m_{11}+\omega_0}\ . \tag{3.10}$$

From (2.12), (2.21), and (3.6), the denominator in the integrand of (3.1),

$$\begin{aligned}\Delta&=\det K\\&=\begin{vmatrix}\lambda-m_{11}-i\Gamma_{11}F & m_{12}-i\Gamma_{12}F\\ m_{12}^{*}-i\Gamma_{12}^{*}F & \lambda-m_{11}-i\Gamma_{11}F\end{vmatrix}\\&\equiv[\lambda-\lambda_S(\lambda)][\lambda-\lambda_L(\lambda)]\ ,\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}\lambda_{S,L}(\lambda)&=m_{11}-i\Gamma_{11}F(\lambda)\pm d(\lambda)\ ,\\ d(\lambda)&=\tfrac{1}{2}[\lambda_S(\lambda)-\lambda_L(\lambda)]\\&=\{[m_{12}-i\Gamma_{12}F(\lambda)][m_{12}^{*}-i\Gamma_{12}^{*}F(\lambda)]\}^{1/2}\ .\end{aligned} \tag{3.12}$$

Substituting (2.21), (3.6), (3.11), and (3.12) into (3.1), we get

$$A_{12}(t)=\frac{i}{2\pi}\int_C d\lambda\, e^{-i\lambda t}\frac{m_{12}-i\Gamma_{12}F(\lambda)}{2d(\lambda)}\times\left[\frac{1}{\lambda-\lambda_S(\lambda)}-\frac{1}{\lambda-\lambda_L(\lambda)}\right]\ . \tag{3.13}$$

The contour C of (3.13) is illustrated in Fig. 1. It is to be deformed according to Fig. 2, such that the integral of (3.13) can be written as the sum of pole contributions and the background contribution.

We shall work in the approximation

$$\frac{\Gamma_{12}}{m_{11}},\ \frac{\Gamma_{11}}{m_{11}},\ \text{and}\ \frac{m_{12}}{m_{11}}\ll 1\ , \tag{3.14}$$

keeping terms only up to first order in these ratios.

A. Pole contribution

In the integrand of (3.13), the poles occur on the second sheet. They are at

$$\begin{aligned}\lambda_S&=m_{11}-i\Gamma_{11}F(\lambda_S)+d(\lambda_S)\ ,\\ \lambda_L&=m_{11}-i\Gamma_{11}F(\lambda_L)-d(\lambda_L)\ .\end{aligned} \tag{3.15}$$

Their contributions lead to

$$A_{12}(t)|_{\text{poles}}=\frac{1}{2}\left[\frac{(m_{12}-i\Gamma_{12}F_S)e^{-i\lambda_S t}}{d_S(1-\lambda_S')}-\frac{(m_{12}-i\Gamma_{12}F_L)e^{-i\lambda_L t}}{d_L(1-\lambda_L')}\right]\ , \tag{3.16}$$

where $F_S=F(\lambda_S)$, $d_S=d(\lambda_S)$, and $\lambda_S'=\lambda_S'(\lambda_S)$, and similarly for quantities with subscript L.

Order-of-magnitude estimates: In order to simplify (3.16), we first estimate the order of magnitude of λ', at $\lambda=\lambda_S$ and λ_L. From (3.12),

$$\lambda_{S,L}'(\lambda)=-i\Gamma_{11}F'(\lambda)\pm d'(\lambda)\ .$$

From (3.9) and (3.10), at $\lambda\sim m_{11},\lambda_S,\lambda_L$,

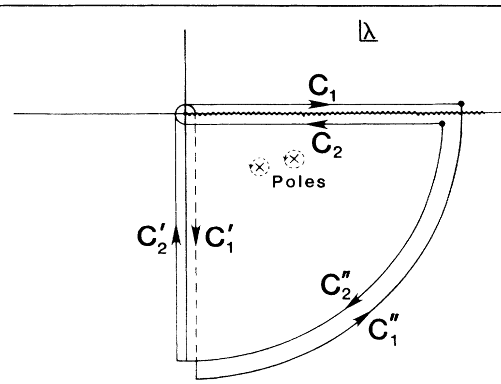

FIG. 2. Illustration of the deformation of contours C_1 and C_2 into the pole contributions plus the background contribution.

$$F'=n\frac{d}{d\lambda}\left[\frac{1}{\sqrt{\lambda}+i\sqrt{\omega_0}}+C\right]$$
$$=-\frac{m_{11}+\omega_0}{2\sqrt{m_{11}}}\frac{1}{\sqrt{m_{11}}(\sqrt{m_{11}}+i\sqrt{\omega_0})^2}$$
$$=O(1/m_{11}) \ . \quad (3.17)$$

To arrive at the last step, we make use of the fact that

$$|F'|^2=\frac{1}{4m_{11}^2} \ . \quad (3.18)$$

While the estimate in Eq. (3.17) depends on the specific form of the form factor assumed, the conclusion that the order of magnitude of the slope of F is characterized by the inverse of a hadronic scale is presumably general. This leads to

$$F',F_S',F_L'\sim 0(1/m_{11}) \ ,$$
$$d',d_S',d_L'=O(\Gamma_{12}/m_{11},m_{12}/m_{11}) \ ,$$

and

$$\lambda_S',\lambda_L',(\lambda_S'-\lambda_L')=O(\Gamma_{11}/m_{11},\Gamma_{12}/m_{11}) \ .$$

In (3.16) the denominator of the $\lambda=\lambda_L$ pole term

$$d_L(1-\lambda_L')=[d_S+d'(\lambda_L-\lambda_S)][1-(\lambda_S'-2d')]$$
$$\approx d_S(1-\lambda_S') \ . \quad (3.19)$$

Substituting (3.19) into (3.16),

$$A_{12}(t)|_{\text{poles}}=\frac{1}{2d_S(1-\lambda_S')}\times[(m_{12}-i\Gamma_{12}F_S)e^{-i\lambda_S t}-(m_{12}-i\lambda_{12}F_L)e^{-i\lambda_L t}] \ . \quad (3.20)$$

B. Background integral

In Fig. 2 the contribution of contours C_1'' and C_2'' vanishes, and so the background integral is along the contour C_1' and C_2'. Introduce the new variable r, such that

$$\text{along } C_1':\ \lambda=e^{-i\pi/2}r^2,\quad \sqrt{\lambda}=e^{-i\pi/4}r \ ,$$
$$\text{along } C_2':\ \lambda=e^{(3\pi/2)i}r^2,\quad \sqrt{\lambda}=-e^{-i\pi/4}r \ . \quad (3.21)$$

The r integrations along C_2' and C_1' are, respectively, from $-\infty$ to 0 and from 0 to ∞. In terms of the r integration, the background integral of (3.13) is given by

$$A_{12}(t)|_{\text{bk}}=\frac{i}{2\pi}\int_{-\infty}^{\infty}2r\,dr\,e^{-r^2 t}\frac{m_{12}-i\Gamma_{12}F(\lambda)}{2d(\lambda)}\times\left[\frac{1}{r^2-i\lambda_S(\lambda)}-\frac{1}{r^2-i\lambda_L(\lambda)}\right] , \quad (3.22)$$

where $\lambda=e^{-i\pi/2}r^2$.

In (3.22) we have dropped terms of order of $(\Gamma_{11}/m_{11})^2$, $(\Gamma_{12}/m_{11})^2$, and $(m_{12}/m_{11})^2$. Furthermore, since the r range is symmetric about $r=0$, only even r terms contribute to the integral and Eq. (3.22) becomes

$$A_{12}(t)|_{\text{bk}}=i\Gamma_{12}J(t) \ , \quad (3.23)$$

where

$$J(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{r\,dr\,e^{-r^2 t}F(\lambda)}{(r^2-im_{11})^2}$$
$$=\frac{m_{11}+\omega_0}{\sqrt{m_{11}}}e^{i\pi/4}I_2(-im_{11},i\omega_0,t) \ . \quad (3.24)$$

The analytic form of I_2 is given by (A3) and (A4) in Appendix A. Thus the background contribution is readily evaluated.

C. $A_{12}(t)$ in the small- and large-t regions

The background contribution near $t=0$ can be read off from (B7) in Appendix B. From (3.23) this leads to

$$A_{12}(t)|_{\text{bk}}=i\Gamma_{12}\left\{F'-\left[\left[\frac{\omega_0}{m_{11}}\right]^{1/2}\frac{\omega_0}{m_{11}+\omega_0}+i\left[\frac{1}{2}+\frac{\omega_0}{m_{11}+\omega_0}\right]\right]t\right\} . \quad (3.25)$$

Expanding the pole contribution of (3.16) gives

$$A_{12}(t)|_{\text{poles}}=\frac{1}{2d_S(1-\lambda_S')}[(m_{12}-i\Gamma_{12}F_S)(1-i\lambda_S t)-(m_{12}-i\Gamma_{12}F_L)(1-i\lambda_L t)]$$
$$\approx-[i\Gamma_{12}F'+im_{12}t+(1+m_{11}F')\Gamma_{12}t] \ . \quad (3.26)$$

For small t, substituting the relation at poles,

$$1+m_{11}F'=\left[\frac{1}{2}+\frac{\omega_0}{m_{11}+\omega_0}\right]+\frac{i\sqrt{\omega_0 m_{11}}}{m_{11}+\omega_0} , \quad (3.27)$$

into Eq. (3.26), together with (3.25), we arrive at

$$A_{12}(t)=A_{12}(t)|_{\text{bk}}+A_{12}(t)|_{\text{poles}}$$
$$=-i\left[m_{12}+\Gamma_{12}\left[\frac{\omega_0}{m_{11}}\right]^{1/2}\right]t \ . \quad (3.28)$$

The large-t behavior of the background contribution can be evaluated using the identity (A9), i.e., for $at,bt\gg 1$:

$$I_2(a,b,t)=\frac{1}{2\sqrt{\pi}a^2 b}\frac{1}{t^{3/2}} \ . \quad (3.29)$$

So for $m_{11}t,\omega_0 t\gg 1$, substituting (3.24) and (3.29) into (3.23), we get

$$A(t)|_{\text{bk}}\simeq\frac{-e^{i\pi/4}}{2\sqrt{\pi}}\left[1+\frac{\omega_0}{m_{11}}\right]\frac{\Gamma_{12}}{\omega_0}\frac{1}{(m_{11}t)^{3/2}} \ . \quad (3.30)$$

IV. IMPLICATIONS OF PRESENT SOLUTION

A. Ratio $r(t)$

Poles only: Consider the pole contribution in Eq. (3.20). First, ignore the form-factor effect, which amounts to setting

$$F_S=F_L=1,\quad \lambda_S'=\lambda_L'=0,\quad d=pq \ , \quad (4.1)$$

where $p^2=m_{12}-i\Gamma_{12}$, $q^2=m_{12}^*-i\Gamma_{12}^*$. In turn, (3.16) becomes

$$A_{12}(t)|_{\text{poles}}=\frac{p}{2q}(e^{-i\lambda_S t}-e^{-i\lambda_L t}) \ . \quad (4.2)$$

This is the same expression as that in LOY theory. It leads to the ratio

$$r(t)|_{\text{poles}}=\left.\frac{A_{12}(t)}{A_{21}(t)}\right|_{\text{poles}}=\frac{p^2}{q^2}=\text{const} . \quad (4.3)$$

where there is CP violation, $|p^2/q^2|\neq 1$. The departure of the ratio $r(t)$ from the value of p^2/q^2 comes about when the form-factor effect is taken into account. This occurs both through the correction to pole contribution of (4.2) and also through the background integral contribution.

First, consider the correction to the pole contribution. Go back to (3.20). We expand the form factor about the normalization point $\lambda=m_{11}$. Here

$$F_L=1-F'd_S, \quad F_S=1+F'd_S . \quad (4.4)$$

Thus

$$A_{12}(t)|_{\text{poles}}=\frac{p}{2q}[(1-\delta)e^{-i\lambda_S t}-(1+\delta)e^{-i\lambda_L t}] , \quad (4.5)$$

where

$$\delta=i\Gamma_{12}F'\frac{q}{p}=O(\Gamma_{12}/m_{11}) . \quad (4.6)$$

Departure from the constancy of the ratio comes about because of the presence of the δ term. Since δ is a first-order small quantity, we can expand the ratio in δ and obtain

$$r(t)|_{\text{poles}}=\left.\frac{A_{12}(t)}{A_{21}(t)}\right|_{\text{poles}} =\frac{p^2}{q^2}\left[1-\Delta\delta\frac{e^{-i\lambda_S t}+e^{-i\lambda_L t}}{e^{-i\lambda_S t}-e^{-i\lambda_L t}}\right] , \quad (4.7)$$

where

$$\Delta\delta=u(\Gamma_{12}\Gamma_{12}^*)^{1/2}F'\sim uO(\Gamma_{12}/m_{12}) , \quad (4.8)$$

with

$$u=i\left[\left[\frac{\Gamma_{12}}{\Gamma_{12}^*}\right]^{1/2}\frac{q}{p}-\left[\frac{\Gamma_{12}^*}{\Gamma_{12}}\right]^{1/2}\frac{p}{q}\right] . \quad (4.9)$$

In the absence of CP violation, m_{12} and Γ_{12} are real, $p^2=q^2$. So $\Delta\delta=0$. In turn, from (4.7), $|r(t)|_{\text{poles}}=1$, which serves as a consistency check.

Back to the CP-violation case. The departure of the ratio $r(t)$ from p^2/q^2 also occurs in the background contribution. From (3.23) and (3.24) it implies that, for the background contribution alone,

$$|r(t)|_{\text{bk}}=\left|\frac{A_{12}(t)}{A_{21}(t)}\right|_{\text{bk}}=\left|\frac{i\Gamma_{12}J(t)}{i\Gamma_{12}^*J(t)}\right|=1 . \quad (4.10)$$

The crucial point here is that $J(t)$ enters in an identical manner for A_{12} as for A_{21}. Since the background integral has a time dependence which differs from that of the pole contribution, the ratio $r(t)$ of the full contributions cannot be constant. Thus Khalfin's theorem, which assumes the constancy of $|r(t)|$, is not applicable to the present solution.

B. Regeneration effect

Next, we demonstrate that there is a regeneration effect in the present solution, which invalidates one of the assumptions stated in Sec. I, leading to the conclusion of the constancy of the magnitude of the ratio r. The presence of the regeneration effect is inferred by the presence of the nondiagonal element in the time-evolution matrix B of (1.6). Based on (1.9) and (2.20),

$$B(t)=R^{-1}A(t)R =\frac{i}{2\pi}\int d\lambda\, e^{-i\lambda t}\frac{R^{-1}NR}{\Delta} , \quad (4.11)$$

with

$$R^{-1}NR=\frac{1}{2pq}\begin{bmatrix}2pqN_{11}-(N_{12}q^2+N_{21}p^2) & N_{12}q^2-N_{21}p^2\\ -N_{12}q^2+N_{21}p^2 & 2pqN_{11}+(N_{12}q^2+N_{21}p^2)\end{bmatrix} . \quad (4.12)$$

We focus our attention on the element B_{12}, which leads to the regeneration of K_S from K_L: i.e.,

$$\begin{aligned}B_{12}&=N_{12}q^2-N_{21}p^2\\ &=(m_{12}-i\Gamma_{12}F)q^2-(m_{12}^*-i\Gamma_{12}^*F)p^2\\ &=-i(F-1)(\Gamma_{12}q^2-\Gamma_{12}^*p^2) . \quad (4.13)\end{aligned}$$

In the last step we have used the relations $p^2=m_{12}-i\Gamma_{12}$ and $q^2=m_{12}^*-i\Gamma_{12}^*$. So

$$B_{12}=v\frac{i}{2\pi}\int_C d\lambda\, e^{-i\lambda t}\frac{F(\lambda)-1}{\Delta} , \quad (4.14)$$

where $v=2\,\text{Im}(\Gamma_{12}m_{12}^*)$. So the regeneration correction occurs only when $v\neq 0$, i.e., when there is CP violation. Deforming the contour, we get

$$B_{12}(t)=B_{12}(t)|_{\text{poles}}+B_{12}(t)|_{\text{bk}} , \quad (4.15)$$

with

$$\begin{aligned}B_{12}(t)|_{\text{poles}}&=\frac{v}{2d}[(F_S-1)e^{-i\lambda_S t}-(F_L-1)e^{-i\lambda_L t}]\\ &=\frac{vF'}{2}(e^{-\lambda_S t}+e^{-i\lambda_L t}) . \quad (4.16)\end{aligned}$$

In a manner similar to going from (3.1) to (3.23), from (4.14),

$$B_{12}(t)|_{\text{bk}}=-vJ(t) . \quad (4.17)$$

In the small-t region,

$$B_{12}(t)|_{\text{poles}}=\frac{vF'}{2}[2-i(\lambda_S+\lambda_L)t]$$
$$=vF'(1-im_{11}t)$$
$$=vF'-ivt\left[-\frac{1}{2}+\frac{\omega_0}{m_{11}+\omega_0}+\frac{i\sqrt{m_{11}\omega_0}}{m_{11}+\omega_0}\right].$$

In the last step, Eq. (3.27) was used. In this region, using (B7), (4.17) becomes

$$B_{12}(t)|_{\text{bk}}=-v\left\{F'-\left[\left[\frac{\omega_0}{m_{11}}\right]^{1/2}\frac{\omega_0}{m_{11}+\omega_0}+i\left[\frac{1}{2}+\frac{m_0}{m+\omega_0}\right]\right]t\right\}.$$

Substituting back into (4.15), in the small-t region the full regeneration amplitude for K_L to K_S is given by

$$B_{12}(t)=2\,\mathrm{Im}(\Gamma_{12}m_{12}^*)\left[\left[\frac{\omega_0}{m_{11}}\right]^{1/2}+i\right]t\ . \quad (4.18)$$

Analogous to (3.30), in the large-t region, we find,

$$B_{12}(t)|_{\text{bk}}=\frac{e^{-(\pi/4)i}}{\sqrt{\pi}}\left[1+\frac{\omega_0}{m_{11}}\right]\frac{\mathrm{Im}(\Gamma_{12}m_{12}^*)}{\omega_0}\times\frac{1}{(m_{11}t)^{3/2}}\ . \quad (4.19)$$

C. Critique on Khalfin's "new *CP*-violation effect"

We recapitulate Khalfin's argument: The time-evolution matrix of K^0 and $\bar{K}^0$ has the spectral representation [see (1.12) and (1.13)]

$$A_{ij}(t)=\int_0^\infty d\lambda\, e^{-i\lambda t}C_{ij}(\lambda)\ ,$$

where

$$C_{ij}(\lambda)=\sum_n\langle i|\lambda n\rangle\langle\lambda n|j\rangle\ .$$

In terms of the spectrum in the K_L and K_S bases,

$$C_{ij}=\sum\langle i|\lambda n\rangle\langle\lambda n|j\rangle$$
$$=\sum_n\langle n|i\rangle^*\langle\lambda n|j\rangle$$
$$=\sum_{n,\alpha,\beta}[\langle\lambda n|\alpha\rangle\langle\tilde{\alpha}|i\rangle]^*\langle\lambda n|\beta\rangle\langle\tilde{\beta}|j\rangle$$
$$=[(R^{-1})^\dagger]_{i\alpha}W_{\alpha\beta}[R^{-1}]_{\beta j}\ , \quad (4.20)$$

where

$$W_{\alpha\beta}=\sum_n\langle\lambda n|\alpha\rangle^*\langle\lambda n|\beta\rangle$$
$$=\sum_n C_{nd}^*C_{n\beta}\ .$$

Carrying out the matrix multiplications

$$C_{12}(\lambda)=\frac{-\zeta^2}{4p^*q}(W_{SS}-W_{SL}+W_{LS}-W_{LL})\ ,$$
$$C_{21}(\lambda)=\frac{-\zeta^2}{4pq^*}(W_{SS}+W_{SL}-W_{LS}-W_{LL})\ . \quad (4.21)$$

Khalfin assumes that

$$W_{\alpha\beta}=\sum_n\left[\frac{\Gamma_{n\alpha}}{\pi}\right]^{1/2}\frac{1}{\lambda-\lambda_\alpha^*}\left[\frac{\Gamma_{n\beta}}{\pi}\right]^{1/2}\frac{1}{\lambda-\lambda_\beta}\ , \quad (4.22)$$

where $\lambda_{\alpha,\beta}=m_{\alpha,\beta}-i\Gamma_{\alpha,\beta}$. The corresponding λ integral is to be evaluated with the approximation

$$\sum_n\sqrt{\Gamma_{\alpha n}\Gamma_{\beta n}}\to\sqrt{\Gamma_\alpha\Gamma_\beta}\ , \quad (4.23)$$

where Γ_α and Γ_β are the total widths of K_α and K_β. This, together with the Weisskopf-Wigner approximation, leads to

$$\int_0^\infty d\lambda\, e^{-i\lambda t}W_{\alpha\beta}(t)$$
$$\approx\frac{1}{\pi}\int_{-\infty}^\infty d\lambda\, e^{-i\lambda t}\frac{\sqrt{\Gamma_\alpha\Gamma_\beta}}{(\lambda-\lambda_\alpha^*)(\lambda-\lambda_\beta)}$$
$$=\begin{cases}e^{-i\lambda_\beta t}, & \text{for } \alpha=\beta\ ,\\ \kappa e^{-i\lambda_S t} & \text{for } \alpha=K_L,\ \beta=K_S\ ,\\ \kappa^* e^{-i\lambda_L t} & \text{for } \alpha=K_S,\ \beta=K_L\ ,\end{cases} \quad (4.24)$$

where

$$\kappa=\frac{-2i\sqrt{\Gamma_S\Gamma_L}}{\lambda_S-\lambda_L^*}\ . \quad (4.25)$$

From (1.12) and (4.21)–(4.25) we arrive at Khalfin's results:

$$A_{12}(t)=-\frac{\zeta^2}{4p^*q}[(1+\kappa)\exp(-i\lambda_S t)-(1+\kappa^*)\exp(-i\lambda_L t)]\ ,$$

and

$$A_{21}(t)=\frac{-\zeta^2}{4pq^*}[(1-\kappa)\exp(-i\lambda_S t)-(1-\kappa^*)\exp(-i\lambda_L t)]\ . \quad (4.26)$$

Numerical estimates: We first present Khalfin's numerical estimates of Refs. 18 and 19. For the neutral-K system, the experimental values of the mean life of K_S and K_L and their mass differences are[20]

$$T_S=0.9\times10^{-10}\ \text{sec},\quad T_L=5.2\times10^{-8}\ \text{sec}\ ,$$
$$\Delta m=m_L-m_S=0.5\times10^{10}\ \text{sec}^{-1}\ .$$

The mean-life values imply the half-widths

$$\Gamma_S=\frac{\gamma_S}{2}=\frac{1}{2T_S}\sim 0.5\times 10^{10}\ \text{sec}^{-1}\ ,$$
$$\Gamma_L=\frac{1}{2T_L}\sim 10^{7}\ \text{sec}^{-1}\ . \tag{4.27}$$

This leads to

$$\kappa=+\frac{2i\sqrt{\Gamma_S\Gamma_L}}{\Delta m+i(\Gamma_S+\Gamma_L)}$$
$$\sim e^{i\pi/4}\left[\frac{2\Gamma_L}{\Gamma_S}\right]^{1/2}\sim 0.06\times e^{i\pi/4}\ . \tag{4.28}$$

For the $D^0\bar{D}^0$ and $B_d^0\bar{B}_d^0$ systems, theoretical estimates give[19,21]

$$\frac{\Delta m(D)}{\Gamma(D)}\sim 10^{-3}\text{–}10^{-2}\ ,$$
$$\Delta\Gamma(D)=\Gamma_S(D)-\Gamma_L(D)\ll\Delta m(D)\ ,$$
$$\frac{\Delta m(B_d^0)}{\Gamma(B_d^0)}\sim 10^{-2}\text{–}10^{-1}\ ,$$
$$\Delta\Gamma(B_d)=\Gamma_S(B_d)-\Gamma_L(B_d)\ll\Delta m(B_d)\ .$$

So

$$\Gamma_L\sim\Gamma_S,\quad \Delta m\ll\Gamma_L,\Gamma_S\ . \tag{4.29}$$

This, together with (4.28), implies that

$$\kappa\sim\frac{2i\Gamma_L}{\Delta m+2i\Gamma_L}\sim 1+O(\Delta m/\Gamma_L)\ . \tag{4.30}$$

Substituting this κ value into (4.26), Khalfin found a substantial difference between $A_{12}(t)$ and $A_{21}(t)$ amplitudes. This leads him to predict "appreciable" *CP*-violation effects in $D^0\bar{D}^0$ and $B^0\bar{B}^0$ systems.

In contrast to Khalfin's estimate, our solution predicts the corresponding deviation from unity to be

$$\delta=O(\Gamma_{12}/m_{11})\ . \tag{4.31}$$

See (4.5) and (4.6). For the neutral-K system, to evaluate δ we take

$$\Gamma_{12}\sim\Gamma_S\sim 5\times 10^{10}\ \text{sec}^{-1}\ . \tag{4.32}$$

The quantity m_{11} is the difference between the kaon mass and the threshold energy of the continuum state, say, $K\to 2\pi$. This difference is 220 MeV. So

$$m_{11}\sim 220\ \text{MeV}\sim 3\times 10^{23}\ \text{sec}^{-1}\ . \tag{4.33}$$

Therefore,

$$\delta\sim 0.16\times 10^{-13}\ . \tag{4.34}$$

From (4.28) and (4.34),

$$\frac{\delta}{\kappa}\sim 3\times 10^{-13}<10^{-12}\ . \tag{4.35}$$

In other words, our solution implies that for the pole contribution, deviation from unity is over 12 orders of magnitude below Khalfin's prediction.

Now we come to the D and B neutral systems; the experimental mean-life values are

$$T_{D^0}=4\times 10^{-13}\ \text{sec},\quad T_{B^0}=14\times 10^{-13}\ \text{sec}\ .$$

So

$$\Gamma_{D^0}=\frac{1}{2T_{D^0}}\sim 1.3\times 10^{12}\ \text{sec}^{-1}\ ,$$
$$\Gamma_{B^0}=\frac{1}{2T_{B^0}}\sim 3.6\times 10^{11}\ \text{sec}^{-1}\ .$$

We assume $m_{11}>220\ \text{MeV}=3.6\times 10^{23}\ \text{sec}^{-1}$, and $\Gamma_{B_d^0}$ is of the same order of magnitude as Γ_{B^0}. Then

$$\delta\sim\frac{\Gamma}{m_{11}}<\frac{3.6\times 10^{11}}{3\times 10^{23}}=1.2\times 10^{-12}<10^{-11}\ , \tag{4.36}$$

which is over 11 orders of magnitude smaller than that given by Khalfin.

We now turn to some theoretical considerations. For each neutral system there are two distinct lifetimes involved; in turn, there are two distinct eigenstates. The energy spectrum matrix $W_{\alpha\beta}$ is nonsingular. It has a positive-definite determinant. For the prediction of a "new *CP*-violation effect," Khalfin made an additional assumption that the off-diagonal matrix elements of $W_{\alpha\beta}$ take the factorizable form of (4.23), which implies the vanishing of the corresponding determinant or an overestimate of the off-diagonal elements. So Khalfin estimates should be regarded as an upper bound. Our analysis above shows that this upper bound is many orders of magnitude larger than what we obtain from our exact solution.

Inspection of (4.25) and (4.26) reveals another deficiency. The new *CP*-violating effect predicted persists for a *CP*-conserving situation, where m_{12} and Γ_{12} are real. For this case the *CP*-violating effect is not to be expected [see Eq. (4.14)].

D. Various t regions of $A_{12}(t)$

The algebraic expression of $A_{12}(t)$ is given by the sum of the pole contribution of (4.5) and the background contribution of (3.23) and (3.24).

At $t=0$: From (3.26) we see the two pole terms cancel out each other, except for a tiny residual value

$$A_{12}(0)|_{\text{poles}}=-i\Gamma_{12}F'=O(\Gamma_{12}/m_{11})\ . \tag{4.37}$$

This extra residual term is canceled by the background contribution of (3.25). This leads to (3.28), which implies

$$A_{12}(0)=0\ . \tag{4.38}$$

So the completeness relation of (2.25) is restored.

Zeno region: Denote $T_1\sim\min(1/m_{11},1/\omega_0)$. In this domain the amplitude rises linearly with t. From (3.28),

$$A_{12}(t)=-i\left[m_{12}+\Gamma_{12}\left[\frac{\omega_0}{m_{11}}\right]^{1/2}\right]t\ . \tag{4.39}$$

We recall the "quantum Zeno effect" of an unstable system[9,10] refers to the situation that if an unstable system is monitored for its existence at a sufficiently small time interval, the lifetime of the system appears to be longer than that obtained by monitoring the same system with a coarser time interval. Recent experiments by Itano *et al.*[13] have confirmed the quantum Zeno effect. The experiment of Itano *et al.* involves the observation of the Zeno effect for the transition from one discrete state into another discrete state. To our knowledge the observation of the Zeno effect from one discrete state to a continuum state has not yet been carried out experimentally. For both discrete-to-discrete and discrete-to-continuum cases, the Zeno effect is due to the fact that the survival probability near $t=0$ is given by

$$P(t)=|A(t)|^2\sim 1-\text{const}\times t^2 . \tag{4.40}$$

So it has a zero slope with respect to t at $t=0$.

Analogously, for the transition process of K^0 to $\bar{K}^0$ or vice versa, because of the orthogonality of the two K states, the transition probability vanishes at $t=0$. On general grounds it can be shown that in the small-t region the transition amplitude is proportional to t; in turn, the corresponding probability is proportional to t^2. So, for the transition probability of $\bar{K}^0$ to K^0 in the small region, we expect

$$P_{12}(t)=\text{const}\times t^2, \quad \frac{dP_{12}}{dt}(0)=0 . \tag{4.41}$$

In other words, there is also the presence of the Zeno effect for the transition from K^0 to $\bar{K}^0$ or vice versa.

In the LOY theory, where a pole approximation is assumed, from (4.2), in the small t region,

$$A_{12}(t)|_{\text{poles}}\to\frac{-ip}{2q}(\lambda_S-\lambda_L)t$$
$$=-ip^2t=-(im_2+\Gamma_{12})t . \tag{4.42}$$

Here again (4.41) is satisfied. While for the unstable decay case the Zeno effect is a paradox in that there is the apparent extension of lifetime due to quantum measurement, for the transition case the Zeno effect is to be expected, since the suppression occurs even within the Weisskopf-Wigner pole approximation.

Actually the pole-approximation result and our results differ in details. For our solution the pole contribution to the transition amplitude alone has a constant term. It takes the combination of the background and pole contributions, which lead to a linear behavior in t in the transition amplitude. Note also the difference between our result in (4.39) and that of the LOY result in (4.42). For instance, the multiplicative phase for the Γ_{12} term is imaginary in our solution, while it is real in the LOY theory.

Pole-dominance region: Denote this region by $T_1<t<T_2$, where T_2 is the time where the pole contribution and background contribution become comparable. Here, in the lowest-order approximation,

$$A_{12}(t)\approx A_{12}(t)|_{\text{poles}}=\frac{p}{2q}(e^{-i\lambda_S t}-e^{-i\lambda_L t}) ,$$
$$\lambda_{S,L}=m_{11}-i\Gamma_{11}\pm pq , \tag{4.43}$$

which is identical to that of LOY theory. Let us estimate T_2 for two different cases.

Case 1: T_2 occurs when $|e^{-i(\lambda_L-\lambda_S)T_2}|\ll 1$. This is the case for the neutral-kaon system. At T_2, the K_S-pole contribution is already vanishingly small. From (4.43) and (3.30), we have

$$\left|-\frac{p}{2q}e^{-i\lambda_L t}\right|\sim\left|\left[1+\frac{\omega_0}{m_{11}}\right]\frac{\Gamma_{12}}{\omega_0}\frac{1}{(mt)^{3/2}}\right| ,$$

or

$$e^{-\Gamma_L t}\sim\frac{\Gamma_{12}}{\omega_0}\frac{1}{(m_{11}t)^{3/2}} . \tag{4.44}$$

Assuming $\omega_0\sim m_{11}$, from (4.32) and (4.33), with $m_{11}\sim 3\times 10^{23}\ \text{sec}^{-1}$, $\Gamma_L\sim 10^7\ \text{sec}^{-1}$, at $t=T_2$,

$$\Gamma_L t-\frac{3}{2}\ln(\Gamma_L t)\sim\frac{5}{2}\ln\frac{m_{11}}{\Gamma_L}\sim 95 ,$$

or

$$T_2\sim 102/\Gamma_L . \tag{4.45}$$

Case 2: T_2 occurs in the region where $|(\lambda_L-\lambda_S)T_2|\ll 1$. We recall that present theoretical estimates of (4.29) give

$$\Delta m(D)\gg\Delta\Gamma(D),\quad \frac{\Delta m(D)}{\Gamma(D)}\sim 10^{-3}-10^{-2} ,$$
$$\Delta m(B_d)\ll\Delta\Gamma(B_d),\quad \frac{\Delta m(B_d)}{\Gamma(B_d)}\sim 10^{-2}-10^{-1} . \tag{4.46}$$

At T_2, using (4.43) and (3.30), we get

$$\left|\frac{p}{2q}e^{-i\lambda_L t}(e^{-i(\lambda_S-\lambda_L)t}-1)\right|\sim\Delta m t e^{-\Gamma_L t}$$
$$\sim\frac{\Gamma_{12}}{\omega_0}\frac{1}{(m_{11}t)^{3/2}} . \tag{4.47}$$

Since Δm, $\Delta\Gamma\ll\Gamma_L$, we expect $\Gamma_{12}\sim\Gamma_L$. As before, assume $\omega_0\sim m_{11}>220$ MeV. Thus (4.47) leads to

$$e^{-\Gamma_L t}\sim\frac{\Gamma_{12}}{\Delta m}\frac{1}{(m_{11}t)^{5/2}}$$
$$=\frac{\Gamma_{12}}{\Delta m}\left[\frac{\Gamma_L}{m_{11}}\right]^{5/2}\frac{1}{(\Gamma_l t)^{5/2}} ,$$

or

$$\Gamma_L t-\frac{5}{2}\ln(\Gamma_L t)=\frac{5}{2}\ln\frac{m_{11}}{\Gamma_L}+\ln\frac{\Gamma_L}{\Delta m} . \tag{4.48}$$

We use the D and B data quoted earlier for evaluation of (4.36):

$$m_{11}>220\ \text{MeV}=3.3\times 10^{23}\ \text{sec}^{-1} ,$$

$$\Gamma_L<1.3\times 10^{12}\ \text{sec}^{-1},\quad \frac{\Gamma_L}{\Delta m}>10 .$$

So T_2 for the D and B systems can be evaluated:

$$\Gamma_L T_2 - \tfrac{5}{2}\ln\Gamma_L T_2 = \tfrac{5}{2}\ln\frac{m_{11}}{\Gamma_L} + \ln\frac{\Gamma_L}{\Delta m} > 67 \ ,$$
$$\Gamma_L T_2 > 78, \quad \text{or } T_2 > 78/\Gamma_L \ . \tag{4.49}$$

Power-law region: For $t > T_2$, from (3.30),

$$A(t) = A(t)|_{\rm bk} \rightarrow -e^{i\pi/4}\left[1+\frac{\omega_0}{m_{11}}\right]\frac{\Gamma_{12}}{\omega_0}\frac{1}{(m_{11}t)^{3/2}} \ . \tag{4.50}$$

We observe that the power-law behavior in the asymptotic t region is very similar to that for the survival amplitude of unstable particle. In fact, they both may be directly correlated to the threshold behavior of the continuum.[4,10]

We see from the analysis in this paper that a quantum system with two metastable states which communicate with each other exhibits interesting phenomena in its time evolution. For its short-time behavior the quantum Zeno effect obtains; for very-long-term behavior there is a regeneration effect even in vacuum unless the long- and short-lived superpositions are strictly orthogonal. In the kaon complex the short-lived particle K_S has passed from the exponential regime to the inverse power regime before appreciable decay of the K_L or regeneration of the K_S takes place. The *CPT* invariance making the diagonal elements of the decay matrix in the $K^0, \bar{K}^0$ basis equal is crucial to the nature of the time evolution. In the study of communicating metastable states in atomic physics, such an additional constraint on *CPT* is not there; consequently the decay exhibits richer features. We will present the general study elsewhere. Suffice it to observe that the asymptotic and Zeno region time dependences are very much the same as with the case of a single metastable state decay: This is not surprising since the generic arguments apply without restriction to the number of channels involved.

ACKNOWLEDGMENTS

We thank Professor Khalfin for discussions on his work. This work was supported in part by the Department of Energy Grant No. DE-FG05-85ER40200.

APPENDIX A: INTEGRAL IDENTITIES

Let

$$I_n(a,b,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{r\,dr\,e^{-r^2 t}}{(r^2+a)^n}\frac{r}{r^2+b} \ . \tag{A1}$$

We want to evaluate this integral for $n=1$ and 2. We recall a useful identity:[22]

$$\int_0^\infty dy\frac{\sqrt{y}\,e^{-yt}}{y+a} = \left[\frac{\pi}{t}\right]^{1/2} - \pi\sqrt{a}\,e^{at}\text{erfc}(\sqrt{at}\,) \ , \tag{A2}$$

where $\text{erfc}(y) = 1-\text{erf}(y)$. So, using (A2), we get

$$I_1(a,b,t) = \frac{1}{\pi(b-a)}\int_0^\infty dy\sqrt{y}\,e^{-yt}\left[\frac{1}{y+a}-\frac{1}{y+b}\right]$$
$$= \frac{1}{a-b}[\sqrt{a}\,e^{at}\text{erfc}(\sqrt{at}\,) - \sqrt{b}\,e^{bt}\text{erfc}(\sqrt{bt}\,)] \ . \tag{A3}$$

For $n=2$, from the definition of (A1),

$$I_2(a,b,t) = -\frac{d}{da}I_1(a,b,t) \ . \tag{A4}$$

Small-t expansion: For small y, i.e., $y<1$,[23]

$$\text{erf}(y) = \frac{2y}{\sqrt{\pi}}\left[1-\frac{y^2}{3}+\cdots\right] \ . \tag{A5}$$

For $at, bt \ll 1$, using (A5) from (A3) and (A4), we get

$$I_1 \approx \frac{1}{\sqrt{a}+\sqrt{b}} - 2\left[\frac{t}{\pi}\right]^{1/2} + \frac{b^{3/2}-a^{3/2}}{b-a}t \ . \tag{A6}$$

Large argument expansion: For large y, i.e., $y>1$,[23]

$$\text{erfc}(y) = \frac{e^{-y^2}}{\sqrt{\pi}y}\left[1-\frac{1}{2y^2}+\cdots\right] \ . \tag{A7}$$

Applying this identity to (A3) and (A4), we get, for $\sqrt{at}, \sqrt{bt} \gg 1$,

$$I_1 \approx \frac{1}{2\sqrt{\pi ab}}\frac{1}{t^{3/2}} \ , \tag{A8}$$

and

$$I_2 \approx \frac{1}{2\sqrt{\pi a^2 b}}\frac{1}{t^{3/2}} \ . \tag{A9}$$

APPENDIX B: $J(t)$ FUNCTION

Recall from (3.24),

$$J(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{r\,dr\,e^{-r^2 t}}{(r^2-im)^2}F(\lambda) \ , \tag{B1}$$

where with $\lambda = e^{-i\pi/2}r^2$ the form-factor function

$$F(\lambda) = n\left[\frac{1}{\sqrt{\lambda}+i\sqrt{\omega_0}}+C\right]$$
$$= ne^{i\pi/4}\left[\frac{r+e^{-i\pi/4}\sqrt{\omega_0}}{r^2+i\omega_0}+e^{-i\pi/4}C\right] \ , \tag{B2}$$

with

$$n = \frac{m+\omega_0}{\sqrt{m}} \quad \text{and} \quad C = \frac{i\sqrt{\omega_0}}{\lambda+\omega_0} \ . \tag{B3}$$

The subscript of m is suppressed in this appendix. Since r range in (B1) is symmetric about $r=0$, only the even part of the integrand contributes to the integral. This leads to

$$J(t)=ne^{i\pi/4}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{r\,dr\,e^{-r^2t}}{(r^2-im)^2}\,\frac{r}{r^2+i\omega_0}\;.$$

$$=ne^{i\pi/4}I_2(-im,i\omega_0,t)\;. \quad \text{(B4)}$$

Small-t behavior of J(t): From (A5) and (A6),

$$J(0)=e^{i\pi/4}n(-1)\frac{d}{d(-im)}\,\frac{1}{\sqrt{-im}+\sqrt{i\omega_0}}$$

$$=e^{-i\pi/4}n\frac{d}{dm}\,\frac{1}{e^{-i\pi/4}\sqrt{m}+e^{i\pi/4}\sqrt{\omega_0}}$$

$$=\frac{d}{d\lambda}\left[\frac{n}{\sqrt{\lambda}+i\sqrt{\omega_0}}+C\right]\Bigg|_{\lambda=m}=F'(m)\;, \quad \text{(B5)}$$

$$J'(0)=e^{i\pi/4}n(-1)\frac{d}{d(-im)}\left[\frac{(-im)^{3/2}-(i\omega_0)^{3/2}}{(-im)-(i\omega_0)}\right]$$

$$=-in\frac{d}{dm}\,\frac{m^{3/2}+i\omega_0^{3/2}}{M+\omega_0}\;. \quad \text{(B6)}$$

From (B5), (B6), and after some algebra, we get, for near $t=0$,

$$J(t)=F'(m)-i\left[\left[\frac{1}{2}+\frac{\omega_0}{m+\omega_0}\right]-\frac{i\omega_0}{m+\omega_0}\left[\frac{\omega_0}{m}\right]^{1/2}\right]t\;. \quad \text{(B7)}$$

Large-t behavior of J(t): From (A9) and (B4), for $\sqrt{mt}\,,\sqrt{\omega_0 t}\gg 1$,

$$J(t)\to e^{i\pi/4}\frac{n}{2\sqrt{\pi}(-im)^2(i\omega_0)t^{3/2}}$$

$$=\frac{e^{3\pi i/4}}{2\sqrt{\pi}}\left[1+\frac{\omega_0}{m}\right]\frac{1}{\omega_0(mt)^{3/2}} \quad \text{(B8)}$$

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