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Shear-free spherically symmetric inhomogeneous cosmological model with heat flow and bulk viscosity

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An exact solution to the Einstein equations with a shear-free imperfect-fluid source is obtained. The solution approaches a locally flat Robertson-Walker one in the large-t limit and thus serves as a viable candidate for a realistic cosmological model. The model built out of this solution is found to be free of horizon, entropy, and flatness problems.

I. INTRODUCTION

The standard Friedmann-Robertson-Walker big-bang model provides a very attractive framework for discussing the large-scale cosmology of the Universe. Nonetheless, the model has some major drawbacks, the most prominent being the horizon, flatness, and entropy problems. These problems arise mainly because of an insistence that the Universe is not only currently in a Robertson-Walker phase but that it had also been in one for almost all of its entire previous history as well. Moreover, this point of view is even reinforced by the very ability of the popular inflationary universe model[1] to provide a candidate solution to all of the standard model problems as a consequence of very early Universe dynamics which occurred even prior to the onset of the conventional radiation-dominated Robertson-Walker phase.

Now though the rationale for and the achievements of the inflationary universe model are, of course, well known (a recent review of its current status may be found in Ref. 1), it is nonetheless useful to recall here some specific features of the familiar standard model problems that inflation potentially resolves. The horizon problem is a problem of the Robertson-Walker geometry itself since once we are given the current degree of isotropy of the microwave background it follows that an earlier time Robertson-Walker universe could not have been causally connected. The problem of understanding the huge entropy of the current Universe is due to the dynamical assumption that the energy-momentum tensor is and always has been that of an adiabatic and hence entropy-conserving perfect fluid. Finally, the flatness problem is a problem which arises because of the structure of the Einstein equations themselves in the standard cosmology, since it requires extreme fine-tuning to have the current energy density of the Universe be so close to the critical energy density for a Universe as old as ours appears to be. This flatness problem is thus a far more detailed dynamical problem than the horizon and entropy problems, which themselves are essentially kinematical in nature. The three standard problems are thus somewhat independent of each other and to seek a resolution of all three problems will thus require a relaxation of quite a few of the standard-model assumptions. As we will see, the imperfect-fluid model that we consider here will exactly do this for us.

To motivate our study we note that while we observe the current Universe to be one which is expanding and which has a matter distribution which is maximally three-space symmetric on the largest scales to good accuracy (this latter feature is actually still open to question), that does not mean that the Universe has always been that way. Nonetheless, it is reasonable to at least assume that the Universe has always been expanding so that there actually was a hot early Universe. However, the question of whether or not it has always been maximally three-space symmetric is not one that we can as readily respond to. Consequently, we can entertain the possibility that the Universe may not always have been as highly symmetric spatially as we now observe it to be, but rather that it only evolved over the course of time into its present highly symmetric form, and then perhaps only quite slowly. In such a situation the horizon, flatness, and entropy problems would all have to be reexamined anew.

In order to try to determine what lower symmetry and what specific dynamics the not-so-recent Universe could possibly have possessed, we shall, in this paper, take into consideration some of the implications of kinetic theory for the fluid dynamics of the Universe. To motivate our approach here we recall some of the characteristic features of standard non-relativistic classical kinetic theory. For our purposes the most significant aspect of Boltzmann transport theory is that the equilibrium Maxwell-Boltzmann distribution, a distribution which is itself independent of the specific collisional forces associated with the system of interest, will be the infinite-time solution to the Boltzmann transport equation provided that there actually are collisions in the first place. Thus while collisions are necessary to thermalize the system, the distribution that is eventually produced is in fact independent of the collisions which produce it. Moreover, at finite times, the system would necessarily not be in a Maxwell-Boltzmann distribution (this distribution not being a finite-time solution to the Boltzmann equation), so that at early times the system would have to be an imperfect, inhomogeneous fluid rather than a homogeneous, perfect one. Thus transport theory allows an imperfect fluid to evolve into a perfect one with the thermalization itself actually removing any inhomogeneities in the fluid as part of the process of establishing thermal equilibrium. Thus the present perfect-fluid energy-momentum tensor of standard cosmology may not have been quite as perfect at earlier times, in which case the geometry would not have been as symmetric as it is now thought to be.

To ascertain what kind of imperfect fluid one might expect we recall the familiar approximate treatment of the nonrelativistic Boltzmann transport equation. Specifically, we can try as a zeroth-order finite-time solution a local Maxwell-Boltzmann distribution only with (slowly) varying particle number density and temperature [$n(r,t)$ and $T(r,t)$, respectively], so that the system, while not actually being in equilibrium, is not too far away from it. With such a distribution we then find that the ideal gas equation of state is still obeyed, only locally rather than globally, so that $P(r,t)$ is still equal to $n(r,t)$ times $T(r,t)$ everywhere even though the pressure, density, and temperature are all constantly changing from point to point. Moreover, if we now insert this zeroth-order form for the distribution function into the Boltzmann transport equation, we can then obtain first-order corrections to the statistical averages of the dynamical quantities which characterize the system. And, indeed, in this way we then obtain the familiar hydrodynamical equations (continuity, Navier-Stokes, and heat conduction) of an imperfect fluid, along with explicit dynamical expressions for its viscosity and heat conduction coefficients. An imperfect viscous fluid with heat flow thus characterizes a system which is evolving toward thermal equilibrium but which has not yet got there.

In order to generalize this discussion to relativistic curved space-time, we note that there are two specific and distinct features to the analysis, one kinematic and the other dynamic. The kinematic aspect of the analysis is the realization that the overall form of the hydrodynamical equations (viz., the three-component Navier-Stokes equation and the one-component heat conduction equation) follows directly from the conservation conditions on the fluid energy-momentum tensor (a recent actual explicit derivation of the nonrelativistic hydrodynamical equations in an external gravitational field via the nonrelativistic reduction of a generally covariant imperfect fluid in a background Schwarzschild geometry may be found in Ref. 2); while the dynamical aspect is the obtaining of explicit expressions for the various transport coefficients which appear in those equations. Thus, even without solving the general-relativistic Boltzmann transport equation we can infer the general form of the equations of motion of an imperfect fluid completely from general covariance considerations alone. The only thing that we would then still lack would be explicit expressions for the viscosity and heat conduction coefficients. To obtain these expressions in general for an expanding universe would require the joint solving of both the Einstein and the general-relativistic Boltzmann transport equations in order to get the associated self-consistent geometry. Rather than try to carry out so prohibitive a procedure we can instead consider a far more limited approach. Specifically, we can introduce a generally covariant imperfect-fluid matter source and try to look for solutions to the Einstein equations which become Robertson-Walker in the long-time limit. As we will see in our study this condition will actually serve to fix the space-time behavior of the transport coefficients, and thus our study can give us some feeling for the kinds of solutions that can be expected when kinetic theory is considered in conjunction with general relativity.

In our study then we propose to consider the Universe to be described by an imperfect fluid in an inhomogeneous geometry. Thus the geometry in general will be taken to be spherically symmetric about a single point only, rather than being isotropic about all points. However, if the fluid does indeed thermalize due to back scattering off the geometry, the geometry will gradually evolve in time into the current homogeneous Universe, and thus naturally become isotropic about all points and hence become maximally three-space symmetric. Today such a universe would not be distinguishable from the standard Friedmann universe, but it would begin to deviate markedly from the standard cosmology as we look backwards in time.

In general we would have to consider an imperfect fluid with shear viscosity, bulk viscosity, and heat conduction, and we would also have to take into consideration the equation of state of the fluid, and allow for the possibility that at earlier times the fluid need not have been comoving with the geometry. To treat this problem in all generality is far too complex, and no exact solutions are known. However, some partial studies have been made in the literature which identify some particular exact solutions to the Einstein equations in a few restricted cases.[3-12] However we are not aware of any studies of inhomogeneous imperfect fluids in which considerations of restrictions due to the equation of state of the fluid and in which considerations regarding the physical implications of the solutions for cosmology have been made. In this paper and an associated companion[13] one we shall study some special exact solutions whose structures have been suggested by the partial studies referred to above, while also finding and studying some new exact ones and determining their implications for cosmology. In this particular paper we specialize to noncomoving, shear-free inhomogeneous cosmological models with bulk viscosity and heat conduction, and in the companion paper we specialize to acceleration-free (and thus comoving), inhomogeneous models with nonvanishing shear viscosity. In both of the papers we shall obtain new families of exact solutions for imperfect-fluid cosmologies which all have the feature that they evolve locally at long times into Robertson-Walker perfect-fluid cosmologies. The models are found to be free of the standard-model problems and require no fine-tuning of parameters.

In Sec. II we present our explicit study of shear-free inhomogeneous cosmologies. Then in Secs. III–V we will show how the horizon, entropy, and flatness problems are, respectively, solved in our model, and finally in Sec. VI we shall make some general comments.

II. AN EXACT INHOMOGENEOUS SOLUTION

In order to first set up the problem we note that for a time-dependent geometry which is spherically symmetric about a single point the line element ds^2 and the (normalized) fluid four-velocity U^μ can always be brought to the form (ν, λ, and Y are all functions of r and t)

$$ds^2=-e^{2\nu}dt^2+e^{2\lambda}dr^2+Y^2(d\theta^2+\sin^2\theta\, d\phi^2) , \quad (2.1)$$

$$U^\mu=(e^{-\nu},0,0,0) \quad (2.2)$$

with no loss of generality. Given a metric and a fluid four-velocity vector, the energy-momentum tensor of an imperfect fluid can be written in the form (using the notation of Refs. 14 and 15 both here and throughout)

$$T_{\mu\nu}=\rho U_\mu U_\nu+\bar{p}H_{\mu\nu}-2\eta\sigma_{\mu\nu}+q_\mu U_\nu+q_\nu U_\mu , \quad (2.3)$$

where we have introduced

$$\begin{aligned} H_{\mu\nu}&=g_{\mu\nu}+U_\mu U_\nu , \\ 2\sigma_{\mu\nu}&=H_\mu{}^\alpha H_\nu{}^\beta(U_{\alpha;\beta}+U_{\beta;\alpha}-\tfrac{2}{3}g_{\alpha\beta}U^\rho{}_{;\rho}) , \\ q_\mu&=-\chi H_\mu{}^\alpha(T_{;\alpha}+TU_{\alpha;\beta}U^\beta) , \\ \bar{p}&=p-\zeta U^\alpha{}_{;\alpha} . \end{aligned} \quad (2.4)$$

In Eqs. (2.3) and (2.4) ρ, p, and T are the energy density, pressure, and temperature, while η, ζ, and χ are the coefficients of shear viscosity, bulk viscosity, and heat conduction, respectively, with all of these transport coefficients being general functions of r and t.

While the above equations describe the structure of the model in general, in order to make it more tractable and actually be able to get an exact solution we shall specialize here to the case where the fluid is shear free, i.e., to the case where $\sigma_{\mu\nu}=0$, so that Y may then be set equal to re^λ in Eq. (2.1). [Note that in this case even if the coefficient of shear viscosity $\eta(r,t)$ is nonzero it will not contribute to $T_{\mu\nu}$ and thus it plays no role in the considerations of this paper. Some cosmological implications due to a nonvanishing shear viscosity contribution are explored in our companion paper[13] and also in Ref. 16.] In the shear-free case the metric reduces to

$$ds^2=-e^{2\nu}dt^2+e^{2\lambda}(dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2) , \quad (2.5)$$

and the Einstein equations take the form

$$\kappa\rho=3\dot{\lambda}^2e^{-2\nu}-e^{-2\lambda}\left[2\lambda''+\frac{4\lambda'}{r}+(\lambda')^2\right] , \quad (2.6)$$

$$\kappa\bar{p}=-e^{-2\nu}(2\ddot{\lambda}-2\dot{\nu}\dot{\lambda}+3\dot{\lambda}^2)+e^{-2\lambda}\left[\lambda'^2+2\nu'\lambda'+\frac{2\lambda'}{r}+\frac{2\nu'}{r}\right] , \quad (2.7)$$

$$\kappa\bar{p}=-e^{-2\nu}(2\ddot{\lambda}-2\dot{\nu}\dot{\lambda}+3\dot{\lambda}^2)+e^{-2\lambda}\left[\nu''+\lambda''+\nu'^2+\frac{\lambda'+\nu'}{r}\right] , \quad (2.8)$$

$$\kappa\chi\frac{\partial}{\partial r}(Te^\nu)=-2\dot{\lambda}'+2\nu'\dot{\lambda} . \quad (2.9)$$

In order to be able to make any sensible physical interpretation for our model we will need an expression for the temperature. For the radiation-dominated universe which we consider in this paper the most simple and natural assumption is to copy the behavior of a lowest-order nonrelativistic imperfect fluid that we described in the Introduction and assume that the finite-time statistical distribution is given by a locally varying form of the equilibrium distribution, so that the energy density is then that of a local blackbody with a locally varying temperature. Thus we set

$$\rho(r,t)=aT^4(r,t) , \quad (2.10)$$

where the parameter a is given by $a=N\pi^2k^4/30c^3\hbar^3$ with N being the relevant number of degrees of freedom. Furthermore we shall, again by analogy with nonrelativistic fluids, take the equation of state of the local blackbody to be the standard

$$\rho(r,t)=3p(r,t) . \quad (2.11)$$

(In passing we note that in Ref. 14, Weinberg also suggests to use the same functional relation between ρ and p for an imperfect fluid as would obtain in the perfect-fluid case.)

In addition to the above equations we need to impose the physical requirement that the transport coefficients ζ and χ be non-negative, a requirement that ensures the positivity of entropy production which can be assumed on general thermodynamic grounds. Also we need to impose our desired boundary condition that the contribution of dissipative effects to the energy-momentum tensor attenuates fast enough (at least as fast as the perfect-fluid components ρ and p do) so that the Universe actually evolves locally into a Robertson-Walker one as $t\to\infty$. This latter requirement is motivated by the experimental observation that the present Universe is remarkably homogeneous on large scales. By imposing these physical restrictions and by making some simplifying calculational assumptions we have actually been able to find an exact solution to our model which meets all of our stated conditions. We will not provide the explicit calculation here (details of the derivation may be found in Ref. 16) but merely state the result. For the metric we obtain the expression

$$ds^2=-\left[\frac{1+\dfrac{\alpha r^2}{t^{2(1-\epsilon)}}}{1+\dfrac{\beta r^2}{t^{2(1-\epsilon)}}}\right]^2dt^2+\frac{t^{2\epsilon}}{\left[1+\dfrac{\beta r^2}{t^{2(1-\epsilon)}}\right]^2}(dr^2+r^2d\Omega) , \quad (2.12)$$

where the parameters α, β, and ϵ are all positive and satisfy the conditions

$$\epsilon=\tfrac{1}{4}\{1+[1+16(\alpha-3\beta)]^{1/2}\} , \quad (2.13)$$

$$\alpha\geq 3\beta , \quad (2.14)$$

$$\alpha^2<\tfrac{3}{4}\beta,\quad \alpha<\tfrac{1}{12}\ , \tag{2.15}$$

and thus ϵ can never be greater than $[1+(\frac{7}{3})^{1/2}]/4$, a number which is less than 1. As will be seen in the next section, the isotropy of the present microwave background requires $\alpha \ll 1$. Thus Eq. (2.15) does not contradict the requirements of Eq. (2.14). We note that these conditions are sufficient conditions but not necessary ones. Hence we are able to demonstrate that there is in fact a wide range of acceptable values of the parameters which satisfy our chosen physical requirements. Thus our solution does not correspond to an isolated (and hence fine-tuned) point in parameter space.

Geometrically we note that while our metric locally becomes flat as time evolves, the topological structure of our metric is actually that of the closed three-sphere S^3, since, as we discuss in more detail in Sec. V, at any fixed time t_0 the spatial part of our metric is precisely that of a three-sphere with effective positive three-space curvature $k=4\beta/t_0^{2(1-\epsilon)}$. Thus our metric describes a closed geometry with a very flat local structure at late times and is thus actually a sort of hybrid Robertson-Walker model. As we will see in Sec. V this feature will enable our model to provide us with a solution to the flatness problem.

With regard to our solution we note further that since $\rho\simeq T^4$ and since $T_{01}\simeq\partial_r(Te^{\nu})\simeq\partial_r(\rho^{1/4}e^{\nu})$ our solution is quite nontrivial in nature. In our solution the density ρ, the heat conduction coefficient χ, and the bulk viscosity coefficient ζ can be obtained from Eqs. (2.6)–(2.12) and take the form

$$\kappa\rho=\frac{3}{t^2}\left[\epsilon+(2-\epsilon)\frac{\beta r^2}{t^{2(1-\epsilon)}}\right]^2\left[1+\frac{\alpha r^2}{t^{2(1-\epsilon)}}\right]^{-2}+\frac{12\beta}{t^2}\ , \tag{2.16}$$

$$\begin{aligned}\chi=\frac{4}{3}\left[\frac{a}{\kappa^3}\right]^{1/4}&\frac{\epsilon\alpha-(2-\epsilon)\beta}{t^{1/2}}\left[1+\frac{\beta r^2}{t^{2(1-\epsilon)}}\right]\\ &\times\left[(\alpha-\beta)\left[1+\frac{\alpha r^2}{t^{2(1-\epsilon)}}\right]^{-1}\left[\epsilon+(2-\epsilon)\frac{\beta r^2}{t^{2(1-\epsilon)}}\right]^2\right.\\ &\left.\quad+2(1-\epsilon)\beta\left[\epsilon+(2-\epsilon)\frac{\beta r^2}{t^{2(1-\epsilon)}}\right]+8\beta(\alpha-\beta)\left[1+\frac{\alpha r^2}{t^{2(1-\epsilon)}}\right]\right]^{-1}\\ &\times\left[3\left[\epsilon+(2-\epsilon)\frac{\beta r^2}{t^{2(1-\epsilon)}}\right]^2\left[1+\frac{\alpha r^2}{t^{2(1-\epsilon)}}\right]^{-2}+12\beta\right]^{3/4}\ ,\end{aligned} \tag{2.17}$$

and

$$\zeta=\frac{1}{3\kappa}\frac{r^2}{t^{3-2\epsilon}}\frac{-X_1-X_2\left[\dfrac{r^2}{t^{2(1-\epsilon)}}\right]-X_3\left[\dfrac{r^2}{t^{2(1-\epsilon)}}\right]^2}{\left[1+\dfrac{\alpha r^2}{t^{2(1-\epsilon)}}\right]^2\left[\epsilon+(2-\epsilon)\dfrac{\beta r^2}{t^{2(1-\epsilon)}}\right]}\ , \tag{2.18}$$

where we have introduced

$$\begin{aligned}X_1&\equiv(12\epsilon^2-28\epsilon+12)\beta-2\epsilon\alpha+4(2\alpha^2-9\alpha\beta+\beta^2)\ ,\\ X_2&\equiv(2\epsilon-4)\beta^2+(12\epsilon^2-20\epsilon+4)\alpha\beta\\ &\quad+4(\alpha^2-9\alpha\beta+2\beta^2)\alpha\ ,\\ X_3&\equiv(-4\epsilon^2+14\epsilon-12)\beta^2\alpha+4(\beta-3\alpha)\beta\alpha^2\ .\end{aligned} \tag{2.19}$$

It can be shown that X_1, X_2, and X_3 are all negative if Eqs. (2.13)–(2.15) are satisfied. Thus ζ and χ are both nonnegative as required and have the feature that they vanish as $t^{-3+2\epsilon}$ and $t^{-1/2}$, respectively, as $t\to\infty$. Note that the time and spatial dependences of ζ and χ are obtained as part of the solution instead of being some given prescribed functions. Actual values for the numerical parameters ϵ, α, and β would require the more detailed knowledge of the structure of the transport coefficients that only actually studying the Boltzmann transport equation could provide [assuming that is that we could ever show that our solution of Eq. (2.12) is in fact a solution to general-relativistic kinetic theory]. Thus we leave them here as free parameters to be constrained by experimental information.

The solution that we obtain here, namely, Eq. (2.12), also possesses an additional interesting feature since it turns out to correspond to a so-called "self-similarity solution," in which the metric transforms as

$$g'_{\mu\nu}(t',r')=\frac{1}{a^2}g_{\mu\nu}(t,r) \tag{2.20}$$

under the transformation

$$t'=at,\quad r'=a^{1-\epsilon}r,\quad \theta'=\theta,\quad \phi'=\phi\ , \tag{2.21}$$

i.e., in which the metric transforms conformally. In other words, the solution admits a homothetic vector field (for discussions on self-similarity and its role in general relativity, see Ref. 5 and, in particular, Ref. 17). This self-similarity property makes it possible to integrate the null geodesic equation (as will be seen in the next section).

The particular expression that we obtain for ϵ as given by Eq. (2.13) calls for some comment. In order to get a solution that asymptotically approaches a radiation-dominated Friedmann model solution, one would initially expect to have to set ϵ equal to $\frac{1}{2}$. However, as we now explain, the explicit structure of the heat conduction term in our model actually prevents ϵ from being exactly $\frac{1}{2}$. To examine this point closely, we note that conservation of the energy-momentum tensor requires that the transport coefficients have to obey

$$\dot{\rho}+3\dot{\lambda}(\rho+\bar{p})=\frac{1}{(-g)^{1/2}}\frac{\partial}{\partial r}\left[(-g)^{1/2}e^{-2\lambda}\chi\frac{\partial}{\partial r}(Te^{\nu})\right], \tag{2.22}$$

where g denotes the determinant of the metric. Because we require that the bulk-viscosity contribution to $\bar{p}$ vanish faster than both ρ and p at large times, the left-hand side of Eq. (2.22) becomes $\dot{\rho}+4\dot{\lambda}\rho$ which behaves like $(-2+4\epsilon)\rho/t$ in the asymptotic limit of our model and thus behaves like t^{-3} asymptotically. On the other hand, the right-hand side of Eq. (2.22) gives a leading term $12[\epsilon\alpha-(2-\epsilon)\beta]t^{-3}$. Thus the contribution from the heat conduction falls off in the same way as that from the perfect-fluid part. Only if the heat conduction were identically zero at all times, i.e., only if $\epsilon\alpha-(2-\epsilon)\beta$ were zero would the value of ϵ be exactly $\frac{1}{2}$. Thus ϵ cannot be exactly $\frac{1}{2}$ in our model once there is nonzero heat conduction at finite times. Hence even while the bulk viscosity and heat conduction terms fall off faster than ρ and p they do not fall off quite fast enough to enforce $\epsilon=\frac{1}{2}$. Thus while the model approaches Robertson-Walker at large times and is even of the Friedmann form for ρ and p, it does not need to be Friedmann for $R(t)$. Since $R(t)$ is power behaved, the model is still Friedmann for the Hubble parameter. As to an actual numerical value for ϵ given in Eq. (2.13), ϵ will nonetheless turn out to be extremely close to $\frac{1}{2}$ since phenomenologically both α and β turn out to be very small as will be shown in the next section.

To find out the extent to which our model approaches a Robertson-Walker geometry asymptotically we define the density contrast function

$$\Delta(r,t)=\frac{\rho(0,t)-\rho(r,t)}{\rho(0,t)} . \tag{2.23}$$

Using Eq. (2.16) we then find that

$$\Delta(r,t)=\frac{\epsilon^2-\left[\epsilon+(2-\epsilon)\frac{\beta r^2}{t^{2(1-\epsilon)}}\right]^2\left[1+\alpha\frac{r^2}{t^{2(1-\epsilon)}}\right]^{-2}}{\epsilon^2+4\beta} \tag{2.24}$$

Therefore for any finite r, we have

$$\Delta(r,\infty)\equiv\lim_{t\to\infty}\Delta(r,t)=0 . \tag{2.25}$$

However at $r=\infty$, which is the antipodal to $r=0$ since the spatial slices globally have S^3 topology, Eq. (2.24) becomes

$$\Delta(\infty,t)\equiv\lim_{r\to\infty}\Delta(r,t) =\frac{\epsilon^2-(2-\epsilon)^2\left[\frac{\beta}{\alpha}\right]^2}{\epsilon^2+4\beta} =\text{positive const} \tag{2.26}$$

and does not approach zero. The above two equations indicate that for arbitrary values of α and β, [subject to the conditions of Eqs. (2.14) and (2.15), of course] our model does approach a flat Robertson-Walker geometry locally even though it does not do so globally where, as we noted before, the geometry is topologically closed. While the metric at the point $r=\infty$ does not quite become homogeneous with that at the point $r=0$, any region with finite radial coordinate (the only region for which we anyway have any observational data) actually does attain homogeneity at large times, namely, it locally approaches Robertson-Walker space-time. Further, even though the density contrast between the points $r=0$ and $r=\infty$ is a constant, the physical distance between the two points grows larger and larger as the Universe expands, indicating that the gradient of the density contrast decreases as a function of time so that the Universe becomes more and more like that of a Friedmann-Robertson-Walker model. Note that the reason why $\Delta(\infty,t)$ does not approach zero is the very same reason that prevents ϵ from being $\frac{1}{2}$. If we take the special case in which $\alpha=3\beta$ [this being an allowed case according to Eq. (2.14)], we then have $\epsilon=\frac{1}{2}$ and $\Delta(\infty,t)=0$. This special case corresponds to the absence of the heat conduction term for all times. In this sense then it is the heat flow that affects the dynamics for all time.

III. NULL GEODESICS AND THE HORIZON PROBLEM

Photons propagate along null geodesics. For those propagating in the radial direction in the geometry of Eq. (2.12), the null geodesic equation reduces to the radial light-cone equation

$$\frac{dt}{dr}=\pm\frac{t^{\epsilon}}{1+\frac{\alpha r^2}{t^{2(1-\epsilon)}}} , \tag{3.1}$$

where the upper sign corresponds to photons going away from the center and the lower sign corresponds to photons going toward the center of the coordinate system. In terms of the parameter $u \equiv t^{1-\epsilon}/(1-\epsilon)r$, the solution of Eq. (3.1) is

$$Ct^{1-\epsilon} = \left| \frac{u \mp u_2}{u \mp u_1} \right|^{1/w}, \tag{3.2}$$

where

$$u_1 \equiv \frac{1}{2}\left[1 + \left[1 - \frac{4\alpha}{(1-\epsilon)^2}\right]^{1/2}\right],$$

$$u_2 \equiv \frac{1}{2}\left[1 - \left[1 - \frac{4\alpha}{(1-\epsilon)^2}\right]^{1/2}\right],$$

$$w \equiv u_1 - u_2 = \left[1 - \frac{4\alpha}{(1-\epsilon)^2}\right]^{1/2}$$

and C is an integration constant.

For photons traveling "outward" (i.e., increasing r) we take the upper sign

$$(Ct^{1-\epsilon})^w = \left| \frac{u - u_2}{u - u_1} \right| . \tag{3.3}$$

There are then three cases.

(i) $u \geq u_1 > u_2$. In this case Eq. (3.3) leads to

$$r = \frac{t^{1-\epsilon}}{1-\epsilon} \frac{(Ct^{1-\epsilon})^w - 1}{u_1 (Ct^{1-\epsilon})^w - u_2} . \tag{3.4}$$

(ii) $u_1 > u \geq u_2$, which yields

$$r = \frac{t^{1-\epsilon}}{1-\epsilon} \frac{(Ct^{1-\epsilon})^w + 1}{u_1 (Ct^{1-\epsilon})^w + u_2} . \tag{3.5}$$

(iii) $u_1 > u_2 > u$, which yields

$$r = \frac{t^{1-\epsilon}}{1-\epsilon} \frac{1 - (Ct^{1-\epsilon})^w}{u_2 - u_1 (Ct^{1-\epsilon})^w} . \tag{3.6}$$

These three families of solutions are plotted in Fig. 1.

For photons traveling "inward" (decreasing r with respect to time), we take the lower sign in Eq. (3.2):

$$(Ct^{1-\epsilon})^w = \frac{u + u_2}{u + u_1}$$

and the solution is

$$r = \frac{t^{1-\epsilon}}{1-\epsilon} \frac{1 - (Ct^{1-\epsilon})^w}{u_1 (Ct^{1-\epsilon})^w - u_2} . \tag{3.7}$$

A plot of Eq. (3.7) is shown in Fig. 2.

The three families of solutions in Fig. 1 are separated by two particular solutions (the dashed lines labeled by $C=0$ and $C=\infty$ in the figure) which correspond to setting $C=0$ and $C=\infty$, respectively, in Eq. (3.5) and are given by

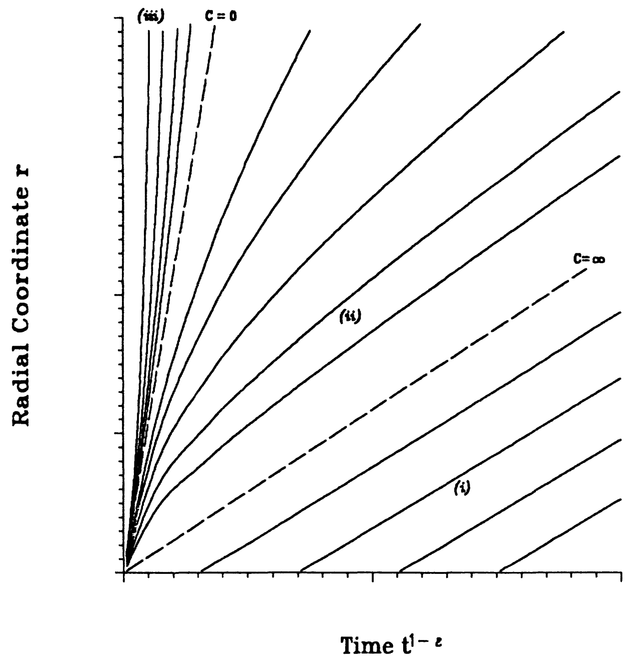

FIG. 1. Outward-going photon geodesics. The figure shows the radial coordinates of photons going outward as functions of time. Note that the horizontal axis is $t^{1-\epsilon}$ so that in the Robertson-Walker limit the curves become straight lines.

$$r = \frac{t^{1-\epsilon}}{(1-\epsilon)u_2}, \tag{3.8}$$

$$r = \frac{t^{1-\epsilon}}{(1-\epsilon)u_1} . \tag{3.9}$$

One special feature of the trajectories in Fig. 1 is that those in regions (ii) and (iii) all start from the origin. No-

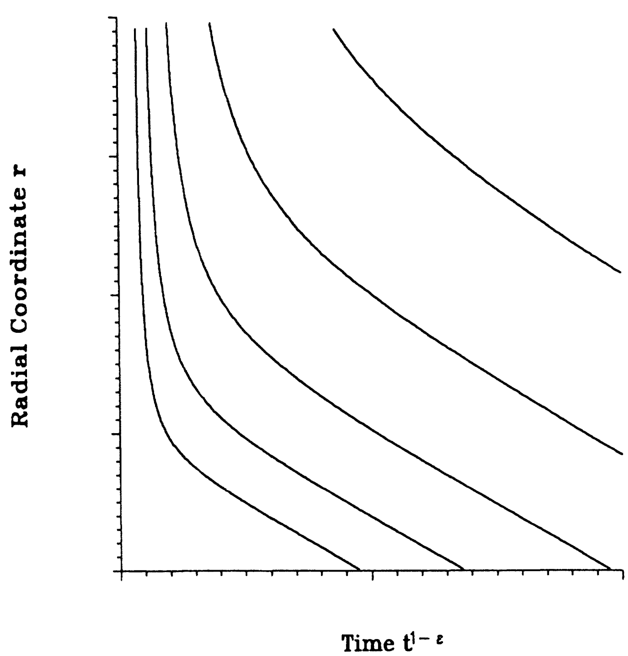

FIG. 2. Inward-going photon geodesics: The figure shows the radial coordinates of photons going inward as functions of time. The curves become parallel straight lines asymptotically.

tice also that in regions (i) and (ii) all curves approach parallel asymptotes (that is they approach their counterparts in the Robertson-Walker case).

Given this general structure for photon trajectories, we now show that we can build a model which is free of the horizon problem if the parameter α is properly chosen. Let us assume that the solar system is located at a place with coordinate r_0 such that

$$r_0 = \frac{t_0^{1-\epsilon}}{(1-\epsilon)u_1} , \tag{3.10}$$

where t_0 is the present age of the Universe. This (simplifying) choice for the value of r_0 such that it falls right on the straight line separating regions (i) and (ii) at the present time t_0 is not essential for our discussion and involves no loss of generality. Let us now examine the trajectory of light that reaches us at t_0 from two opposite directions. The light coming toward us from $r > r_0$ follows Eq. (3.7) (path AS in Fig. 3). The light coming toward us from $r < r_0$ follows Eq. (3.9) (path BS in Fig. 3). An upper limit on the value of the parameter α can be determined from the requirement of isotropy of the current cosmic-microwave-background radiation while a lower limit may be set by requiring causality, that is, by the actual resolving of the horizon problem. We now discuss both of these limits.

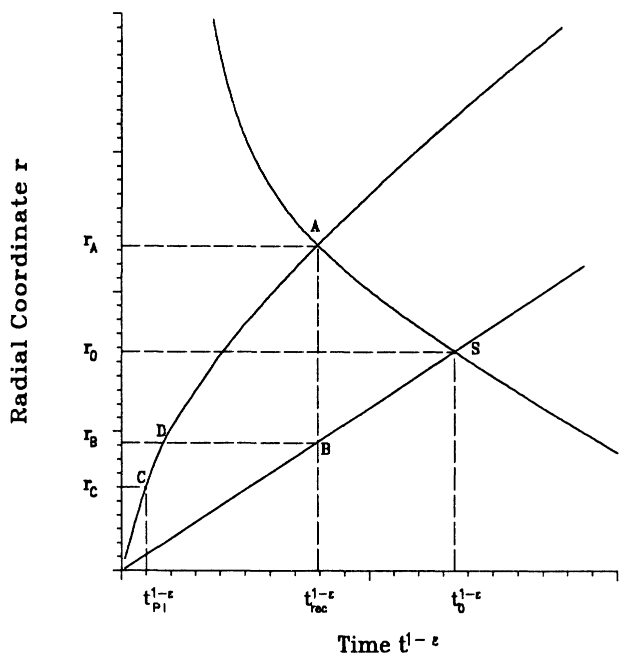

FIG. 3. Photon paths from the Planck epoch to present. Shown here are the geodesics of photons coming from two opposite directions (labeled A and B) and received by a present-day observer at point S. Also shown is the geodesic CA of a photon that causally connects points r_A and r_B (since $r_C < r_B$) prior to the recombination time (i.e., light leaving from point C in the figure at the Planck time arrives at point D and then continues on and reaches point A at t_{rec}, while point D itself becomes point B at t_{rec}). The scales in the figure are arbitrary and have been drawn out of proportion to give a clearer physical picture.

A. The isotropy of the microwave-background radiation

Since the model we present here is inhomogeneous, i.e., since the energy density (and thus the temperature) depend explicitly on the spatial coordinate r, and since we assume the solar system is at $r = r_0 \neq 0$, the temperatures, frequencies and intensities of light signals that we see in the two directions described above are in general different. This means there is always an in-principle anisotropy in the microwave-background radiation. The degree of anisotropy depends on the value of the parameter α. In order to be consistent with the experimental fact that the microwave background is isotropic to a high degree, we need to set a limit to the value of α such that the obtained anisotropy is below the experimental limit. There are two contributions to the anisotropy. The first one arises because, due to the fact that the temperature depends on the spatial coordinate, different points in space have different temperatures. The second contribution is due to the fact that photons emitted from different places have different redshifts by the time they reach the solar system. We will discuss these two points separately.

The cosmic-background radiation that we see today is thought to have been emitted at the time of "recombination" t_{rec} since the Universe would have become opaque immediately thereafter and photons would have propagated freely since then. Hence we have to calculate the coordinates of the points associated with the photons that we see today as of when they were emitted at t_{rec}. Referring to Fig. 3, and taking the radial coordinates of the points A and B from which photons were emitted at t_{rec} to be r_A and r_B, we then obtain

$$r_B = \frac{t_{rec}^{1-\epsilon}}{(1-\epsilon)u_1} , \tag{3.11}$$

$$r_A = \frac{t_{rec}^{1-\epsilon}}{(1-\epsilon)u_1} \frac{u_1 - \frac{1}{2}\left[\left[\frac{t_{rec}}{t_0}\right]^{1-\epsilon}\right]^{w}}{\frac{1}{2}\left[\left[\frac{t_{rec}}{t_0}\right]^{1-\epsilon}\right]^{w} - u_2} . \tag{3.12}$$

Note that Eq. (3.12) is just Eq. (3.7) with the constant C now being determined by the requirement that $r = r_0$ at $t = t_0$. Anticipating α (and hence β) to be very small compared to unity, we simplify the above expressions by expanding in terms of α and keeping only the lowest relevant order:

$$\epsilon \simeq \tfrac{1}{2} + 2(\alpha - 3\beta) \simeq \tfrac{1}{2} , \tag{3.13}$$

$$u_1 \simeq 1 - \frac{\alpha}{(1-\epsilon)^2} \simeq 1 - 4\alpha , \tag{3.14}$$

$$u_2 \simeq 4\alpha , \tag{3.15}$$

$$w = 1 - 8\alpha . \tag{3.16}$$

Substituting these parameters into Eq. (3.11) and (3.12) yields

$$\frac{r_B}{t_{rec}^{1-\epsilon}} \simeq 2 , \tag{3.17}$$

and

$$\frac{r_A}{t_{\rm rec}^{1-\epsilon}} \simeq 4 \left[\frac{t_{\rm rec}}{t_0} \right]^{-1/2} \simeq 6\times 10^3 \; , \tag{3.18}$$

where in the last step we have used the fact that

$$\left[\frac{t_{\rm rec}}{t_0} \right]^{1/2} \simeq \frac{T(t_0)}{T(t_{\rm rec})} \tag{3.19}$$

with this last equality being due to the expression for the temperature which follows from Eqs. (2.10) and (2.16):

$$\begin{aligned} T(r,t) &= \left[\frac{3}{\kappa a t^2} \right]^{1/4} \left[\epsilon^2 \left[1 + \frac{2-\epsilon}{\epsilon} \frac{\beta r^2}{t^{2(1-\epsilon)}} \right]^2 \left[1 + \frac{\alpha r^2}{t^{2(1-\epsilon)}} \right]^{-2} + 4\beta \right]^{1/4} \\ &= T_0(t) \left[\left[1 + \frac{2-\epsilon}{\epsilon} \frac{\beta r^2}{t^{2(1-\epsilon)}} \right]^2 \left[1 + \frac{\alpha r^2}{t^{2(1-\epsilon)}} \right]^{-2} + \frac{4\beta}{\epsilon^2} \right]^{1/4} . \end{aligned} \tag{3.20}$$

Experimentally the quantity $T(t_0)/T(t_{\rm rec})$ is of order $1/1500$. The temperatures at r_A and r_B at the time of recombination are then obtained by substituting Eqs. (3.18) and (3.19) into Eq. (3.20):

$$T_A \equiv T(r_A, t_{\rm rec}) \simeq T_0(t_{\rm rec})[1 + 1.8\times 10^7 (3\beta - \alpha)] \; , \tag{3.21}$$

and

$$T_B \equiv T(r_B, t_{\rm rec}) \simeq T_0(t_{\rm rec})[1 + 2(5\beta - \alpha)] \; . \tag{3.22}$$

Therefore the temperature anisotropy at the recombination time between points A and B is given by

$$\frac{T_A - T_B}{T_0(t_{\rm rec})} \simeq 1.8\times 10^7 (3\beta - \alpha) \; . \tag{3.23}$$

Since $\alpha \geq 3\beta$ the temperature at point A was lower than that at point B.

The second contribution to the anisotropy, namely, the redshift anisotropy, can be calculated as follows. A photon emitted from the point $r = r_A$ at a slightly later time $t = t_{\rm rec} + \Delta t_{\rm rec}$ will be received at the time $t = t_0 + \Delta t_0$ at $r = r_0$. The ratio of the time delay $\Delta t_0 / \Delta t_{\rm rec}$ is equal to the ratio of the frequency of the photon at the place of emission to that at the place of reception, this being the conventional gravitational redshift. The photon emitted at $t = t_{\rm rec} + \Delta t_{\rm rec}$ from $r = r_A$ again follows the trajectory described by Eq. (3.7) with the constant C being determined by the condition that it reaches us at $t = t_0 + \Delta t_0$. Thus we get

$$r_A = \frac{(t_{\rm rec} + \Delta t_{\rm rec})^{1-\epsilon}}{(1-\epsilon) u_1} \times \left[\frac{ 1 - \dfrac{ \dfrac{u_2}{u_1} + \left[1 + \dfrac{\Delta t_0}{t_0} \right]^{1-\epsilon} }{ 1 + \left[1 + \dfrac{\Delta t_0}{t_0} \right]^{1-\epsilon} } \left[\dfrac{t_{\rm rec} + \Delta t_{\rm rec}}{t_0 + \Delta t_0} \right]^{(1-\epsilon) w} }{ \dfrac{ \dfrac{u_2}{u_1} + \left[1 + \dfrac{\Delta t_0}{t_0} \right]^{1-\epsilon} }{ 1 + \left[1 + \dfrac{\Delta t_0}{t_0} \right]^{1-\epsilon} } \left[\dfrac{t_{\rm rec} + \Delta t_{\rm rec}}{t_0 + \Delta t_0} \right]^{(1-\epsilon) w} - \dfrac{u_2}{u_1} } \right] \tag{3.24}$$

Equating the above expression with that of Eq. (3.12) gives a relation between Δt_0 and $\Delta t_{\rm rec}$. Expanding in terms of Δt_0 and $\Delta t_{\rm rec}$ to first order, we obtain, after some arithmetic, the relation

$$\left. \frac{\Delta t_0}{\Delta t_{\rm rec}} \right|_A = \frac{t_0}{t_{\rm rec}} \frac{ \left[\dfrac{t_{\rm rec}}{t_0} \right]^{2(1-\epsilon) w} - 8 u_1 u_2 \left[\dfrac{t_{\rm rec}}{t_0} \right]^{(1-\epsilon) w} + 4 u_1 u_2 }{ (u_1 - u_2)^2 \left[\dfrac{t_{\rm rec}}{t_0} \right]^{(1-\epsilon) w} } \; . \tag{3.25}$$

Similarly for a photon emitted from $r = r_B$ we have

$$\left.\frac{\Delta t_0}{\Delta t_{\rm rec}}\right|_{\rm B} = \frac{t_0}{t_{\rm rec}} \left[\frac{t_{\rm rec}}{t_0}\right]^{(1-\epsilon)w} . \tag{3.26}$$

Note that if one sets $\alpha=0$ in Eq. (3.25) or (3.26) one obtains

$$\left.\frac{\Delta t_0}{\Delta t_{\rm rec}}\right|_{\alpha=0} = \left[\frac{t_0}{t_{\rm rec}}\right]^{1/2} , \tag{3.27}$$

the standard-model result. Therefore the difference in redshifts of the relevant photons is

$$\begin{aligned}\Delta\left[\frac{\nu_{\rm rec}}{\nu_0}\right] &= \left.\frac{\nu_{\rm rec}}{\nu_0}\right|_{\rm A} - \left.\frac{\nu_{\rm rec}}{\nu_0}\right|_{\rm B} \\ &= \left.\frac{\Delta t_0}{\Delta t_{\rm rec}}\right|_{\rm A} - \left.\frac{\Delta t_0}{\Delta t_{\rm rec}}\right|_{\rm B} \\ &= \frac{t_0}{t_{\rm rec}}\frac{4u_1u_2}{(u_1-u_2)^2}\left[\left[\frac{t_{\rm rec}}{t_0}\right]^{(1-\epsilon)w} - 2 + \left[\frac{t_{\rm rec}}{t_0}\right]^{-(1-\epsilon)w}\right] ,\end{aligned} \tag{3.28}$$

so that the relative difference in redshifts is

$$\begin{aligned}\frac{\Delta\left[\dfrac{\nu_{\rm rec}}{\nu_0}\right]}{\left[\dfrac{\nu_{\rm rec}}{\nu_0}\right]_{\alpha=0}} &= \frac{4u_1u_2}{(u_1-u_2)^2}\left[\frac{t_0}{t_{\rm rec}}\right]^{1/2}\left[\left[\frac{t_{\rm rec}}{t_0}\right]^{(1-\epsilon)w} - 2 + \left[\frac{t_{\rm rec}}{t_0}\right]^{-(1-\epsilon)w}\right] \\ &\simeq 16\alpha\times 1500\left[\frac{1}{1500} - 2 + 1500\right] \simeq 3.6\times 10^7\alpha .\end{aligned} \tag{3.29}$$

Comparing Eqs. (3.23) and (3.29), we see that they are of the same order of magnitude unless β is very close to $\alpha/3$ in which case the contribution due to Eq. (3.23) becomes insignificant (this corresponds to the special case where the heat conduction can be neglected). The total temperature anisotropy predicted by the theory is the sum of two contributions:

$$\begin{aligned}\left.\frac{\Delta T}{T}\right|_{\rm total} &= 3.6\times 10^7\alpha + |1.8\times 10^7(3\beta-\alpha)| \\ &= 3.6\times 10^7\alpha + 1.8\times 10^7(\alpha-3\beta) \\ &< 5.4\times 10^7\alpha .\end{aligned} \tag{3.30}$$

Experimentally, the anisotropy is

$$\frac{\Delta T}{T} \le 7\times 10^{-5} . \tag{3.31}$$

Therefore the upper limit on α is

$$\alpha \le 1.3\times 10^{-12} . \tag{3.32}$$

It is indeed very small compared to unity.

B. Causality and the horizon problem

There is a lower limit on α set by the requirement of causality, i.e., by the resolution of the horizon problem. In essence this is the requirement that the points $r=r_{\rm A}$ and $r=r_{\rm B}$ be causally connected with each other before the Universe evolves into the recombination era. On first looking at Fig. 1 and 3, it would seem that there is automatically no causality problem since there are null geodesics in region (iii) of Fig. 1 that reach $r=\infty$ in a finite period of time before the recombination time $t_{\rm rec}$ and thus all such points should be causally connected. However, this is only so if we can apply our classical solution to the entire history of the Universe, including the period prior to the Planck time. This is not the case since in the very early Universe we expect quantum effects to be important and our classical solution no longer applies. Therefore we can only meaningfully study our classical model for the history of the Universe after some natural time scale of the order of the Planck time $t_{\rm Pl}$ itself. So we should instead ask what the condition on α is so that the points $r=r_{\rm A}$ and $r=r_{\rm B}$ would be causally connected during the period from $t_{\rm Pl}$ until $t_{\rm rec}$. In other words, we need to obtain a condition on α such that a photon emitted from a point C for which $r\le r_{\rm B}$ at the Planck time $t_{\rm Pl}$ can reach the point $r=r_{\rm A}$ by the recombination time $t_{\rm rec}$. Conversely, we need to require that a photon received at $t_{\rm rec}$ at $r_{\rm A}$ must have been emitted from a point $r=r_{\rm C}\le r_{\rm B}$ assuming, of course, that there were no collisions along its path.

According to Eq. (3.18) point A is well below the C=0 straight line given by Eq. (3.8). Thus the photon emitted at point C follows the path of Eq. (3.5). Since this CA path is always below the line given by Eq. (3.8) we have

$$r_C < \bar{r} \equiv \frac{t_{\rm Pl}^{1-\epsilon}}{(1-\epsilon)u_2} . \quad (3.33)$$

Thus in order to have $r_C \le r_B$ it is more than sufficient to require the right-hand side of the above inequality to be smaller than r_B: i.e.,

$$\frac{t_{\rm Pl}^{1-\epsilon}}{(1-\epsilon)u_2} \le \frac{t_{\rm rec}^{1-\epsilon}}{(1-\epsilon)u_1} . \quad (3.34)$$

This immediately gives

$$\frac{u_2}{u_1} \ge \left[\frac{t_{\rm Pl}}{t_{\rm rec}}\right]^{1-\epsilon} . \quad (3.35)$$

Using Eqs. (3.13)–(3.15) we find that α is constrained according to

$$\alpha \ge \frac{1}{4}\left[\frac{t_{\rm Pl}}{t_{\rm rec}}\right]^{1/2} \simeq \frac{1}{4}\left[\frac{10^{-44}}{10^{11}}\right]^{1/2} \simeq 10^{-28} . \quad (3.36)$$

This lower limit is small enough to be completely acceptable for our purposes here.

In this section we thus find a range of acceptable values for α by requiring that the model be in agreement with the experimentally established isotropy of the current microwave-background radiation and by requiring that the model be causal (free of the horizon problem). The first requirement gives the upper limit and the second gives the lower one. These limits not only do not contradict each other but actually give the extremely wide (and thus not fine-tuned) range:

$$10^{-28} \le \alpha \le 1.3\times 10^{-12} . \quad (3.37)$$

IV. ENTROPY PRODUCTION

In addition to the physical conditions stated in the preceding two sections, a physical model has to satisfy the following thermodynamic conditions.

(i) Conservation of baryon number

$$(nU^\mu)_{;\mu}=0 , \quad (4.1)$$

where n is the particle number density and U^μ is the fluid four-velocity vector.

(ii) Gibbs's relation

$$Td\left[\frac{S}{n}\right]=d\left[\frac{\rho}{n}\right]+pd\left[\frac{1}{n}\right] , \quad (4.2)$$

where T is the temperature and S is the entropy density.

(iii) Positivity of entropy production

$$S^\mu{}_{;\mu}\ge 0 , \quad (4.3)$$

where $S^\mu = SU^\mu + q^\mu/T$ is the entropy flux and q^μ is given by Eq. (2.4).

Since, for the general case of Eqs. (2.1)–(2.4),

$$S^\mu{}_{;\mu}=\frac{\zeta}{T}(3\dot{\lambda}e^{-\nu})^2+\frac{\chi}{T^2}\left[e^{-\lambda-\nu}\frac{\partial}{\partial r}(Te^{\nu})\right]^2 + \frac{4\eta}{3T}e^{-2\nu}\left[\dot{\lambda}-\frac{\dot{Y}}{Y}\right]^2 \quad (4.4)$$

the positivity of entropy production is indeed guaranteed if the transport coefficients χ, ζ, and η are all nonnegative, conditions which we have in fact imposed in all of the solutions considered both here and in our companion paper. The increase of total entropy obtained below is then a direct manifestation of this property.

With regard to the baryon number density, we note that with U^μ given by Eq. (2.2), Eq. (4.1) becomes

$$\partial_t[ne^{-\nu}(-g)^{1/2}]=0 \quad (4.5)$$

which can then be solved for n. For the shear-free geometry of interest studied in this paper [viz. Eq. (2.12)] this yields

$$n(r,t)=\frac{n_0(r)}{r^2t^{3\epsilon}}\left[1+\frac{\beta r^2}{t^{2(1-\epsilon)}}\right]^3 , \quad (4.6)$$

where $n_0(r)$ is a convenient integration function. Substituting $\rho=aT^4$ into Eq. (4.2) we obtain for the entropy-to-baryon ratio for our model:

$$s\equiv\frac{S}{n}=\frac{4}{3}a\frac{T^3}{n}=\left[\frac{4}{3}\left[\frac{a(3\epsilon^2)^3}{\kappa^3}\right]^{1/4}\frac{r^2}{n_0(r)}\right] \times\frac{t^{3(\epsilon-1/2)}\left[\left[1+\frac{2-\epsilon}{\epsilon}\frac{\beta r^2}{t^{2(1-\epsilon)}}\right]^2\left[1+\frac{\alpha r^2}{t^{2(1-\epsilon)}}\right]^{-2}+\frac{4\beta}{\epsilon^2}\right]^{3/4}}{\left[1+\frac{\beta r^2}{t^{2(1-\epsilon)}}\right]^3} \quad (4.7)$$

It is easy to check that this expression is an increasing function of t. Actually it is zero at $t=0$ and monotonically increases to infinity as $t\to\infty$. For the region of the presently observed Universe, i.e., $r_B\le r\le r_A$, the quantity $\alpha r^2/t_0^{2(1-\epsilon)}$ is always negligible compared to unity. Thus

$$s_{\rm present}=s(t_0)=\left[\frac{4}{3}\left[\frac{a(3\epsilon^2)^3}{\kappa^3}\right]^{1/4}\frac{r^2}{n_0(r)}\right]t_0^{3(\epsilon-1/2)} . \quad (4.8)$$

Equation (4.7) can thus be written as

$$s=s_{\text{present}}\left[\frac{t}{t_0}\right]^{3(\epsilon-1/2)}\frac{\left[\left[1+\frac{2-\epsilon}{\epsilon}\frac{\beta r^2}{t^{2(1-\epsilon)}}\right]^2\left[1+\frac{\alpha r^2}{t^{2(1-\epsilon)}}\right]^{-2}+\frac{4\beta}{\epsilon^2}\right]^{3/4}}{\left[1+\frac{\beta r^2}{t^{2(1-\epsilon)}}\right]^3} . \tag{4.9}$$

We now calculate the entropy per baryon at the Planck time for the place where the solar system currently is, i.e., at $r=r_0=t_0^{1-\epsilon}/(u_1-u_1\epsilon)$,

$$s(r_0,t_{\text{Pl}})=s_{\text{present}}(r_0)\left[\frac{t_{\text{Pl}}}{t_0}\right]^{3(\epsilon-1/2)}\frac{\left[\left[1+\frac{2-\epsilon}{\epsilon}\frac{\beta t_0^{2(1-\epsilon)}}{(1-\epsilon)^2u_1^2t^{2(1-\epsilon)}}\right]^2\left[1+\frac{\alpha t_0^{2(1-\epsilon)}}{(1-\epsilon)^2u_1^2t^{2(1-\epsilon)}}\right]^{-2}+\frac{4\beta}{\epsilon^2}\right]^{3/4}}{\left[1+\frac{\beta t_0^{2(1-\epsilon)}}{(1-\epsilon)^2u_1^2t^{2(1-\epsilon)}}\right]^3}$$

$$\simeq s_{\text{present}}\left[\frac{3}{16\alpha\beta}\right]^{3/2}\left[\frac{t_{\text{Pl}}}{t_0}\right]^3 . \tag{4.10}$$

Thus we see that by choosing the values of the parameters the ratio $s_{\text{present}}/s_{\text{Pl}}$ can be made as large as we want it to be. Phenomenologically we would like this ratio to be at least of order 10^{85} in order to solve the standard entropy problem. By substituting this value and the values $t_{\text{Pl}}\simeq 10^{-44}$ sec and $t_0\simeq 10^{17}$ sec, Eq. (4.10) then gives the following constraint on the parameters α and β:

$$\alpha\beta\geq 10^{-66} . \tag{4.11}$$

As long as Eq. (4.11) is satisfied the present entropy per baryon is at least 10^{85} times that of the Planck epoch. Given the condition of Eq. (4.11) the present extremely large value of entropy per baryon is then indeed produced after the Planck epoch in our model. Therefore the entropy problem that exists in the standard model does not exist here. The relation given in Eq. (4.11) can be considered yet another constraint on the value of β in addition to the ones given in Eqs. (2.14) and (2.15). These relations can now be written in the combined form

$$\frac{\alpha}{3}\geq\beta\geq\beta_m(\alpha) , \tag{4.12}$$

where $\beta_m(\alpha)$ is the larger of $10^{-66}/\alpha$ and $4\alpha^2/3$. In other words, if α is in the range $1.3\times10^{-12}\geq\alpha\geq 9.1\times10^{-23}$ the allowed range for β is $\alpha/3\geq\beta\geq 4\alpha^2/3$, while if α is in the range $9.1\times10^{-23}\geq\alpha\geq 10^{-28}$ the range for β is $\alpha/3\geq\beta\geq 10^{-66}/\alpha$. Thus the range of allowed values for β is just as acceptable for us as the range of allowed values for α already found in Eq. (3.37).

We can also look at the entropy problem in a slightly different way. Specifically, we calculate the total entropy in the Universe:

$$\mathcal{S}_{\text{total}}=\int S^0(-g)^{1/2}d^3x , \tag{4.13}$$

where S^0 is the zeroth component of the entropy current S^μ. To simplify the arithmetic we take the upper limit in Eq. (4.12) and consider the special case $3\beta=\alpha$. This corresponds to the zero heat conduction case. In this case the integral in Eq. (4.13) can readily be performed, and we obtain

$$\mathcal{S}_{\text{total}}=\mathcal{S}_{\text{total}}(t_0)\left[\frac{t}{t_0}\right]^{3/2} . \tag{4.14}$$

The total entropy of the Universe increases in time as $t^{3/2}$ and is finite. The finiteness of the total entropy is due to the finiteness of the volume of the Universe in our model, a feature which we shall show momentarily in Sec. 5. From Eq. (4.14) we can also calculate the total entropy at the Planck time, and obtain

$$\mathcal{S}_{\text{total}}(t_{\text{Pl}})=\mathcal{S}_{\text{total}}(t_0)\left[\frac{t_{\text{Pl}}}{t_0}\right]^{3/2}=10^{-91}\mathcal{S}_{\text{total}}(t_0) . \tag{4.15}$$

so it is indeed as small as we want it be.

V. THE FLATNESS PROBLEM

In the preceding section we indicated that the Universe has a finite volume. We now elaborate on this point a little further. In our model the spatial part of the metric is given as

$$dl^2=\frac{t^{2\epsilon}}{\left[1+\frac{\beta r^2}{t^{2(1-\epsilon)}}\right]^2}(dr^2+r^2d\Omega) . \tag{5.1}$$

and its associated spatial volume element is given by

$$dV=\frac{t^{3\epsilon}}{\left[1+\frac{\beta r^2}{t^{2(1-\epsilon)}}\right]^3}r^2\sin\theta dr\, d\theta\, d\phi . \tag{5.2}$$

The total volume is thus

$$V=4\pi t^{3\epsilon}\int_0^\infty\frac{r^2dr}{\left[1+\frac{\beta r^2}{t^{2(1-\epsilon)}}\right]^3}=\frac{\pi^2}{4}\frac{t^3}{\beta^{3/2}} . \tag{5.3}$$

Comparing with the spatial part of a closed Robertson-Walker metric written in isotropic coordinates

$$dl^2_{\rm RW}=\frac{R^2}{\left[1+\frac{k}{4}r^2\right]^2}(dr^2+r^2d\Omega) \tag{5.4}$$

we find that the spatial metric of Eq. (5.1) represents a geometry with an effective positive curvature "constant" $k(t)\equiv 4\beta/t^{2(1-\epsilon)}$ which decreases with time and only approaches zero as $t\to\infty$. To further clarify this point we introduce the dimensionless quantity[18]

$$\Omega=\frac{3\kappa\rho}{\Theta^2} \tag{5.5}$$

which would correspond to the quantity ρ/ρ_c in the Friedmann model ($\Theta=U^\alpha{}_{;\alpha}$ may be identified with the conventionally defined fluid expansion parameter). In the geometry of Eq. (2.12) we find that Ω is given by

$$\Omega=1+4\beta\left[\epsilon+(2-\epsilon)\frac{\beta r^2}{t^{2(1-\epsilon)}}\right]^{-2}\left[1+\frac{\alpha r^2}{t^{2(1-\epsilon)}}\right]^2 . \tag{5.6}$$

We thus see that, as $t\to\infty$,

$$\Omega(t\to\infty)=1+\frac{4\beta}{\epsilon^2} \tag{5.7}$$

which is very close to 1 since β is small, indicating that the Universe is very flat for late times. On the other hand, at $t=0$ we have

$$\Omega(t=0)=1+4\beta\frac{\alpha^2}{(2-\epsilon)^2\beta^2} \tag{5.8}$$

which is in general much larger than $\Omega(t\to\infty)$ and, even in the special case of $\alpha=3\beta$ would only be as close to 1 as $\Omega(t\to\infty)$ is. The second term on the right-hand side of Eq. (5.6) is the dimensionless curvature scalar K introduced in Ref. 18. In our model K is, in general, a decreasing function of t in sharp contrast to the situation in the standard model where it actually increases with time. In our model K decreases from an initial value of $4\beta\{\alpha/[(2-\epsilon)\beta]\}^2$ [which can be as large as $\frac{4}{3}$ according to Eq. (2.15)] to a very small final value of $4\beta/\epsilon^2$. Our ability then to obtain a behavior for Ω which is so different from that of the standard model is because, as indicated in the Introduction, our Eqs. (2.6)–(2.9) differ more and more radically from the standard Friedmann model equations the further back in time we look. Additionally, Eq. (5.3) shows that the volume of the Universe in our model increases without limit as a function of time in sharp contrast to the closed Friedmann-Robertson-Walker case where the volume eventually starts to decrease after reaching a maximum value (see Sec. VI for further discussion on this point). Therefore the Universe in our model is a finite volume one at any finite time (and hence has finite total entropy) with an effective positive curvature which only approaches zero asymptotically. The Universe becomes flatter and flatter as time goes on and the volume gets bigger and bigger. The flatness problem is thus naturally solved in the model inasmuch as no fine-tuning of parameters is needed.

VI. SOME ADDITIONAL REMARKS

We have constructed a model of an expanding universe that is free of three of the major problems of the standard hot big-bang model, namely, the horizon, entropy and flatness problems. Even though the model is intrinsically inhomogeneous, we can nonetheless select the free parameters without fine-tuning so that the presently observed Universe is able to be homogeneous enough today so as not to contradict current observational facts. Because the parameters α and β are so small, the spatial dependence of the temperature is negligible at any time after the recombination time. In other words, the temperature after the recombination time is practically spatially independent and is given by the same expression as that given in the standard model. That is why we were able to take the same estimate for the values of $t_{\rm rec}$ and t_0 as in the standard model.

We now make some final closing remarks that are relevant to our model.

(1) Throughout the paper we have discussed only the radiation-dominated case. A more realistic model has to include a matter-dominated phase since we know that the Universe must undergo a transition from the radiation-dominated phase to the matter-dominated one. A matter-dominated model can be built in the same way as the one we built for the radiation-dominated area. In this case the pressure p is zero, so that $\bar{p}=-\zeta U^\alpha{}_{;\alpha}$ has to be negative semidefinite. It can be shown that if α and β obey a set of conditions similar to those of Eq. (2.15), this requirement can be satisfied. The value of ϵ obtained is very close to $\frac{2}{3}$, the value that it has in the standard matter-dominated era. All the rest of the discussion of this paper can be applied to this matter-dominated phase, including the null geodesic equation which is integrable regardless of what specific value the parameter ϵ takes. Thus we can build a more realistic model by replacing the metric with the matter-dominated one for the time period subsequent to the recombination time. All of our discussion about the horizon, entropy, and flatness problems can be carried out in the same manner and the conclusions are not substantially changed.

(2) It is of interest to examine the heat flow vector:

$$\begin{aligned} q^1 &= -\chi e^{-\nu}e^{-2\lambda}\frac{\partial}{\partial r}(Te^\nu) \\ &= -2e^{-\nu}e^{-2\lambda}(\nu'\dot{\lambda}-\dot{\lambda}')/\kappa . \end{aligned} \tag{6.1}$$

In our model we find that q^1 turns out to be negative. This means that heat flows from the large-r region toward the center. However, because of Eq. (2.14), the energy density (and thus likewise the temperature) is a decreasing function of r. Therefore the heat flows from a low-T region to a high-T region. The mathematical reason for this is as follows. The heat-flow vector depends not only on the gradient of T but also on the acceleration vector $\dot{U}_\alpha\equiv U_{\alpha;\beta}U^\beta$. In our solution the acceleration contribution is opposite to and larger in magnitude than the gradient of T (i.e., the spatial derivatives of T and Te^{ν} have opposite signs in our solution). Thus the heat flow is dominated by the acceleration. The fact that the heat flows against the temperature gradient may at first seem counterintuitive. However, our notion of heat flowing in the direction of the temperature gradient has been obtained in standard nonrelativistic laboratory experiments. Once we are in an accelerating frame with the fluid not being comoving with the geometry (this being the case in our model), we do not possess as much intuition. Specifically, what we actually extract out from the familiar nonrelativistic correlation between heat flow and temperature gradient is only the fact that the general covariant scalar heat conductivity coefficient χ of Eq. (2.4) is positive. This is then the intuitive feature of heat flow which we retain in the general-relativistic case, with Eq. (2.4) then correlating the sign of the heat-flow vector q^{μ} with the vector $H^{\mu\alpha}(T_{;\alpha}+TU_{\alpha;\beta}U^{\beta})$ rather than with the temperature gradient itself. Since the sign of the radial component of the (spacelike in our case) vector q^{μ} is not a general-relativistic invariant, there thus appears to be no explicit correlation between the directions of heat flow and temperature gradient in the general case.

(3) It can be readily shown that our solution also satisfies the desirable so-called dominant energy condition[19] as long as the ratio α/β is not too large. However, since $\bar{p}$ is negative and large in magnitude at early times due to the contribution from the bulk viscosity, the so-called strong energy condition[19] is not satisfied in our model. Now unlike the dominant energy condition, the strong energy condition not as well established as a physical criterion, and it is interesting to note that it is not in fact satisfied in the familiar inflationary universe model, for instance. Moreover, as we shall see immediately, it may even be advantageous to our model that the strong energy condition not in fact obtain.

(4) As mentioned before, Eq. (5.1) indicates that the constant-t slices of the space-time manifold have S^3 topology, and yet the model expands indefinitely. This brings to mind the interesting closed-universe-recollapse conjecture[20] which states that a closed universe (one with S^3 spatial topology, for instance) will under certain physical conditions stop expanding and recollapse just like a closed Friedmann-Robertson-Walker universe does. The conjecture in the form given in Ref. 20 requires that both the strong energy condition and the positive pressure criterion be obeyed. As discussed in point (3) our model does not in fact satisfy these conditions. Thus our model does not constitute a counterexample to the conjecture given in Ref. 20.

To conclude then we see that in this paper we have found a new exact solution to imperfect-fluid cosmologies (a result which is of interest in and of itself), and have shown that inhomogeneous universes may even be of relevance to real cosmologies.

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