Electromagnetic form factors of spin-$\frac{3}{2}$ baryons

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Relationships are established among current matrix elements, covariant vertex functions, and multipole form-factor decompositions for the case of electromagnetic interactions of spin-$\frac{3}{2}$ systems. The electromagnetic current matrix element for spin-$\frac{3}{2}$ baryons is defined in terms of the minimum required four independent covariant vertex functions. Explicit Lorentz-invariant relations for the multipole form factors are derived in terms of the previously defined covariant vertex function coefficients. The derivation does not involve any nonrelativistic approximations. Finally, the multipole form factors are isolated and expressed in terms of the electromagnetic current matrix elements. These results are particularly useful in lattice QCD calculations.

I. INTRODUCTION

The electromagnetic form factors of hadrons are an indispensable tool for exploring the underlying structure of hadrons. Both model-dependent and model-independent nonperturbative approaches to quantum chromodynamics such as lattice gauge theory or the QCD sum-rule approach have exploited the well-known electromagnetic current matrix elements[1] to calculate quantities such as magnetic moments, electric and magnetic charge radii, and magnetic transition moments of low-lying hadronic states.

The study of the electromagnetic form factors of spin-$\frac{3}{2}$ systems is a well-explored field. Aspects such as the multipole decomposition of current matrix elements and the general covariant vertex functions have received a great deal of attention. The purpose of this paper is to provide relationships among the current matrix elements, covariant vertex functions, and multipole form-factor decompositions for the case of electromagnetic interactions of spin-$\frac{3}{2}$ systems. In this paper we define the electromagnetic current matrix element for spin-$\frac{3}{2}$ baryons in terms of the minimum required four independent covariant vertex functions. The main focus of this paper is to derive explicit Lorentz-invariant relations for the multipole form factors in terms of the previously defined covariant vertex function coefficients. A previous attempt[2] to do this using a nonrelativistic approximation contains some errors, and the results are incomplete. Our derivation does not involve any nonrelativistic approximations. The results are presented in a general framework such that they may be immediately and easily applied to a wide range of analytical calculations such as the QCD sum-rule approach, electroproduction of mesons off spin-$\frac{3}{2}$ baryons, pion-nucleon bremsstrahlung processes involving intermediate spin-$\frac{3}{2}$ baryons, etc. Finally, we invert the multipole decomposition of the current matrix element to isolate and express the multipole form factors in terms of the electromagnetic current matrix elements. These results are particularly useful in lattice gauge calculations, and therefore we present these results with specific reference to lattice QCD.

With recent attempts to measure the Ω^- magnetic moment,[3] theoretical investigations of the electromagnetic properties of the low-lying spin-$\frac{3}{2}$ baryons are rather timely and appropriate. Furthermore, nonperturbative approaches such as lattice QCD or QCD sum rules are at present generally limited to the investigations of the lowest-lying baryon state for a given total angular momentum. Hence, in the investigation of the excited-state spectrum of baryons, this work is only one step in a trend toward the analysis of higher angular momentum systems.

II. MULTIPOLE FORM FACTORS

In defining the electromagnetic form factors of spin-$\frac{3}{2}$ baryons, we consider the following electromagnetic current matrix element:

$$\langle p',s'|j^{\mu}(0)|p,s\rangle=\bar{u}_{\alpha}(p',s')\mathcal{O}^{\alpha\mu\beta}u_{\beta}(p,s) \ . \tag{1}$$

Here p,p' denote momenta, s,s' spins, and $u_{\alpha}(p,s)$ is a spinvector in the Rarita-Schwinger[4] formalism. We have obtained the following Lorentz-covariant form for the tensor:

$$\mathcal{O}^{\alpha\mu\beta}=g^{\alpha\beta}\left[a_1\gamma^{\mu}+\frac{a_2}{2M_B}P^{\mu}\right]+\frac{q^{\alpha}q^{\beta}}{(2M_B)^2}\left[c_1\gamma^{\mu}+\frac{c_2}{2M_B}P^{\mu}\right] , \tag{2}$$

where $P=p'+p$, $q=p'-p$, and M_B is the mass of the baryon considered. Here and in the following we follow the Dirac representation for the γ matrices illustrated in Ref. 1. In (2) the parameters a_1, a_2, c_1, and c_2 are independent covariant vertex function coefficients which will be related to the multipole form factors.

Equation (2) satisfies the standard requirements of invariance under time reversal (T), parity (P), G parity, and gauge invariance. In deriving (2) we have also used the following subsidiary conditions for the on-shell spin-$\frac{3}{2}$ baryons:

$$\gamma^\mu u_\mu(p,s)=0, \quad p^\mu u_\mu(p,s)=0 , \tag{3a}$$

$$\bar{u}_\mu(p',s')\gamma^\mu=0, \quad \bar{u}_\mu(p',s')p'^\mu=0 . \tag{3b}$$

Note that in Ref. 2 the expression analogous to (2) contains two additional terms proportional to $(q^\alpha g^{\mu\beta}-q^\beta g^{\mu\alpha})$ and $i\epsilon^{\alpha\mu\beta\nu}q_\nu\gamma_5$. However, it is known[5-7] that the number of independent amplitudes should be four in the vertex of (1). Indeed, the following nontrivial identity relations show that the additional two terms in Ref. 2 are *not* linearly independent of the other terms in (2). One can show

$$(q^\alpha g^{\mu\beta}-q^\beta g^{\mu\alpha})=2M_B\left[1-\frac{q^2}{4M_B^2}\right]g^{\alpha\beta}\gamma^\mu -g^{\alpha\beta}P^\mu+\frac{1}{M_B}q^\alpha q^\beta\gamma^\mu \tag{4a}$$

and

$$-i\epsilon^{\alpha\beta\mu\nu}a_\mu b_\nu\gamma_5=(\not{a}\not{b}-a\cdot b)i\sigma^{\alpha\beta}+\not{b}(\gamma^\alpha a^\beta-\gamma^\beta a^\alpha) -\not{a}(\gamma^\alpha b^\beta-\gamma^\beta b^\alpha) +(a^\alpha b^\beta-a^\beta b^\alpha) , \tag{4b}$$

where a_μ and b_μ are arbitrary four-vectors. Note that (4a) is valid only for on-shell baryons.[8]

In our calculations we use a well-known explicit form of the Rarita-Schwinger spinor $u_\mu(p,s)$ in terms of the spin-1 vector $\epsilon_\mu(p,m)$ and the Dirac spinor $u(p,\bar{s})$; i.e.,

$$u_\mu(p,s)=\sum_{m,\bar{s}}(1m\tfrac{1}{2}\bar{s}|1\tfrac{1}{2}\tfrac{3}{2}s)\epsilon_\mu(p,m)u(p,\bar{s}) , \tag{5}$$

where the coefficient in (5) is a Clebsch-Gordan coefficient in Condon-Shortley phase convention.[9] The polarization vector $\epsilon_\mu(p,m)$ is given by

$$\epsilon^0(p,m)=\frac{\hat{\boldsymbol{\epsilon}}_m\cdot\mathbf{p}}{M_B} , \qquad \boldsymbol{\epsilon}(p,m)=\hat{\boldsymbol{\epsilon}}_m+\frac{\hat{\boldsymbol{\epsilon}}_m\cdot\mathbf{p}}{M_B}\frac{\mathbf{p}}{E+M_B} \quad (m=\pm1,0) , \tag{6}$$

where $E=(|\mathbf{p}|^2+M_B^2)^{1/2}$ and $\hat{\boldsymbol{\epsilon}}_m$ is a spherical unit vector which satisfies $\hat{\boldsymbol{\epsilon}}_m^*\cdot\hat{\boldsymbol{\epsilon}}_{m'}=\delta_{mm'}$. Note that the spin-$\frac{3}{2}$ spinor $u_\mu(p,s)$ defined by (5) satisfies (3) as well as the Dirac equation.

The multipole expansion of the electromagnetic current matrix element is well known.[2,6,7] In the laboratory or Breit frame, the current matrix element is

$$\langle p',s'|j^0(0)|p,s\rangle=-A\langle\tfrac{3}{2}s'|G_{E0}(q^2)+2\sqrt{5}\tau G_{E2}(q^2)[\Sigma^{(2)}\times(\hat{\mathbf{q}}\times\hat{\mathbf{q}})^{(2)}]^{(0)}|\tfrac{3}{2}s\rangle , \tag{7a}$$

$$\langle p',s'|\mathbf{j}(0)|p,s\rangle=-\sqrt{\tau}\langle\tfrac{3}{2}s'|\{G_{E0}(q^2)+2\sqrt{5}\tau G_{E2}(q^2)[\Sigma^{(2)}\times(\hat{\mathbf{q}}\times\hat{\mathbf{q}})^{(2)}]^{(0)}\}\hat{\mathbf{P}} +i\{\tfrac{1}{3}G_{M1}(q^2)\Sigma^{(1)}+3\tau G_{M3}(q^2)[\Sigma^{(3)}\times(\hat{\mathbf{q}}\times\hat{\mathbf{q}})^{(2)}]^{(1)}\}\times\hat{\mathbf{q}}|\tfrac{3}{2}s\rangle , \tag{7b}$$

where $\tau=-q^2/(2M_B)^2$ (≥0), and $\hat{\mathbf{P}}$ and $\hat{\mathbf{q}}$ are unit vectors. In (7), A is a factor dependent on the kinematics; i.e., $A=\sqrt{1+\tau}$ for a laboratory frame ($\mathbf{p}=0$) and $A=1$ for a baryon Breit frame ($\mathbf{P}=\mathbf{p}'+\mathbf{p}=0$). Note that the term proportional to $\hat{\mathbf{P}}$ vanishes in the baryon Breit frame. The spin matrix elements in (7) are defined by Clebsch-Gordan coefficients:

$$\langle\tfrac{3}{2}s'|\tfrac{3}{2}s\rangle=\delta_{s's} , \tag{8a}$$

$$\langle\tfrac{3}{2}s'|\Sigma_m^{(1)}|\tfrac{3}{2}s\rangle=\sqrt{15}(\tfrac{3}{2}s'1m|\tfrac{3}{2}1\tfrac{3}{2}s) , \tag{8b}$$

$$\langle\tfrac{3}{2}s'|\Sigma_m^{(2)}|\tfrac{3}{2}s\rangle=-\sqrt{\tfrac{5}{6}}(\tfrac{3}{2}s'2m|\tfrac{3}{2}2\tfrac{3}{2}s) , \tag{8c}$$

$$\langle\tfrac{3}{2}s'|\Sigma_m^{(3)}|\tfrac{3}{2}s\rangle=-\tfrac{7}{6}\sqrt{\tfrac{2}{3}}(\tfrac{3}{2}s'3m|\tfrac{3}{2}3\tfrac{3}{2}s) . \tag{8d}$$

In (7), G_{E0}, G_{E2}, G_{M1}, and G_{M3} are called charge ($E0$), electroquadrupole ($E2$), magnetic-dipole ($M1$) and magnetic-octupole ($M3$) multipole form factors, respectively.

By inserting (2) and (5) into (1), and working out a lengthy but straightforward calculation, we obtain the following expressions for the multipole form factors in terms of the covariant vertex function coefficients a_1, a_2, c_1, and c_2:

$$G_{E0}(q^2)=(1+\tfrac{2}{3}\tau)[a_1+(1+\tau)a_2]-\tfrac{1}{3}\tau(1+\tau)[c_1+(1+\tau)c_2] , \tag{9a}$$

$$G_{E2}(q^2)=[a_1+(1+\tau)a_2]-\tfrac{1}{2}(1+\tau)[c_1+(1+\tau)c_2] ,$$

$$G_{M1}(q^2)=(1+\tfrac{4}{5}\tau)a_1-\tfrac{2}{5}\tau(1+\tau)c_1 , \tag{9c}$$

$$G_{M3}(q^2)=a_1-\tfrac{1}{2}(1+\tau)c_1 . \tag{9d}$$

Note that (9) is a Lorentz-invariant expression in which all orders of τ have been kept. It should also be noted that the $E2$ and $M3$ form factors are nonzero even in the case where only the a_1 and a_2 terms are kept, although they are linearly dependent to the $E0$ and $M1$ form factors, respectively. The nonrelativistic forms may be easily extracted in the Breit frame by keeping terms to first order in τ.

We define the spin-independent hadron tensor as

$$W^{\mu\nu}=\frac{1}{4}\sum_{ss'}\langle p',s'|j^{\mu}(0)|p,s\rangle\langle p',s'|j^{\nu}(0)|p,s\rangle^{*}$$
$$=W_1\left[-g^{\mu\nu}+\frac{q^{\mu}q^{\nu}}{q^2}\right]+\frac{W_2}{M_B^2}\left[p^{\mu}-\frac{p\cdot q}{q^2}q^{\mu}\right]\left[p^{\nu}-\frac{p\cdot q}{q^2}q^{\nu}\right]. \tag{10}$$

W_1 and W_2 are related to the experimental cross section for the elastic scattering of an electron in a one-photon-exchange approximation through the well-known formula

$$\frac{d\sigma}{d\Omega_e}=\frac{Z_B^2\alpha^2\cos^2(\theta_e/2)}{4E_e^2\sin^4(\theta_e/2)[1+(2E_e/m_e)\sin^2(\theta_e/2)]}\left[W_2+2W_1\tan^2\frac{\theta_e}{2}\right], \tag{11}$$

where $Z_B=2$ for doubly charged baryons, and otherwise $Z_B=1$. Using (1), (2), and (9), we obtain W_1 and W_2 in terms of the multipole form factors as follows:

$$W_1=\tfrac{5}{9}\tau G_{M1}^2(q^2)+\tfrac{8}{15}\tau^3G_{M3}^2(q^2), \tag{12a}$$

$$W_2=\frac{1}{1+\tau}\left[G_{E0}^2(q^2)+\frac{5}{9}\tau G_{M1}^2(q^2)+\frac{4}{9}\tau^2G_{E2}^2(q^2)+\frac{8}{15}\tau^3G_{M3}^2(q^2)\right]. \tag{12b}$$

By (7), (9), and (12), the electromagnetic form factors of spin-$\frac{3}{2}$ baryons are well defined at the hadronic level.

The final part of this paper is devoted to extracting the electromagnetic multipole form factors in full relativistic form from the current matrix elements. These results are particularly useful in lattice gauge calculations, and therefore we present these results with specific reference to lattice QCD. The study of electromagnetic form factors in lattice QCD centers around the three-point function[10]

$$\langle G_{Nj^{\mu}N}(t_2,t_1;\mathbf{p}',\mathbf{p};\Gamma)\rangle_{\alpha\beta}=\sum_{\mathbf{x}_2,\mathbf{x}_1}e^{-i\mathbf{p}'\cdot\mathbf{x}_2}e^{+i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}_1}\mathrm{Tr}\{\Gamma\langle\Omega|T[\chi_\alpha(x_2)j^{\mu}(x_1)\bar{\chi}_\beta(0)]|\Omega\rangle\}. \tag{13}$$

$\chi_\alpha(x)$ is an interpolating field coupling to spin-$\frac{3}{2}$ baryons, Ω represents the QCD vacuum, and Γ is a 4×4 matrix in Dirac space, which will be discussed in detail. For large Euclidean time separations $t_2-t_1\gg1$ and $t_1\gg1$, the three-point function at the hadronic level takes the limit

$$\langle G_{Nj^{\mu}N}(t_2,t_1;\mathbf{p}',\mathbf{p};\Gamma)\rangle_{\alpha\beta}=\sum_{s,s'}e^{-E_{p'}(t_2-t_1)}e^{-E_pt_1}\mathrm{Tr}[\Gamma\langle\Omega|\chi_\alpha(0)|p',s'\rangle\langle p',s'|j^{\mu}(0)|p,s\rangle\langle p,s|\bar{\chi}_\beta(0)|\Omega\rangle]. \tag{14}$$

At the hadronic level one defines

$$\langle\Omega|\chi_\alpha(0)|p,s\rangle=\lambda u_\alpha(p,s), \tag{15}$$

where λ represents the coupling strength of $\chi_\alpha(0)$ to baryon state $|p,s\rangle$. Hence the electromagnetic form factors are to be extracted from the tensor $M^{\mu}_{\alpha\beta}$ (which is also a 4×4 matrix in Dirac space) defined as

$$M^{\mu}_{\alpha\beta}\stackrel{\text{def}}{=}\sum_{s,s'}u_\alpha(p',s')\langle p',s'|j^{\mu}(0)|p,s\rangle\bar{u}_\beta(p,s). \tag{16}$$

The exponential energy dependence in (14) and the coupling dependence (λ) in (15) of the three-point function may be eliminated with the use of the two-point function.[10] Hence we are left with the evaluation of

$$\mathrm{Tr}(\Gamma M^{\mu}_{\alpha\beta})=\sum_{s,s',\bar{a}\bar{b}}[\Gamma]_{\bar{a}\bar{b}}[u_\alpha(p',s')]_{\bar{b}}\langle p',s'|j^{\mu}(0)|p,s\rangle[\bar{u}_\beta(p,s)]_{\bar{a}}$$
$$=\sum_{s,s'}\bar{u}_\beta(p,s)\Gamma u_\alpha(p',s')\langle p',s'|j^{\mu}(0)|p,s\rangle$$
$$=\sum_{s,s'}\Gamma_{\beta\alpha}(s,s')\langle p',s'|j^{\mu}(0)|p,s\rangle, \tag{17}$$

where we have defined (a c number)

$$\Gamma_{\beta\alpha}(s,s')\stackrel{\text{def}}{=}\bar{u}_\beta(p,s)\Gamma u_\alpha(p',s'). \tag{18}$$

Equation (17) is determined by the hadron matrix element multipole decomposition of (7). On the other hand, the three-point function of (13) may be calculated at the quark level. The task is to define the correct choice of the matrix Γ and combinations of the Lorentz indices α and β which make it possible to isolate and extract the electromagnetic multipole form factors G_{E0}, G_{M1}, G_{E2}, and G_{M3} from the quark-level calculations of the three-point function. The most general form of the matrix Γ may be written

$$\Gamma=\begin{bmatrix} a_0 I+\mathbf{a}\cdot\boldsymbol{\sigma} & b_0 I+\mathbf{b}\cdot\boldsymbol{\sigma} \\ c_0 I+\mathbf{c}\cdot\boldsymbol{\sigma} & d_0 I+\mathbf{d}\cdot\boldsymbol{\sigma}\end{bmatrix}, \tag{19}$$

where a_μ, b_μ, c_μ, and d_μ are 16 independent coefficients, and $I(\boldsymbol{\sigma})$ is a unit (Pauli) matrix.

We present our results in two frames of reference. We select the Breit frame (BF) in which the extraction of the multipole form factors is simple and elegant, and the laboratory frame (LF) where additional kinematical factors are required. The LF is particularly interesting in lattice calculations where the minimum momentum transfer is limited by the largest spatial dimension of the lattice. Charge radii and magnetic moments are determined by the properties of the form factors at $q^2=0$. In the BF the minimum q^2 available is 4 times that of the LF.

Let us first start in the BF. We specify, for simplicity, the direction of the initial baryon momentum as the x axis; i.e., $\mathbf{p}=(|\mathbf{p}|,0,0)$. By choosing $a_0=1$, $d_0=-1$, and all other coefficients to be zeros, i.e.,

$$\Gamma_0=\begin{bmatrix} I & 0 \\ 0 & -I\end{bmatrix}=\gamma^0 , \tag{20}$$

we then obtain, for (18),

$$\Gamma_{\beta\alpha}(s,s')=\sum_{m,m',\bar{s}}(1m\tfrac{1}{2}\bar{s}|1\tfrac{1}{2}\tfrac{3}{2}s)(1m'\tfrac{1}{2}\bar{s}|1\tfrac{1}{2}\tfrac{3}{2}s')\times\epsilon^*_\beta(\mathbf{p},m)\epsilon_\alpha(-\mathbf{p},m') . \tag{21}$$

We find that the $E0$ and $E2$ form factors can be projected out by the following combination of (21). Namely, for $E0$,

$$\Gamma_{E0}(s,s')\overset{\text{def}}{=}\Gamma_{00}(s,s')+\Gamma_{11}(s,s')+\Gamma_{22}(s,s')+\Gamma_{33}(s,s') \tag{22a}$$

$$=\langle\tfrac{3}{2}s|\tfrac{3}{2}s'\rangle , \tag{22b}$$

and, for $E2$,

$$\Gamma_{E2}(s,s')\overset{\text{def}}{=}\Gamma_{00}(s,s')+\Gamma_{11}(s,s')+\Gamma_{22}(s,s')-2\Gamma_{33}(s,s') \tag{23a}$$

$$=-\sqrt{6}\langle\tfrac{3}{2}s|\Sigma_0^{(2)}|\tfrac{3}{2}s'\rangle . \tag{23b}$$

Applying these operators to the time component of the quark current matrix elements, we have, for (17),

$$\mathrm{Tr}[\Gamma_0(M^0_{00}+M^0_{11}+M^0_{22}+M^0_{33})]=-4G_{E0}(q^2) \tag{24a}$$

and

$$\mathrm{Tr}[\Gamma_0(M^0_{00}+M^0_{11}+M^0_{22}-2M^0_{33})]=-\tfrac{4}{3}\tau G_{E2}(q^2) . \tag{24b}$$

Similarly, by choosing $\mathbf{a}=\mathbf{d}=\hat{\boldsymbol{\epsilon}}^y=(0,1,0)$ and all other coefficients to be zeros in (19), i.e.,

$$\Gamma_y=\begin{bmatrix} \hat{\boldsymbol{\epsilon}}^y\cdot\boldsymbol{\sigma} & 0 \\ 0 & \hat{\boldsymbol{\epsilon}}^y\cdot\boldsymbol{\sigma}\end{bmatrix}=\gamma^0\gamma^2\gamma_5 , \tag{25}$$

one can project out the $M1$ and $M3$ form factors. We now have

$$\Gamma_{\beta\alpha}(s,s')=\sum_{m,m',\bar{s},\bar{s}'}(1m\tfrac{1}{2}\bar{s}|1\tfrac{1}{2}\tfrac{3}{2}s)(1m'\tfrac{1}{2}\bar{s}'|1\tfrac{1}{2}\tfrac{3}{2}s')\times\epsilon^*_\beta(\mathbf{p},m)\epsilon_\alpha(-\mathbf{p},m')\times\langle\tfrac{1}{2}\bar{s}|\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\epsilon}}^y|\tfrac{1}{2}\bar{s}'\rangle . \tag{26}$$

Using (26), we find the following combinations. For $M1$,

$$\Gamma_{M1}(s,s')\overset{\text{def}}{=}\Gamma_{00}(s,s')+\Gamma_{11}(s,s')+\Gamma_{22}(s,s')+\Gamma_{33}(s,s') \tag{27a}$$

$$=\tfrac{1}{3}\langle\tfrac{3}{2}s|\boldsymbol{\Sigma}^{(1)}\cdot\hat{\boldsymbol{\epsilon}}^y|\tfrac{3}{2}s'\rangle , \tag{27b}$$

and, for $M3$,

$$\Gamma_{M3}(s,s')\overset{\text{def}}{=}\Gamma_{00}(s,s')+\Gamma_{11}(s,s')+\Gamma_{22}(s,s')-\tfrac{3}{2}\Gamma_{33}(s,s') . \tag{28}$$

Applying these operators to the z component of the quark matrix elements, we have

$$\mathrm{Tr}[\Gamma_y(M^z_{00}+M^z_{11}+M^z_{22}+M^z_{33})]=-\tfrac{20}{9}\sqrt{\tau}iG_{M1}(q^2) \tag{29a}$$

and

$$\mathrm{Tr}[\Gamma_y(M^z_{00}+M^z_{11}+M^z_{22}-\tfrac{3}{2}M^z_{33})]=-\tfrac{2}{3}\tau\sqrt{\tau}iG_{M3}(q^2) . \tag{29b}$$

Equations (24) and (29) summarize the correct combination of Dirac and Lorentz indices for extracting the electromagnetic multipole moments of spin-$\frac{3}{2}$ baryons in the BF.

The combinations of Lorentz indices in (24) and (29) are not in a Lorentz-invariant form such as $g^{\alpha\beta}M^\mu_{\alpha\beta}$. Therefore, the expressions for extracting the form factors are frame dependent. In the LF we specify the direction of the final baryon momentum in the x direction; i.e., $\mathbf{p}'=(|\mathbf{p}'|,0,0)$. We choose the same forms for Γ as in the BF case, i.e., Γ_0 for $E0$ and $E2$, and Γ_y for $M1$ and $M3$. Note that contributions from the coefficients d_0 and $\mathbf{d}$ vanish in this frame. Using Γ_0, we now obtain

$$\Gamma_{\beta\alpha}(s,s')=\sqrt{1+\tau}\sum_{m,m',\bar{s}}(1m\tfrac{1}{2}\bar{s}|1\tfrac{1}{2}\tfrac{3}{2}s)\times(1m'\tfrac{1}{2}\bar{s}|1\tfrac{1}{2}\tfrac{3}{2}s')\times\epsilon^*_\beta(0,m)\epsilon_\alpha(\mathbf{p}',m') . \tag{30}$$

For $E0$, we define

$$\Gamma_{E0}(s,s')\overset{\text{def}}{=}\frac{1}{1+2\tau}\Gamma_{11}(s,s')+\Gamma_{22}(s,s')+\Gamma_{33}(s,s') \tag{31a}$$

$$=\sqrt{1+\tau}\langle\tfrac{3}{2}s|\tfrac{3}{2}s'\rangle \tag{31b}$$

and, for $E2$,

$$\Gamma_{E2}(s,s') \overset{\text{def}}{=} \frac{1}{1+2\tau}\Gamma_{11}(s,s')+\Gamma_{22}(s,s')-2\Gamma_{33}(s,s') \quad (32a)$$

$$= -\sqrt{6}\sqrt{1+\tau}\langle \tfrac{3}{2}s|\Sigma_0^{(2)}|\tfrac{3}{2}s'\rangle \ . \quad (32b)$$

Applying these operators to the time component of the quark matrix elements, we have

$$\mathrm{Tr}\left[\Gamma_0\left[\frac{1}{1+2\tau}M_{11}^0+M_{22}^0+M_{33}^0\right]\right] = -4(1+\tau)G_{E0}(q^2) \quad (33a)$$

and

$$\mathrm{Tr}\left[\Gamma_0\left[\frac{1}{1+2\tau}M_{11}^0+M_{22}^0-2M_{33}^0\right]\right] = -\tfrac{4}{3}\tau(1+\tau)G_{E2}(q^2) \ . \quad (33b)$$

Similarly, for the $M1$ and $M3$ form factors, we now have

$$\Gamma_{\beta\alpha}(s,s') = \sum_{m,m',\bar{s},\bar{s}'}(1m\tfrac{1}{2}\bar{s}|1\tfrac{1}{2}\tfrac{3}{2}s)(1m'\tfrac{1}{2}\bar{s}'|1\tfrac{1}{2}\tfrac{3}{2}s') \times\epsilon_\beta^*(0,m)\epsilon_\alpha(\mathbf{p}',m') \times\langle \tfrac{1}{2}\bar{s}|\sigma\cdot\hat{\epsilon}^y|\tfrac{1}{2}\bar{s}'\rangle \ . \quad (34)$$

We define, for $M1$,

$$\Gamma_{M1}(s,s') \overset{\text{def}}{=} \frac{1}{1+2\tau}\Gamma_{11}(s,s')+\Gamma_{22}(s,s')+\Gamma_{33}(s,s') \quad (35a)$$

$$= \tfrac{1}{3}\langle \tfrac{3}{2}s|\Sigma^{(1)}\cdot\hat{\epsilon}^y|\tfrac{3}{2}s'\rangle \quad (35b)$$

and, for $M3$,

$$\Gamma_{M3}(s,s') \overset{\text{def}}{=} \frac{1}{1+2\tau}\Gamma_{11}(s,s')+\Gamma_{22}(s,s')-\tfrac{3}{2}\Gamma_{33}(s,s') \ . \quad (36)$$

By applying these operators to the z component of the quark matrix elements, we have

$$\mathrm{Tr}\left[\Gamma_y\left[\frac{1}{1+2\tau}M_{11}^z+M_{22}^z+M_{33}^z\right]\right] = \tfrac{20}{9}\sqrt{\tau(1+\tau)}\,iG_{M1}(q^2) \quad (37a)$$

and

$$\mathrm{Tr}\left[\Gamma_y\left[\frac{1}{1+2\tau}M_{11}^z+M_{22}^z-\frac{3}{2}M_{33}^z\right]\right] = \tfrac{2}{3}\tau\sqrt{\tau(1+\tau)}\,iG_{M3}(q^2) \ . \quad (37b)$$

Equations (33) and (37) summarize the correct combination of Dirac and Lorentz indices for extracting the electromagnetic multipole moments of spin-$\frac{3}{2}$ baryons in the LF. A calculation of these equations at the quark level in lattice QCD is currently in progress.

III. SUMMARY

This completes our analysis of spin-$\frac{3}{2}$ electromagnetic multipole form factors. Equations (1) and (2) define the electromagnetic current matrix element for spin-$\frac{3}{2}$ baryons in terms of the minimum required four independent covariant vertex functions. In (7) the current matrix elements are expressed in terms of the multipole form factors G_{E0}, G_{E2}, G_{M1}, and G_{M3}. The explicit Lorentz-invariant structures of these form factors are illustrated in (9) in terms of the covariant vertex function coefficients a_1, a_2, c_1, and c_2. Finally, a formalism to invert the relationship of (7) and express the multipole form factors in terms of the current matrix elements was presented with reference to lattice QCD. The results in the laboratory frame are summarized in (33) and (37). We have established relationships among current matrix elements, covariant vertex functions, and multipole form-factor decompositions for the case of electromagnetic interactions of spin-$\frac{3}{2}$ systems. It is hoped that this work will assist in future investigations of the electromagnetic properties of spin-$\frac{3}{2}$ baryons.

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