Gravitational waves emitted from infinite strings

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Gravitational radiation from an infinite string with a helicoidal standing wave is studied in the weak-field approximation. The radiation power and the spectrum are calculated. The implications of the results for gravitational damping of small-scale structure on the long strings are briefly discussed.

I. INTRODUCTION

A large number of current particle-physics models predict spontaneous symmetry-breaking processes as the early Universe expanded and cooled. Such processes can leave behind topological defects, as relics of the old symmetric phase.[1] The case of remnants in the form of one-dimensional strings or vortex lines[2] presents a particular interest, providing the necessary density fluctuations for galaxy and large-scale structure formation[3,4] and leading to presently observable effects.[5]

According to the standard picture, 80% of the energy in a cosmic-string network is initially in very long (longer than the horizon) Brownian strings and only 20% in a distribution of closed loops. During the expansion of the Universe, long strings chop off loops, which self-intersect until a class of non-self-intersecting loop trajectories is reached.

The energy-loss mechanism for oscillating loops is mainly the emission of gravitational waves.[6,7] However, a straight infinite string does not radiate, even when modulated by traveling waves.[8] Thus, the expansion of the Universe straightened out the long strings, which being unable to decay by gravitational radiation, could survive indefinitely.

Recent numerical simulation results[9,10] question the so-far accepted one-scale hypothesis, yielding an evolution of the string network, which deviates from the standard-model picture.[4-11] As string segments intercommute, discontinuities in their velocity and direction (kinks) are developed, which result in the presence of significant substructures on the long strings. Such small-scale wiggles are likely to play an important role in the energy distribution of the produced loops. It has been suggested[10] that the characteristic scale of the wiggles may be determined by gravitational radiation from long strings. In order to assess this possibility, we shall study the spectrum of gravitational waves emitted by a particular class of *modulated* long strings. We analytically calculate, in the weak-field approximation, the power radiated by an infinite string with a helicoidal standing wave.

It would be fair to mention that the numerical results of Refs. 9 and 10 are still under debate. The small-scale structure they yield is not generally accepted and contradicts other recent numerical simulations.[12]

Throughout the paper we use the system of units in which $\hbar=c=1$; we choose $(-,+,+,+)$ to be the signature for the metric and we denote by μ the mass per unit length of a string.

II. PRELIMINARIES

The space-time trajectory of a macroscopic string, with dimension much greater than its thickness, is described by a vector function $\mathbf{f}(\sigma,t)$, where σ can be thought of as a parameter along the string trajectory. Here we consider an infinite string having the form of a helix along the z axis. We write $\mathbf{f}$ as

$$\mathbf{f}(\sigma,t)=\boldsymbol{\xi}(\sigma,t)+\sqrt{1-\epsilon^2}\sigma\hat{\mathbf{z}}\ , \tag{1}$$

where ξ is a modulation orthogonal to the z axis. The actual form of ξ can then be obtained as follows.

The action for a macroscopic string is the Nambu action.[2-13] Thus, the equation of motion in a flat nonexpanding background is[14]

$$\ddot{\mathbf{f}}-\mathbf{f}''=0\ , \tag{2}$$

with the constraints[14]

$$\dot{\mathbf{f}}\cdot\mathbf{f}'=0\ ,\qquad \dot{\mathbf{f}}^{\,2}+\mathbf{f}'^{2}=1\ , \tag{3}$$

where dots and primes stand for derivatives with respect to t and σ, respectively. The general solution to Eq. (2) reads

$$\mathbf{f}(\sigma,t)=\tfrac{1}{2}[\mathbf{a}(\sigma-t)+\mathbf{b}(\sigma+t)]\ , \tag{4}$$

where the otherwise arbitrary functions $\mathbf{a}$ and $\mathbf{b}$ must satisfy

$$\mathbf{a}'^2=\mathbf{b}'^2=1\ , \tag{5}$$

due to the constraints of Eq. (3).

Taking into account the definition of ξ [see Eq. (1)], we are led to

$$\boldsymbol{\xi}(\sigma,t)=\tfrac{1}{2}[\boldsymbol{\alpha}(\sigma-t)+\boldsymbol{\beta}(\sigma+t)]\ , \tag{6}$$

with $\boldsymbol{\alpha}'^2=\boldsymbol{\beta}'^2=\epsilon^2$. We shall consider a solution of the form

$$\boldsymbol{\alpha}(\sigma-t)=\left[\frac{\epsilon}{\Omega}\cos[\Omega(\sigma-t)],\frac{\epsilon}{\Omega}\sin[\Omega(\sigma-t)],0\right], \quad (7)$$

$$\boldsymbol{\beta}(\sigma+t)=\left[\frac{\epsilon}{\Omega}\cos[\Omega(\sigma+t)],\frac{\epsilon}{\Omega}\sin[\Omega(\sigma+t)],0\right]. \quad (8)$$

Then, the full string trajectory is

$$\mathbf{f}(\sigma,t)=\left[\frac{\epsilon}{\Omega}\cos(\Omega\sigma)\cos(\Omega t),\frac{\epsilon}{\Omega}\sin(\Omega\sigma)\times\cos(\Omega t),\sqrt{1-\epsilon^2}\sigma\right]. \quad (9)$$

Clearly, Ω is the *breathing* frequency of the helix and the parameter ϵ ($0\le\epsilon<1$) determines its *winding number* per unit length.

The energy-momentum tensor is obtained by varying the string action with respect to the metric and the result is[7]

$$T_{\mu\nu}(\mathbf{x},t)=\mu\int d\sigma(\dot f_\mu \dot f_\nu - f'_\mu f'_\nu)\delta^{(3)}(\mathbf{x}-\mathbf{f}(\sigma,t)), \quad (10)$$

where $\mathbf{x}$ is the field point and we take $f_0=t$. Because of the symmetry of the problem, we find it convenient to work in cylindrical coordinates. Thus, $\mathbf{x}=(\rho\cos\theta,\rho\sin\theta,\zeta)$ and

$$\delta^{(3)}(\mathbf{x}-\mathbf{f}(\sigma,t))=\frac{1}{\rho}\delta\left[\rho-\frac{\epsilon}{\Omega}|\cos(\Omega t)|\right]\delta\left[\theta-\Omega\sigma-\frac{\pi}{2}\{1-\mathrm{sgn}[\cos(\Omega t)]\}\right]\delta(\zeta-\sqrt{1-\epsilon^2}\sigma). \quad (11)$$

We notice that for our particular string trajectory, the energy-momentum tensor $T_{\mu\nu}(\mathbf{x},t)$ turns out to be periodic along the z axis, with period $2\pi\sqrt{1-\epsilon^2}/\Omega$, and periodic in time, with period $2\pi/\Omega$.

III. GRAVITATIONAL RADIATION FROM AN INFINITELY LONG SOURCE

In the weak-field approximation, the gravitational field $h_{\mu\nu}(\mathbf{x},t)$ is given by the well-known *retarded potential* formula[15]

$$h_{\mu\nu}(\mathbf{x},t)=4G\int d\mathbf{x}'\frac{S_{\mu\nu}(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}, \quad (12)$$

where the source density $S_{\mu\nu}(\mathbf{x},t)$ is simply related to the energy-momentum tensor $T_{\mu\nu}(\mathbf{x},t)$ by

$$S_{\mu\nu}=T_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}T^\lambda_\lambda. \quad (13)$$

If one is interested in the energy transmitted through a surface at large distance from the source, the formula relating $h_{\mu\nu}$ to $S_{\mu\nu}$ can be considerably simplified, when the symmetry of the source is explicitly taken into account. The standard case discussed in textbooks[15] is that of a localized source. Here, we analyze the case of an infinitely long source in some detail. We stress that by *infinitely long* we mean a source which is infinite in one direction, say the z direction, and has a finite size in the orthogonal plane.

Quite generally the source density will be given by a sum of Fourier components:

$$S_{\mu\nu}(\mathbf{x},t)=\sum_\omega e^{-i\omega t}S_{\mu\nu}(\mathbf{x},\omega)+\text{c.c.} \quad (14)$$

In addition, to fully take advantage of the present symmetry, it is convenient to Fourier analyze $S_{\mu\nu}(\mathbf{x},\omega)$ with respect to z. Thus,

$$S_{\mu\nu}(\mathbf{x},\omega)=\sum_\kappa S_{\mu\nu}(\mathbf{r},\kappa,\omega)e^{i\kappa z}, \quad (15)$$

where $\mathbf{r}$ is the component of $\mathbf{x}$ orthogonal to the z axis, i.e., $\mathbf{x}\equiv(\mathbf{r},z)$. We shall first calculate the field which originates from a single-frequency source. Such a field is given by[15]

$$h_{\mu\nu}(\mathbf{r},z,t)=4Ge^{-i\omega t}\sum_\kappa\int\frac{dz'd\mathbf{r}'}{D}S_{\mu\nu}(\mathbf{r}',\kappa,\omega)\times e^{i(\omega D+\kappa z')}+\text{c.c.}, \quad (16)$$

where $D=\sqrt{(z-z')^2+(\mathbf{r}-\mathbf{r}')^2}$. One can easily show that only Fourier components of $S_{\mu\nu}$ with $\kappa<\omega$ will contribute to the radiation of energy, since otherwise the field decays exponentially as one moves away from the source. Thus, for $\kappa<\omega$,

$$h_{\mu\nu}(\mathbf{r},z,t)=4Gi\pi e^{-i\omega t}\sum_{\kappa<\omega}e^{i\kappa z}\int d\mathbf{r}\,S_{\mu\nu}(\mathbf{r}',\kappa,\omega)H_0^{(1)}(\sqrt{\omega^2-\kappa^2}|\mathbf{r}-\mathbf{r}'|)+\text{c.c.} \quad (17)$$

and, for r much bigger than the orthogonal size of the source, this simplifies to

$$h_{\mu\nu}(\mathbf{r},z,t)\simeq 4Gi\pi e^{-i\omega t}\left[\frac{2}{\pi}\right]^{1/2}e^{-i\pi/4}\sum_{\kappa<\omega}e^{i\kappa z}\int d\mathbf{r}'S_{\mu\nu}(\mathbf{r}',\kappa,\omega)\frac{\exp(i|\mathbf{r}-\mathbf{r}'|\sqrt{\omega^2-\kappa^2})}{(\omega^2-\kappa^2)^{1/4}\sqrt{|\mathbf{r}-\mathbf{r}'|}}+\text{c.c.}$$
$$\simeq 4G\sqrt{2\pi}e^{i\pi/4}\sum_{\kappa<\omega}\frac{\exp[i(-\omega t+\kappa z'+\mathbf{q}\cdot\mathbf{r})]}{\sqrt{r}\,(\omega^2-\kappa^2)^{1/4}}\int d\mathbf{r}'S_{\mu\nu}(\mathbf{r}',\kappa,\omega)e^{-i\mathbf{q}\cdot\mathbf{r}'}+\text{c.c.}, \quad (18)$$

where, to obtain the last equation, we have replaced $|\mathbf{r}-\mathbf{r}'|$ by r in the denominator and by $r-(\mathbf{r}/r)\cdot\mathbf{r}'$ in the exponential, on account of the finite extent of the source. Also,

$$\mathbf{q}\equiv\sqrt{\omega^2-\kappa^2}\frac{\mathbf{r}}{r} \ . \tag{19}$$

From Eq. (18) it is apparent that, for large r, the field is a superposition of plane waves of the form

$$\exp[i(\mathbf{Q}\cdot\mathbf{x}-\omega t)]e_{\mu\nu}(r,\mathbf{Q},\omega) \ , \tag{20}$$

where the "wave vector" $\mathbf{Q}$ and the "polarization tensor" $e_{\mu\nu}$ are explicitly given by

$$\mathbf{Q}=(\mathbf{q},\kappa) \ , \tag{21}$$

$$e_{\mu\nu}=4G\sqrt{2\pi}\frac{e^{i\pi/4}}{\sqrt{r}}\frac{S_{\mu\nu}(\mathbf{Q},\omega)}{(\omega^2-\kappa^2)^{1/4}} \ . \tag{22}$$

The total gravitational field corresponding to the source of Eq. (14) is obtained by summing over the frequency components:

$$h_{\mu\nu}(\mathbf{r},t)=\sum_{\omega}\sum_{\kappa<\omega}\exp[i(\mathbf{Q}\cdot\mathbf{x}-\omega t)]e_{\mu\nu}(r,\mathbf{Q},\omega)+\text{c.c.} \tag{23}$$

We notice that when the source has periodicity in z, as is in the case examined here, the Fourier component of $S_{\mu\nu}$ appearing in Eqs. (18) and (22) is

$$S_{\mu\nu}(\mathbf{Q},\omega)=\frac{1}{\Delta}\int_{0<z\le\Delta}d\mathbf{x}\,S_{\mu\nu}(\mathbf{x},\omega)e^{-i\mathbf{Q}\cdot\mathbf{x}} \ , \tag{24}$$

where Δ is the period along z. The present symmetry suggests to consider the power radiated through a cylindrical surface centered on the source and having a radius much larger than the source's orthogonal size. The power radiated through such a surface in the angle between φ and $\varphi+d\varphi;z$ and $z+dz$ is

$$dP=d\varphi\, r\, dz\,\hat{\mathbf{r}}^i\langle t^{i0}\rangle \ , \tag{25}$$

where $\hat{\mathbf{r}}^i$ is the vector orthogonal to the surface and $\langle t^{i0}\rangle$ is the average of the energy flux vector of the plane wave, with the average being taken over space-time dimensions large compared to $(1/\omega)$.[15] Since the average energy-momentum tensor of a plane wave is[15]

$$\langle t_{\mu\nu}\rangle=\frac{k_\mu k_\nu}{16\pi G}(e^{\lambda\rho *}e_{\lambda\rho}-\tfrac{1}{2}|e^{\lambda}_{\lambda}|^2) \ , \tag{26}$$

with $k\equiv(\omega,\mathbf{Q})$, Eq. (25) becomes

$$dP=d\varphi\, dz\, 2G\sum_{\omega}\sum_{\kappa<\omega}\omega[S^{\lambda\rho *}(\mathbf{Q},\omega)S_{\lambda\rho}(\mathbf{Q},\omega_n)-\tfrac{1}{2}|S^{\lambda}_{\lambda}(\mathbf{Q},\omega_n)|^2] \tag{27}$$

$$=d\varphi\, dz\, 2G\sum_{\omega}\sum_{\kappa<\omega}\omega[T^{\lambda\rho *}(\mathbf{Q},\omega)T_{\lambda\rho}(\mathbf{Q},\omega_n)-\tfrac{1}{2}|T^{\lambda}_{\lambda}(\mathbf{Q},\omega_n)|^2] \ , \tag{28}$$

where the last equation has been obtained using the relation between the source $S_{\mu\nu}$ and the energy-momentum tensor $T_{\mu\nu}$, given in Eq. (13). Thus, the gravitational energy radiated, in the weak-field approximation, through a cylinder of unit length per unit time, reads

$$P=2G\sum_{\omega}\sum_{\kappa<\omega}\omega\int_0^{2\pi}d\varphi[T^{\lambda\rho *}(\mathbf{q},\kappa,\omega)T_{\lambda\rho}(\mathbf{q},\kappa,\omega)-\tfrac{1}{2}|T^{\lambda}_{\lambda}(\mathbf{q},\kappa,\omega)|^2] \ . \tag{29}$$

IV. GRAVITATIONAL RADIATION FROM AN INFINITE STRING WITH A HELICOIDAL STANDING WAVE

In this section we will calculate the gravitational radiation for the string trajectory specified by Eq. (9) in the weak-field approximation. In the present case, the combination of components of $T_{\mu\nu}$ appearing in Eqs. (28) and (29) has no dependence on φ (see below). Therefore, using Eq. (28) or (29) is completely equivalent. We will first calculate the components of the energy-momentum tensor from Eqs. (9) and (10). We define some new parameters which will enable us to write the final formulas for the energy-momentum components in concise form:

$$\tilde{z}\equiv\frac{z\Omega}{\sqrt{1-\epsilon^2}} \ , \tag{30}$$

$$\Delta_1\equiv\delta\left[\rho-\frac{\epsilon}{\Omega}|\cos(\Omega t)|\right]\times\delta\left[\theta-\tilde{z}-\frac{\pi\{1-\text{sgn}[\cos(\Omega t)]\}}{2}\right] \ . \tag{31}$$

After some algebra, we obtain, for the components of the energy-momentum tensor, $T_{\mu\nu}(\mathbf{x},t)$:

$$T_{00}=\frac{\mu}{\rho\sqrt{1-\epsilon^2}}\Delta_1 \ , \tag{32}$$

$$T_{01}=T_{10}=-\frac{\mu}{\rho\sqrt{1-\epsilon^2}}\epsilon\sin(\Omega t)\cos\tilde{z}\Delta_1 \ , \tag{33}$$

$$T_{02}=T_{20}=-\frac{\mu}{\rho\sqrt{1-\epsilon^2}}\epsilon\sin(\Omega t)\sin\tilde{z}\Delta_1 \ , \tag{34}$$

$$T_{03}=T_{30}=0 \ , \tag{35}$$

$$T_{11}=\frac{\mu}{\rho\sqrt{1-\epsilon^2}}\epsilon^2[\cos^2\tilde{z}\sin^2(\Omega t)-\sin^2\tilde{z}\cos^2(\Omega t)]\Delta_1 \ , \tag{36}$$

$$T_{22}=\frac{\mu}{\rho\sqrt{1-\epsilon^2}}\epsilon^2[\sin^2\tilde{z}\sin^2(\Omega t)-\cos^2\tilde{z}\cos^2(\Omega t)]\Delta_1 \ , \tag{37}$$

$$T_{33}=\frac{-\mu\sqrt{1-\epsilon^2}}{\rho}\Delta_1 \ , \tag{38}$$

$$T_{12}=T_{21}=\frac{\mu}{\rho\sqrt{1-\epsilon^2}}\epsilon^2\cos\tilde{z}\,\sin\tilde{z}\,\Delta_1\;, \tag{39}$$

$$T_{13}=T_{31}=\frac{\mu\epsilon}{\rho}\sin\tilde{z}\,\cos(\Omega t)\Delta_1\;, \tag{40}$$

$$T_{23}=T_{32}=\frac{-\mu\epsilon}{\rho}\cos\tilde{z}\,\cos(\Omega t)\Delta_1\;. \tag{41}$$

We next calculate the Fourier transform of the energy-momentum tensor $T_{\mu\nu}(\mathbf{x},t)$ in two steps. We first compute $T_{\mu\nu}(\mathbf{x},\omega)$ and then $T_{\mu\nu}(\mathbf{q},\kappa,\omega)$. Because of the periodicity of the energy-momentum tensor, in both time and z variables, we denote the discrete values of ω and κ by ω_n and κ_m, with $\omega_n\equiv n\Omega$ and $\kappa_m=m\Omega/\sqrt{1-\epsilon^2}$. Clearly,

$$T_{\mu\nu}(\mathbf{x},t)=\sum_{n=0}^{n=\infty}T_{\mu\nu}(\mathbf{x},\omega_n)e^{in\Omega t}+\text{c.c.}\;, \tag{42}$$

with

$$T_{\mu\nu}(\mathbf{x},\omega_n)=(1+\delta_{n,0})^{-1}\frac{\Omega}{2\pi}\int_{-\pi/\Omega}^{\pi/\Omega}T_{\mu\nu}(\mathbf{x},t)e^{-in\Omega t}dt\;. \tag{43}$$

Again, it is convenient to define a couple of intermediate parameters to present the result of the integral of Eq. (43):

$$t_\rho\equiv\arccos(\Omega\rho/\epsilon)\;, \tag{44}$$

$$\Delta_2^{\pm}\equiv(1+\delta_{n,0}^{-1})[\delta(\theta-\tilde{z})\pm(-1)^n\delta(\theta-\tilde{z}-\pi)]\;. \tag{45}$$

Thus, the components $T_{\mu\nu}(\mathbf{x},\omega_n)$ are given by

$$T_{00}=\frac{\Omega\mu}{\pi\rho\sqrt{1-\epsilon^2}}\frac{\cos(nt_\rho)}{\sqrt{\epsilon^2-\Omega^2\rho^2}}\Delta_2^+\;, \tag{46}$$

$$T_{01}=T_{10}=\frac{i\Omega\epsilon\mu}{\pi\rho\sqrt{1-\epsilon^2}}\cos\tilde{z}\,\sin(nt_\rho)\Delta_2^-\;, \tag{47}$$

$$T_{02}=T_{20}=\frac{i\Omega\epsilon\mu}{\pi\rho\sqrt{1-\epsilon^2}}\sin\tilde{z}\,\sin(nt_\rho)\Delta_2^-\;, \tag{48}$$

$$T_{03}=T_{30}=0\;, \tag{49}$$

$$T_{11}=\frac{\Omega\epsilon^2\mu}{\pi\rho\sqrt{1-\epsilon^2}}\frac{\cos(nt_\rho)}{\sqrt{\epsilon^2-\Omega^2\rho^2}}\left[\cos^2\tilde{z}-\frac{\Omega^2\rho^2}{\epsilon^2}\right]\Delta_2^+\;, \tag{50}$$

$$T_{22}=\frac{\Omega\epsilon^2\mu}{\pi\rho\sqrt{1-\epsilon^2}}\frac{\cos(nt_\rho)}{\sqrt{\epsilon^2-\Omega^2\rho^2}}\left[\sin^2\tilde{z}-\frac{\Omega^2\rho^2}{\epsilon^2}\right]\Delta_2^+\;, \tag{51}$$

$$T_{33}=\frac{-\Omega\mu\sqrt{1-\epsilon^2}}{\pi\rho}\frac{\cos(nt_\rho)}{\sqrt{\epsilon^2-\Omega^2\rho^2}}\Delta_2^+\;, \tag{52}$$

$$T_{12}=T_{21}=\frac{\Omega\mu\epsilon^2}{\pi\rho\sqrt{\epsilon^2-\Omega^2\rho^2}}\cos\tilde{z}\,\sin\tilde{z}\,\cos(nt_\rho)\Delta_2^+\;, \tag{53}$$

$$T_{13}=T_{31}=\frac{\Omega^2\mu}{\pi\sqrt{\epsilon^2-\Omega^2\rho^2}}\sin\tilde{z}\,\cos(nt_\rho)\Delta_2^-\;, \tag{54}$$

$$T_{23}=T_{32}=\frac{-\Omega^2\mu}{\pi\sqrt{\epsilon^2-\Omega^2\rho^2}}\cos\tilde{z}\,\cos(nt_\rho)\Delta_2^-\;. \tag{55}$$

The spatial Fourier transform $T_{\mu\nu}(\mathbf{q},\kappa_m,\omega_n)$ of the energy-momentum tensor are calculated from

$$T_{\mu\nu}(\mathbf{q},\kappa_m,\omega_n)=\frac{\Omega}{2\pi\sqrt{1-\epsilon^2}}\int_0^{\epsilon/\Omega}r\,dr\int_0^{2\pi}d\theta\,e^{-iqr\cos(\theta-\varphi)}\int_0^{2\pi\sqrt{1-\epsilon^2}/\Omega}dz\,e^{-i\kappa_m z}T_{\mu\nu}(\mathbf{x},\omega_n)\;, \tag{56}$$

using Eqs. (46)–(55), with θ and φ being the polar angles of the vectors $\mathbf{r}$ and $\mathbf{q}$. After some lengthy manipulations, relatively simple expressions are obtained in terms of Bessel functions. These are best presented introducing a few definitions:

$$l\equiv(m+n)/2\;, \tag{57}$$

$$v\equiv(m-n)/2\;, \tag{58}$$

$$\tilde{q}\equiv\epsilon q/(2\Omega)\;, \tag{59}$$

$$E=0\quad\text{if } m+n=\text{odd}\;,$$
$$E=\frac{\mu}{\sqrt{1-\epsilon^2}}\exp[-im(\varphi+\pi/2)],\quad\text{if } m+n=\text{even}\;, \tag{60}$$

$$I_{i,j}\equiv J_i(\tilde{q})J_j(\tilde{q})\;, \tag{61}$$

$$I_{i,j}^{\pm}\equiv J_i(\tilde{q})J_j(\tilde{q})\pm J_{i+1}(\tilde{q})J_{j-1}(\tilde{q})\;. \tag{62}$$

Hence the Fourier components of $T_{\mu\nu}(\mathbf{q},\kappa_m,\omega_n)$ read

$$T_{00}=EI_{l,v}\;, \tag{63}$$

$$T_{01}=T_{10}=-\frac{E\epsilon}{4}(e^{i\varphi}I_{l,v}^-+e^{-i\varphi}I_{v,l}^-)\;, \tag{64}$$

$$T_{02}=T_{20}=\frac{iE\epsilon}{4}(e^{i\varphi}I_{l,v}^--e^{-i\varphi}I_{v,l}^-)\;, \tag{65}$$

$$T_{03}=T_{30}=0\;, \tag{66}$$

$$T_{11}=-\frac{E\epsilon^2}{4}(e^{2i\varphi}I_{l-1,v-1}+e^{2i\varphi}I_{l+1,v+1}+I_{l+1,v-1}+I_{l-1,v+1})\;, \tag{67}$$

$$T_{22}=-\frac{E\epsilon^2}{4}(-e^{2i\varphi}I_{l-1,v-1}-e^{2i\varphi}I_{l+1,v+1}+I_{l+1,v-1}+I_{l-1,v+1})\;, \tag{68}$$

$$T_{33}=-(1-\epsilon^2)EI_{l,v}\;, \tag{69}$$

$$T_{12}=T_{21}=\frac{iE\epsilon^2}{4}(e^{2i\varphi}I_{l-1,v-1}-e^{-2i\varphi}I_{l+1,v+1})\;, \tag{70}$$

$$T_{13}=T_{31}=\frac{\epsilon E\sqrt{1-\epsilon^2}}{4}(e^{-i\varphi}I_{v,l}^++e^{i\varphi}I_{l,v}^+)\;, \tag{71}$$

$$T_{23}=T_{32}=-\frac{i\epsilon E\sqrt{1-\epsilon^2}}{4}(-e^{-i\varphi}I_{v,l}^++e^{i\varphi}I_{l,v}^+)\;. \tag{72}$$

Using the Fourier components of the energy-momentum tensor given in the above equations and the relation $J_{-n}(z)=(-1)^nJ_n(z)$, we easily get the emitted gravitational radiation from Eq. (29). The power emitted within a cylinder of unit length, whose axis coincides with the string axis, is

$$P=4\pi G\Omega\mu^2\frac{\epsilon^2}{1-\epsilon^2}\sum_n n \sum_{m<n\sqrt{1-\epsilon^2}}{}' [(1-\epsilon^2/2)J_v J_l (J_{l+1}-J_{l-1})(J_{v+1}-J_{v-1}) + \tfrac{1}{4}\epsilon^2(J_{l+1}^2-J_l^2)(J_{v+1}^2-J_v^2)+\tfrac{1}{4}\epsilon^2(J_l^2-J_{l-1}^2)(J_v^2-J_{v-1}^2)] \,, \tag{73}$$

where the prime in the sum over m further restricts m to those values for which $m+n$ is even, and the argument of Bessel functions is

$$\tilde{q}=(\epsilon/2)\sqrt{n^2-m^2/(1-\epsilon^2)} \,.$$

V. RESULTS AND DISCUSSION

A careful inspection of Eq. (73) reveals that the power radiated by the infinite string with a helicoidal standing wave we have considered here diverges as $\epsilon\to 1$. The factor $\epsilon^2/(1-\epsilon^2)$, which appears in the total power is clearly diverging in this limit, whereas the frequency sum, which it multiplies, saturates to a finite value. The reason for this is simply understood by realizing that in such a limit the amount (length) of string per unit distance along z goes also to infinity. This can be immediately seen by writing the energy per unit length ($\tilde{E}$) along the z axis. From Eq. (9), we get $\Delta z=\sqrt{1-\epsilon^2}\Delta E/\mu$, where $\Delta E=\mu\Delta\sigma$. Hence,

$$\tilde{E}=\frac{\mu}{\sqrt{1-\epsilon^2}} \,. \tag{74}$$

Clearly, the *winding number* of the helix $\Omega/(2\pi\sqrt{1-\epsilon^2})$, defined as the number of revolutions per unit length along its axis, is also diverging in the limit of $\epsilon\to 1$. On the other hand, for $\epsilon=0$ the emitted radiation vanishes, since in such a case the trajectory reduces to that of a static infinite straight string.

We have numerically evaluated the radiated power as a function of ϵ. The results are given in Fig. 1. One clearly sees the increase of the emitted power with ϵ. From Eq. (73) it is clear that the emitted power has a strong dependence upon ϵ, through the factor $\epsilon^2/(1-\epsilon^2)$. Therefore, it may be useful to rewrite

$$P\equiv G\Omega\mu^2\frac{\epsilon^2}{1-\epsilon^2}\tilde{P}(\epsilon) \,, \tag{75}$$

where the residual dependence on ϵ is contained in the *reduced* power $\tilde{P}$ and its explicit expression is obtained by comparing Eq. (76) with Eq. (73). In fact, from Fig. 2, it is clear that $\tilde{P}$ is a smooth function of ϵ, which vanishes at $\epsilon=0$ and remains otherwise of order 1.

We have also analyzed for given ϵ the frequency spectrum of the gravitational radiation. In Figs. 3(a)–3(c) we report such a spectrum for few values of the parameter ϵ. Of course, the radiated power decreases with increasing frequency. It is also apparent that the contribution from odd frequencies becomes less and less important as ϵ approaches 1. In fact, for our string with a helicoidal

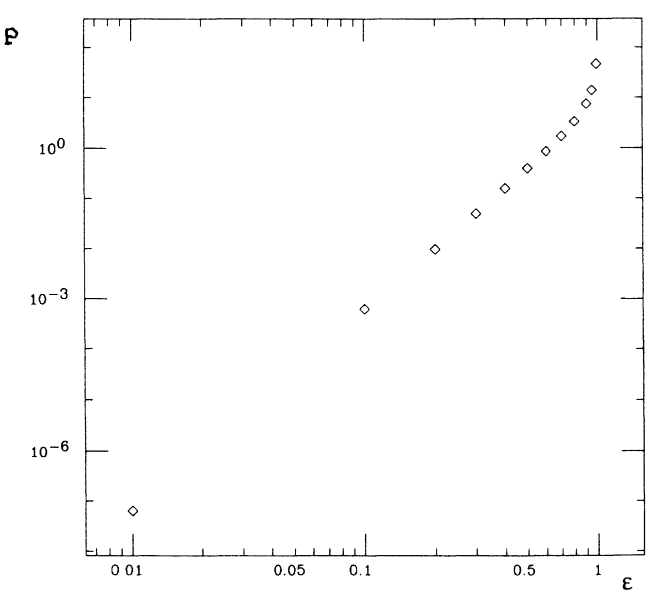

FIG. 1. Power P radiated by an infinite string with a helicoidal standing wave, in the weak-field approximaition, vs ϵ. ϵ is the parameter determining the *winding* number of the helix and P is in units of $G\mu^2\Omega$.

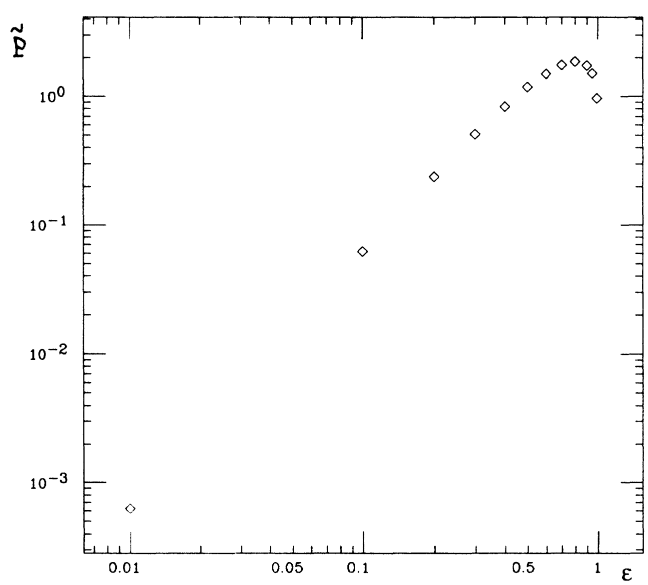

FIG. 2. ϵ dependence of the *reduced* power $\tilde{P}$ [see text and Eqs. (73) and (75)].

standing wave, the lowest radiating odd frequency is bigger than $\Omega/\sqrt{(1-\epsilon^2)}$, which diverges in the limit $\epsilon \rightarrow 1$. On the other hand, the lowest even frequency which appears in the spectrum of the emitted power is 2Ω. Figure 4 shows the ratio between the power emitted at 2Ω and the total power, as a function of ϵ. We find that for small ϵ essentially all the power is emitted at the lowest allowed frequency. In such a case ($\epsilon \ll 1$), $\tilde{q} \approx \epsilon$ and expanding the Bessel functions in Eq. (73) we get $\tilde{P} \approx 2\pi\epsilon^2$. This can also be seen in Fig. 2; for small ϵ, the graph represents a straight line with slope 2. The radiation power in this limit can be written as

$$P \approx 4\pi^2 G\mu^2\epsilon^4\lambda^{-1} , \tag{76}$$

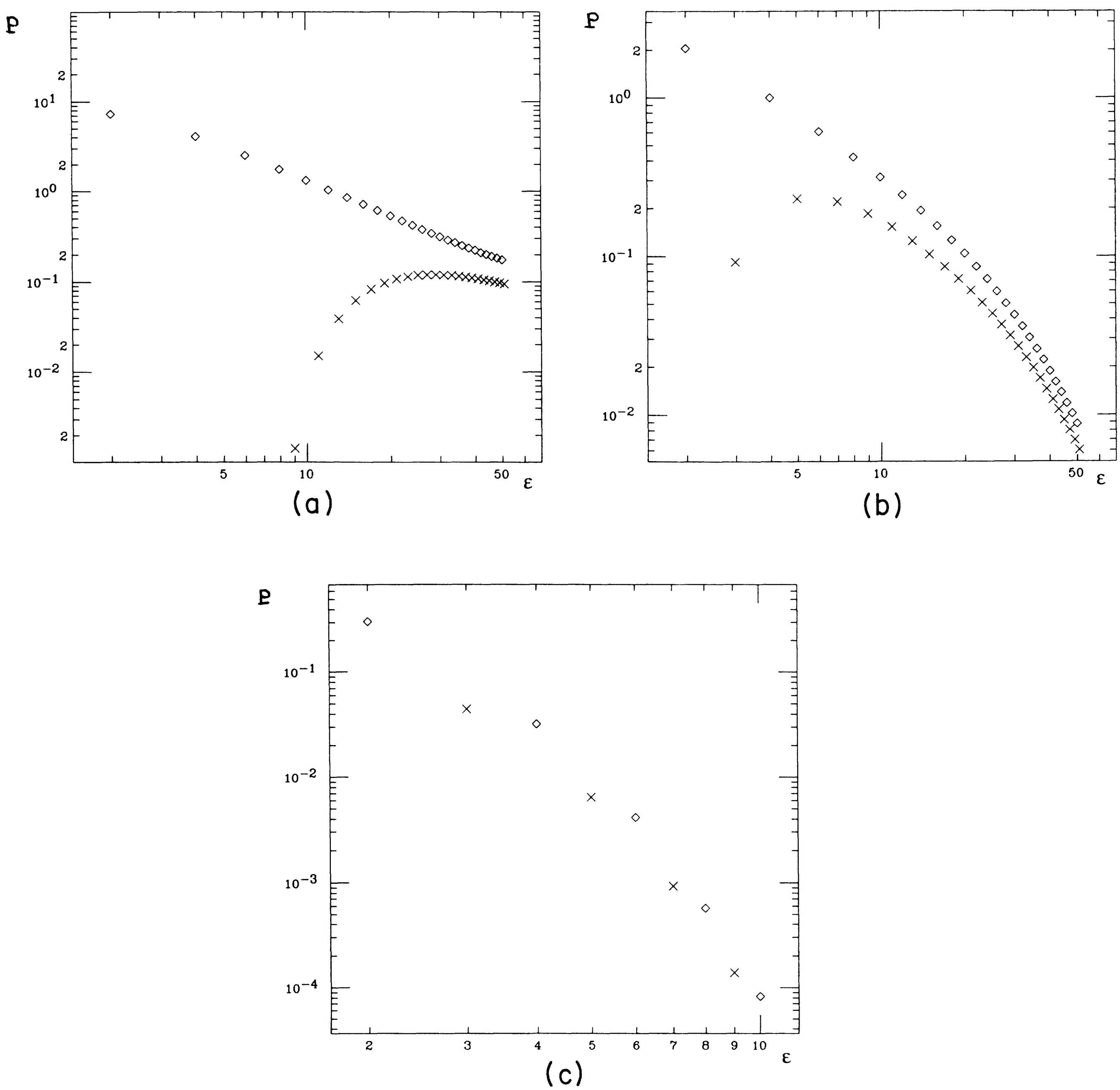

FIG. 3. Frequency dependence of the power radiated by an infinite string with a helicoidal standing wave, in the weak-field approximation. ω is in units of the *natural* frequency Ω. The cases $\epsilon=0.99$, $\epsilon=0.9$, $\epsilon=0.5$ are shown in (a)–(c), respectively. The *diamonds* give the power radiated at even frequencies and the *crosses* that at odd frequencies.

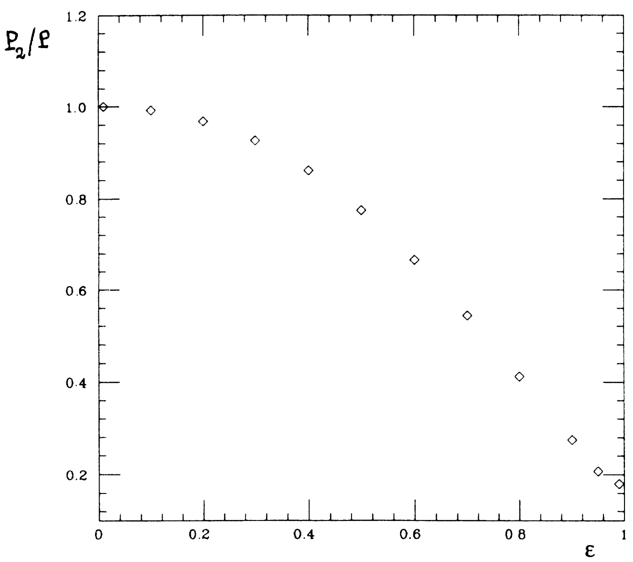

FIG. 4. Ratio of the power radiated at frequency $2\Omega(P_2)$ to the total radiated power (P) vs ϵ.

where $\lambda=2\pi/\Omega$ is the wavelength.

To assess the effect of the gravitational back reaction on the oscillating string, we compare the radiation power [Eqs. (75) and (76)] with the oscillation energy

$$\tilde{E}_{\text{osc}}=\mu\left[\frac{1}{\sqrt{1-\epsilon^2}}-1\right] . \qquad (77)$$

The characteristic damping time of the oscillations can be estimated from

$$\tau\sim\frac{\tilde{E}_{\text{osc}}}{P} . \qquad (78)$$

In the small-ϵ limit this gives

$$\tau\sim\frac{\lambda}{4\pi G\mu\epsilon^2} . \qquad (79)$$

For $\epsilon\approx 1$ this equation should also give the right order of magnitude.

We see from Eq. (79) that gravitational damping is rather efficient for large-amplitude waves ($\epsilon\sim 1$), but becomes less and less efficient as ϵ decreases. In the simulations of Ref. 9 the small-scale structure is spread over many different scales, and the amplitude of each logarithmic range of wavelengths may in fact be small. This indicates that gravitational radiation may be less effective in damping the small-scale structure as it was assumed in Ref. 10. Another suggestion to resolve the discrepancy of the numerical simulations has been recently introduced.[16] The radiation from a string with a wide spectrum of waves is now being investigated.

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