Classical and quantal nonrelativistic Chern-Simons theory

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We construct a nonrelativistic field theory for the second-quantized N-body system of point particles with Chern-Simons interactions. Various properties of this model are discussed: its obvious and hidden symmetries, its relation to a relativistic field theory, and its supersymmetric formulation. We present classical, static solutions—solitons—that satisfy a self-dual equation, which is equivalent to the Liouville equation; hence, it is completely solvable. The dynamical role of the Chern-Simons interaction is demonstrated: the interaction does not merely change statistics but also provides the forces that bind the classical solitons.

I. INTRODUCTION

Dynamics for a collection of N point particles moving nonrelativistically on a plane and interacting with each other through the mediation of a U(1) gauge field, with Chern-Simons kinetic action,[1,2] is described by a Hamiltonian consisting solely of the particle kinetic energy:

$$H=\sum_{p=1}^{N}\tfrac{1}{2}m_p v_p^2 \ . \tag{1.1}$$

Here m_p is the particle mass and $\mathbf{v}_p$ its velocity—the time derivative of the position vector $\mathbf{r}_p$: $\mathbf{v}_p=\dot{\mathbf{r}}_p$. As is the case for magnetic systems, the interaction is hidden in the nontrivial relation between the particle's kinetic momentum $m_p\mathbf{v}_p$ and its canonical momentum $\mathbf{p}_p$, conjugate to $\mathbf{r}_p$:

$$m_p\mathbf{v}_p=\mathbf{p}_p-\frac{e_p}{c}\mathbf{a}_p(\mathbf{r}_1,\ldots,\mathbf{r}_N) \ . \tag{1.2a}$$

The vector potential $\mathbf{a}_p$ seen by particle p describes point vortices located at all the other particles:

$$a_p^i(\mathbf{r}_1,\ldots,\mathbf{r}_N)=\frac{1}{2\pi\kappa}\epsilon^{ij}\sum_{q\neq p}^{N}e_q\frac{r_p^j-r_q^j}{|\mathbf{r}_p-\mathbf{r}_q|^2} \ . \tag{1.2b}$$

Each particle carries the "charge" e_p, c is the velocity of light, and κ is a measure of the interaction. The vector potential involves the curl of the Laplacian Green's function:

$$\frac{1}{2\pi}\epsilon^{ij}\frac{r^j}{r^2}=\epsilon^{ij}\partial_j\frac{1}{2\pi}\ln r \ , \tag{1.3a}$$

$$\nabla^2\frac{1}{2\pi}\ln r=\delta^2(\mathbf{r}) \ . \tag{1.3b}$$

[In the plane the curl of a vector $\mathbf{V}$ is a scalar S, and the curl of a scalar is a vector: in components $S=\epsilon^{ij}\partial_i V^j$, $(\nabla\times S)^i=\epsilon^{ij}\partial_j S$.] The "magnetic" field arising from (1.2b), $\nabla\times\mathbf{a}_p$, is a sum of δ functions, vanishing almost everywhere. Consequently, the potential $\mathbf{a}_p$ may be presented as a singular pure gauge:

$$\frac{1}{2\pi}\epsilon^{ij}\frac{r^j}{r^2}=-\frac{1}{2\pi}\partial_i\arctan\frac{y}{x}=-\frac{1}{2\pi}\partial_i\theta \ , \qquad \mathbf{r}=(x,y)=(r\cos\theta,r\sin\theta) \ . \tag{1.3c}$$

The singularity at $\mathbf{r}=0$ is also seen from the identity

$$\epsilon^{ij}\partial_i\partial_j\theta=2\pi\delta(\mathbf{r}) \ , \tag{1.3d}$$

which is a consequence of (1.3a), (1.3b), (1.3c).

The above action-at-a-distance description follows from a local formulation based on the Lagrangian

$$L=L_{\text{matter}}+L_{\text{CS}}+L_{\text{interaction}} \ . \tag{1.4}$$

The matter Lagrangian is conventional,

$$L_{\text{matter}}=\sum_{p=1}^{N}\tfrac{1}{2}m_p v_p^2 \ , \tag{1.5a}$$

while the Chern-Simons Lagrangian provides the kinetic term for the gauge fields:

$$L_{\text{CS}}=\frac{\kappa}{2}\int d^2\mathbf{r}\,\epsilon^{\alpha\beta\gamma}\partial_\alpha A_\beta A_\gamma = \frac{\kappa}{2c}\int d^2\mathbf{r}\,\partial_t\mathbf{A}\times\mathbf{A}-\kappa\int d^2\mathbf{r}\,A^0B \ . \tag{1.5b}$$

Here the fields depend on time t and on the field point $\mathbf{r}$, i.e., on the three-vector $x^\mu=(ct,\mathbf{r})$; B is the "magnetic" field $B=\nabla\times\mathbf{A}$; also we define the "electric" field by $\mathbf{E}=-\nabla A^0-(1/c)\partial_t\mathbf{A}$. Finally, the interaction Lagrangian may be alternatively presented by particle variables

$$L_{\text{interaction}}=\frac{1}{c}\sum_{p=1}^{N}e_p\mathbf{v}_p\cdot\mathbf{A}(t,\mathbf{r}_p) -\sum_{p=1}^{N}e_pA^0(t,\mathbf{r}_p) \ , \tag{1.5c}$$

or by field variables in terms of a conserved current $j^\mu=(c\rho,\mathbf{j})$, $\partial_t\rho+\nabla\cdot\mathbf{j}=0$,

$$L_{\text{interaction}}=-\frac{1}{c}\int d^2\mathbf{r}\, A_\mu j^\mu$$
$$=\frac{1}{c}\int d^2\mathbf{r}\, \mathbf{A}\cdot\mathbf{j}-\int d^2\mathbf{r}\, A^0\rho\ , \tag{1.5d}$$

with the current depending on field and matter coordinates:

$$j^\mu(t,\mathbf{r})=\sum_{p=1}^{N} e_p v_p^\mu \delta(\mathbf{r}-\mathbf{r}_p)\ ,$$
$$v_p^\mu=(c,\mathbf{v}_p)\ . \tag{1.6}$$

The previous Hamiltonian description (1.1) and (1.2) is derived by eliminating the gauge-field variables in favor of matter variables by use of the gauge-field equation of motion, which in the present case reduces to a field-current identity:[1-3]

$$\frac{\kappa}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta}=\kappa\epsilon^{\mu\alpha\beta}\partial_\alpha A_\beta=\frac{1}{c}j^\mu\ , \tag{1.7a}$$

$$B=-\frac{1}{\kappa}\rho\ , \tag{1.7b}$$

$$E^i=\frac{1}{c\kappa}\epsilon^{ij}j^j\ . \tag{1.7c}$$

(One finds that self-interactions are absent.[4])

Various symmetries are present. First, there are the obvious space translation and rotation symmetries, with associated momentum $\mathbf{P}$ and angular momentum J constants of motion (generators):

$$\mathbf{P}=\sum_{p=1}^{N} m_p\mathbf{v}_p\ , \tag{1.8}$$

$$J=\sum_{p=1}^{N} \mathbf{r}_p\times m_p\mathbf{v}_p\ . \tag{1.9}$$

Moreover, there is a hidden SO(2,1) conformal invariance:[4] redefining time by a translation $t\to t-a$, by a dilation $t\to at$, or by a special conformal transformation $1/t\to 1/t+a$, is an invariance provided the particle coordinate is transformed, respectively, as $\mathbf{r}_p(t)\to\mathbf{r}_p(t+a)$, $\mathbf{r}_p(t)\to\sqrt{a}\,\mathbf{r}_p(t/a)$, and $\mathbf{r}_p(t)\to(1-at)\mathbf{r}_p[t/(1-at)]$. The respective generators are the Hamiltonian (1.1), the dilation generator

$$D=tH-\tfrac{1}{4}\sum_{p=1}^{N} m_p(\mathbf{r}_p\cdot\mathbf{v}_p+\mathbf{v}_p\cdot\mathbf{r}_p)\ , \tag{1.10}$$

and the conformal generator

$$K=-t^2H+2tD+\tfrac{1}{2}\sum_{p=1}^{N} m_p r_p^2\ . \tag{1.11}$$

Finally, our system is invariant against Galileo boosts $\mathbf{r}\to\mathbf{r}+\mathbf{V}t$, generated by

$$\mathbf{G}=t\mathbf{P}-\sum_{p=1}^{N} m_p\mathbf{r}_p\ . \tag{1.12}$$

In all the generators the interaction is again hidden in the relation (1.2) between kinetic and canonical momenta.

The quantum-mechanical problem requires solving the N-body Schrödinger equation for the wave function $\psi(t;\mathbf{r}_1,\ldots,\mathbf{r}_N)$:

$$i\hbar\partial_t\psi=\sum_{p=1}^{N}\frac{\hbar^2}{2m_p}\left[\nabla_p-i\frac{e_p}{\hbar c}\mathbf{a}_p\right]^2\psi\ . \tag{1.13}$$

Time may be separated in the usual way,

$$\psi=e^{-iEt/\hbar}u_E\ , \tag{1.14}$$

and u_E solves an eigenvalue problem:

$$Eu_E=\sum_{p=1}^{N}-\frac{\hbar^2}{2m_p}\left[\nabla_p-i\frac{e_p}{\hbar c}\mathbf{a}_p\right]^2 u_E\ . \tag{1.15}$$

Owing to the pure-gauge nature of $\mathbf{a}_p$, the interaction may be removed by redefining the phase of u_E. From (1.2b) and (1.3c) it follows that

$$a_p^i(\mathbf{r}_1,\ldots,\mathbf{r}_N)=\frac{1}{2\pi\kappa}e^{ij}\frac{\partial}{\partial r_p^j}\sum_{q\neq p}^{N} e_q\ln|\mathbf{r}_p-\mathbf{r}_q|$$
$$=-\frac{1}{2\pi\kappa}\frac{\partial}{\partial r_p^i}\sum_{q\neq p} e_q\theta_{pq}\ , \tag{1.16}$$
$$\tan\theta_{pq}=\frac{y_p-y_q}{x_p-x_q}\ .$$

Hence, by defining

$$u_E=e^{-i\Theta}u_E^0\ , \tag{1.17}$$

$$\Theta=\sum_{p<q}\nu_{pq}\theta_{pq}\ , \tag{1.18a}$$

$$\nu_{pq}=\frac{e_pe_q}{hc\kappa},\quad h=2\pi\hbar\ , \tag{1.18b}$$

we see that the wave function u_E^0 solves the *free* N-body Schrödinger equation

$$Eu_E^0=H^0u_E^0=\sum_{p=1}^{N}-\frac{\hbar^2}{2m_p}\nabla_p^2u_E^0\ , \tag{1.19}$$

while the interaction is hidden in the complicated boundary conditions satisfied by u_E^0: Since u_E is single valued but Θ is not, u_E^0 must be appropriately multivalued so that $u_E=e^{-i\Theta}u_E^0$ remains single valued (for noninteger ν_{pq}, which measure the magnetic flux due to particle q and felt by particle p; integer ν_{pq} is invisible). As a consequence, even though H^0 is a sum of free one-body Hamiltonians, u_E^0 is *not* a product of free one-body wave functions (plane waves). Rather it must be an appropriate superposition of one-body wave functions (orbitals), where the superposition is chosen so that the correct multivaluedness of u_E^0 is achieved.

The nontrivial task of constructing a superposition of free wave functions so that a complicated boundary condition is satisfied has been solved so far only for the two-body problem.[4] (Owing to the absence of self-interactions, the one-body problem is trivial.) The general case is reminiscent of the N-body problem on a line, with the interaction potential comprising a sum of one-

dimensional δ functions. There too the many-body wave function is obtained by superposing one-body, free wave functions in a fashion prescribed by the Bethe *Ansatz*. While the two-dimensional generalization of the δ-functional potential problem has resisted solution, it may be that our dynamics, with velocity-dependent δ functions contributing to the Lorentz force [in $\mathbf{E}(\mathbf{r}_p)+(\mathbf{v}_p/c)\times B(\mathbf{r}_p)$, $\mathbf{E}$ and B are sums of δ functions according to (1.6) and (1.7)] is more tractable.

An alternative and equivalent approach to the N-body problem with identical particles ($m_p=m$, $e_p=e$) is through second quantization: The Schrödinger problem is replaced by a nonrelativistic field theory. By the addition of a magnetic interaction, we obtain a two-dimensional, gauged nonlinear Schrödinger equation. This paper is devoted to a study of such a field theory and to an elaboration of the self-dual soliton solutions that we have recently found in the classical version of the model.[5]

Of course, the second-quantized one-dimensional problem with δ-function potentials also leads to the nonlinear Schrödinger equation in one dimension, whose analysis on the classical level was an important achievement in the complete integrability program for nonlinear partial differential equations.[6] Moreover, finding solitons of the nonlinear one-dimensional Schrödinger equation, quantizing them, and reproducing the Bethe solution was a milestone in the semiclassical, nonperturbative analysis of nonlinear quantum field theories.[7] No useful results are known for the *ungauged*, two-dimensional, nonlinear Schrödinger equation. However, adding a Chern-Simons gauge interaction evidently opens new possibilities for solution.

In Sec. II we present our field theory. Owing to particle number conservation, the model may be analyzed in the N-particle sector, where we reproduce the Schrödinger equation (1.15), with an additional magnetic interaction. The classical field theory is analyzed in Sec. III, and static, self-dual, zero-energy solutions are presented. Section IV is devoted to concluding remarks about several further features of the model. In particular, relativistic and supersymmetric generalizations of our model are discussed, and it is demonstrated that the Chern-Simons term does not merely alter statistics, but in general gives rise to dynamical interactions as well.

II. NONRELATIVISTIC QUANTUM FIELD THEORY

A. Hamiltonian and equation of motion

We consider a quantum field operator Ψ and its Hermitian conjugate $\Psi^\dagger$, obeying (bosonic) commutation relations at equal times (a time argument of our operators is suppressed):

$$[\Psi(\mathbf{r}),\Psi(\mathbf{r}')]=0,\quad [\Psi^\dagger(\mathbf{r}),\Psi^\dagger(\mathbf{r}')]=0\ ,$$
$$[\Psi(\mathbf{r}),\Psi^\dagger(\mathbf{r})]=\delta(\mathbf{r}-\mathbf{r}')\ . \tag{2.1}$$

(For definiteness and simplicity we take a bosonic algebra; similar analysis can be given with a fermionic algebra.) We posit a Hamiltonian, which governs time evolution. Ordering of noncommutating operators is important; the normal-ordering prescription of putting all $\Psi^\dagger$ to the left of Ψ is made and is denoted by colons. To describe the Hamiltonian accurately, we begin by defining the operator covariant derivative

$$\mathbf{\Pi}(\mathbf{r})=\left[\nabla-i\frac{e}{\hbar c}\mathbf{A}(\mathbf{r})\right]\Psi(\mathbf{r})\equiv\mathbf{D}\Psi(\mathbf{r}) \tag{2.2}$$

and its conjugate $\mathbf{\Pi}^\dagger\equiv(\mathbf{D}\Psi)^\dagger$, where the Hermitian vector potential operator $\mathbf{A}$ is constructed from the number density operator

$$\rho=\Psi^\dagger\Psi \tag{2.3}$$

by a formula that solves the Chern-Simons relation between magnetic field $B=\nabla\times\mathbf{A}$ and charge density $e\rho$; compare (1.7b):

$$B=-\frac{e}{\kappa}\rho\ . \tag{2.4}$$

We take

$$\mathbf{A}(\mathbf{r})=\nabla\times\frac{e}{\kappa}\int d^2\mathbf{r}'G(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\ , \tag{2.5}$$

with G the Green's function for the Laplacian, as in (1.3). Hence, as explained there, $\nabla\times G(\mathbf{r}-\mathbf{r}')$ may also be presented as a gradient:

$$\nabla\times G(\mathbf{r}-\mathbf{r}')=-\frac{1}{2\pi}\nabla\Theta(\mathbf{r}-\mathbf{r}')\ ,$$
$$\tan\Theta(\mathbf{r}-\mathbf{r}')=\frac{y-y'}{x-x'}\ . \tag{2.6}$$

Note that the Θ is multivalued, since $\Theta=0$ and $\Theta=2\pi$ are identified.

From the commutation relations it follows that

$$[\mathbf{A}(\mathbf{r}),\Psi(\mathbf{r}')]=-\frac{e}{\kappa}\nabla\times G(\mathbf{r}-\mathbf{r}')\Psi(\mathbf{r}')\ . \tag{2.7}$$

$\nabla\times G(\mathbf{r})$ is ill defined at the origin; we shall prescribe that it vanishes there; i.e., we have in mind a regularization that preserves the antisymmetry of $\nabla\times G$ under space reflection. Therefore, $\Psi(\mathbf{r})$ and $\Psi^\dagger(\mathbf{r})$ commute with $\mathbf{A}(\mathbf{r})$, and no ordering ambiguity afflicts $\mathbf{\Pi}$ and $\mathbf{\Pi}^\dagger$. Also $\rho(\mathbf{r})$ commutes with $\rho(\mathbf{r}')$ and so do the vector potential components with each other.

The posited Hamiltonian is[5]

$$H=\int d^2\mathbf{r}\,\mathcal{H}\ , \tag{2.8a}$$

$$\mathcal{H}(\mathbf{r})=\frac{\hbar^2}{2m}\mathbf{\Pi}^\dagger(\mathbf{r})\cdot\mathbf{\Pi}(\mathbf{r})-\frac{g}{2}:[\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})]^2:\ . \tag{2.8b}$$

In view of (2.3) and (2.4), this may alternatively be written as

$$\mathcal{H}(\mathbf{r})=\frac{\hbar^2}{2m}\mathbf{\Pi}^\dagger(\mathbf{r})\cdot\mathbf{\Pi}(\mathbf{r})+\frac{g\kappa}{2e}:B(\mathbf{r})\rho(\mathbf{r}):\ , \tag{2.8c}$$

which shows that the quartic term in (2.8b) describes a magnetic-field–charge-density interaction of strength $g\kappa/2e^2$. Note that although $\mathbf{\Pi}$ and $\mathbf{\Pi}^\dagger$ are separately normal ordered, $\mathbf{\Pi}^\dagger\cdot\mathbf{\Pi}$ is not.

The field equation of motion follows by commutation:

$$i\hbar\partial_t\Psi(\mathbf{r})=[\Psi(\mathbf{r}),H]$$
$$=-\frac{\hbar^2}{2m}D^2\Psi(\mathbf{r})+eA^0(\mathbf{r})\Psi(\mathbf{r})-g\rho(\mathbf{r})\Psi(\mathbf{r})$$
$$+\frac{e^4}{2mc^2\kappa^2}\int d^2\mathbf{r}'\nabla G(\mathbf{r}-\mathbf{r}')$$
$$\cdot\nabla G(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\Psi(\mathbf{r}) . \quad (2.9)$$

The scalar potential A^0 in (2.9) is given by

$$A^0(\mathbf{r})=-\frac{e}{c\kappa}\int d^2\mathbf{r}'G(\mathbf{r}-\mathbf{r}')\nabla\times\mathbf{j}(\mathbf{r}') , \quad (2.10)$$

where $\mathbf{j}$ is the number current-density operator:

$$\mathbf{j}=\frac{\hbar}{2mi}(\Psi^\dagger\boldsymbol{\Pi}-\boldsymbol{\Pi}^\dagger\Psi) . \quad (2.11)$$

The third term on the right-hand side of (2.9) comes from the magnetic-quartic interaction, while the last term

$$\frac{e^4}{2mc^2\kappa^2}\int\frac{d^2\mathbf{r}'}{(2\pi)^2}\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\rho(\mathbf{r}')\Psi(\mathbf{r})$$

is a quantum correction arising from reordering; it will play a crucial role in the following. The magnetic next-to-last term in (2.9) may also be written as $-g\int d\mathbf{r}'\delta(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\Psi(\mathbf{r})$; hence it is of the same dimensionality as the reordering term, since $\delta(\mathbf{r})$ scales as r^{-2}.

Because the matter density ρ satisfies the continuity equation

$$\partial_t\rho=\frac{i}{\hbar}[H,\rho]=-\nabla\cdot\mathbf{j} , \quad (2.12)$$

the number operator $N=\int d^2\mathbf{r}\rho$ commutes with the Hamiltonian. Also, with the help of (2.12) we see that the formula (2.10) for the scalar potential A^0 solves the remaining Chern-Simons field-current identity between the electric field and the current density [compare (1.7c)]:

$$E^i=-\partial_i A^0-\frac{1}{c}\partial_t A^i=\frac{e}{c\kappa}\epsilon^{ij}j^j . \quad (2.13)$$

We record one further commutator that will be used below:

$$\frac{i}{\hbar}[\rho(\mathbf{r}),\mathbf{j}(\mathbf{r}')]=\frac{1}{m}\rho(\mathbf{r}')\nabla\delta(\mathbf{r}-\mathbf{r}') . \quad (2.14)$$

(In this section and below, in contrast with Sec. I, ρ and $\mathbf{j}$ are defined without the gauge coupling constant e; they are *matter* densities; the *charge* densities of Sec. I are obtained upon multiplication by e.)

B. State space and the Schrödinger equation

The two commuting operators H and N can be simultaneously diagonalized, and we label eigenstates by their eigenvalues $|E,N\rangle$:

$$H|E,N\rangle=E|E,N\rangle , \quad (2.15)$$

$$N|E,N\rangle=N|E,N\rangle . \quad (2.16)$$

Also, we posit the existence of the "vacuum" zero state $|\Omega\rangle$, annihilated by Ψ,

$$\Psi(\mathbf{r})|\Omega\rangle=0=\langle\Omega|\Psi^\dagger(\mathbf{r}) , \quad (2.17)$$

and also by N and H:

$$N|\Omega\rangle=0=H|\Omega\rangle . \quad (2.18)$$

However, $\Psi^\dagger$ operating on $|\Omega\rangle$ produces another state, and in particular we define

$$\langle\Omega|\Psi(\mathbf{r}_1)\cdots\Psi(\mathbf{r}_N)|E,N\rangle\equiv u_E(\mathbf{r}_1,\ldots,\mathbf{r}_N) . \quad (2.19)$$

It is clear that exactly N field operators are needed to connect $|E,N\rangle$ to $|\Omega\rangle$. Also, as a consequence of its definition (2.19) and the commutativity of the operators Ψ, $u_E(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ is a bosonic wave function, symmetric under position interchange.

By considering the commutator matrix element

$$\langle\Omega|[\Psi(\mathbf{r}_1)\cdots\Psi(\mathbf{r}_N),H]|E,N\rangle=Eu_E(\mathbf{r}_1,\ldots,\mathbf{r}_N) , \quad (2.20)$$

evaluating the commutators from (2.9), and reexpressing the resultant matrix elements in terms of u_E again, one arrives at the Schrödinger equations obeyed by the N-body wave functions u_E.

For example, in the one-body problem we have

$$u_E(\mathbf{r})=\langle\Omega|\Psi(\mathbf{r})|E,1\rangle , \quad (2.21a)$$

$$Eu_E(\mathbf{r})=\langle\Omega|[\Psi(\mathbf{r}),H]|E,1\rangle . \quad (2.21b)$$

The commutator, given by (2.9), involves operators that contain $\Psi^\dagger$ standing on the left and annihilating $\langle\Omega|$, except for the Laplacian portion of $D^2\Psi$:

$$\langle\Omega|[\Psi(\mathbf{r}),H]=-\frac{\hbar^2}{2m}\nabla^2\langle\Omega|\Psi(\mathbf{r}) .$$

Hence the one-body problem is free—there are no self-interactions:

$$-\frac{\hbar^2}{2m}\nabla^2u_E(\mathbf{r})=Eu_E(\mathbf{r}) . \quad (2.21c)$$

For two bodies we begin with

$$u_E(\mathbf{r}_1,\mathbf{r}_2)=\langle\Omega|\Psi(\mathbf{r}_1)\Psi(\mathbf{r}_2)|E,2\rangle , \quad (2.22a)$$

$$Eu_E(\mathbf{r}_1,\mathbf{r}_2)=\langle\Omega|[\Psi(\mathbf{r}_1)\Psi(\mathbf{r}_2),H]|E,2\rangle$$
$$=-\frac{\hbar^2}{2m}\nabla_1^2\langle\Omega|\Psi(\mathbf{r}_1)\Psi(\mathbf{r}_2)|E,2\rangle$$
$$+\langle\Omega|\Psi(\mathbf{r}_1)[\Psi(\mathbf{r}_2),H]|E,2\rangle . \quad (2.22b)$$

Here the matrix element of the remaining commutator is no longer trivial, since the $\Psi(\mathbf{r}_1)$ operator is interposed. The evaluation is straightforward, with the result

$$Eu_E(\mathbf{r}_1,\mathbf{r}_2)=\left[-\frac{\hbar^2}{2m}\left[\nabla_1-\frac{ie^2}{\hbar c\kappa}\nabla_1\times G(\mathbf{r}_1-\mathbf{r}_2)\right]^2\right.$$
$$-\frac{\hbar^2}{2m}\left[\nabla_2-\frac{ie^2}{\hbar c\kappa}\nabla_2\times G(\mathbf{r}_2-\mathbf{r}_1)\right]^2$$
$$\left.-g\delta(\mathbf{r}_1-\mathbf{r}_2)\right]u_E(\mathbf{r}_1,\mathbf{r}_2) . \quad (2.22c)$$

It is instructive to follow the steps leading to (2.22c), since the reordering last term in (2.9) plays a crucial role: it is needed to reconstruct the full covariant Laplacian $[\nabla_1-(ie^2/\hbar c\kappa)\nabla\times\mathbf{G}]^2$ from the free Laplacian ∇_1^2 seen in (2.22b). Equation (2.22c) is just the two-body equation of (1.15) for identical, bosonic particles, with an additional δ-function interaction. Note that this further contact interaction, which as we remarked can be viewed as an interaction with the magnetic field, does not affect wave functions that vanish at $\mathbf{r}_1=\mathbf{r}_2$; however, careful analysis of the Schrödinger operator is required to ensure self-adjointness.[8] It will be shown below that the additional contact interaction, with a definite value for its strength, g, is a very natural modification of the minimal theory.

We do not here pursue further the quantum-mechanical problem. Also, we leave for future analysis the subject of divergences and renormalization in our quantum field theory.

C. Removing the gauge field

We stated in Sec. I that the interaction with the gauge potential $\mathbf{a}_p$ can be removed by using the fact that $\mathbf{a}_p$ is locally a pure gauge, which disappears after a phase redefinition of the wave function [see (1.16)–(1.19)]. This is again seen explicitly (2.22c). According to (2.6),

$$-\frac{ie^2}{\hbar c\kappa}\nabla_{1,2}\times G(\mathbf{r}_1-\mathbf{r}_2)=\frac{ie^2}{\hbar c\kappa}\nabla_{1,2}\Theta(\mathbf{r}_1-\mathbf{r}_2)\ .$$

Therefore, by redefining the single-valued wave function u_E according to

$$u_E(\mathbf{r}_1,\mathbf{r}_2)=e^{-i\nu\theta}u_E^0(\mathbf{r}_1,\mathbf{r}_2)\ , \tag{2.23}$$

where $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2=(r\cos\theta,r\sin\theta)$ and

$$\nu=\frac{e^2}{hc\kappa}\ , \tag{2.24}$$

the gauge-field interaction is eliminated:

$$Eu_E^0(\mathbf{r}_1,\mathbf{r}_2)=\left[-\frac{\hbar^2}{2m}\nabla_1^2-\frac{\hbar^2}{2m}\nabla_2^2 -g\delta(\mathbf{r}_1-\mathbf{r}_2)\right]u_E^0(\mathbf{r}_1,\mathbf{r}_2)\ , \tag{2.25}$$

but u_E^0 satisfies a nontrivial boundary condition that is sensitive to magnetic flux with noninteger ν:

$$u_E^0(\mathbf{r}_1,\mathbf{r}_2)|_{\theta=2\pi}=e^{i2\pi\nu}u_E^0(\mathbf{r}_1,\mathbf{r}_2)|_{\theta=0}\ . \tag{2.26a}$$

Note that rotating the relative separation $\mathbf{r}$ by 2π corresponds to a *double* interchange: $(\mathbf{r}_1,\mathbf{r}_2)\to(\mathbf{r}_2,\mathbf{r}_1)\to(\mathbf{r}_1,\mathbf{r}_2)$. Since $u_E(\mathbf{r}_1,\mathbf{r}_2)=u_E(\mathbf{r}_2,\mathbf{r}_1)$, it follows that

$$u_E^0(r_1,r_2)=e^{i\pi\nu}u_E^0(\mathbf{r}_2,\mathbf{r}_1)\ . \tag{2.26b}$$

The question we wish to address is how this phenomenon manifests itself in the quantum-field-theoretic formalism. The discussion involves mathematically delicate manipulation, and care must be exercised to avoid blunders.

At issue is whether the vector potential operator (2.5), which occurs in the covariant derivative (2.2), is in some sense a pure gauge. As a consequence of (2.6), we may certainly write $\mathbf{A}$ as

$$\mathbf{A}(\mathbf{r})=-\frac{e}{2\pi\kappa}\int d^2\mathbf{r}'\nabla\Theta(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\ . \tag{2.27}$$

However, moving the gradient with respect to $\mathbf{r}$ out of the integral and thus displaying $\mathbf{A}$ as a pure gauge is in general not correct; the integration cannot be interchanged with the differentiation. The reason for this is that Θ is multivalued and integrating Θ over the two-dimensional $\mathbf{r}'$ plane requires specifying a cut in $\mathbf{r}'$, which begins at $\mathbf{r}$. The details of the cut are immaterial, but now the range of the $\mathbf{r}'$ integration depends on $\mathbf{r}$, and consequently, moving the $\mathbf{r}$ derivative outside the $\mathbf{r}'$ integral gives an additional contribution.

There is an exceptional situation where we need not exercise the above-described care: The $\mathbf{r}$ gradient *can* be moved outside the integral with impunity and the vector potential *is* a pure gauge. This happens when the matter density ρ is localized at points, i.e., when it is a superposition of δ functions. Of course, in the present context ρ is an operator, but in nonrelativistic quantum field theory the eigenvalues of this operator are indeed δ functions. So for the problem at hand, we *can* write

$$\mathbf{A}(\mathbf{r})=\nabla\left[-\frac{e}{2\pi\kappa}\int d^2\mathbf{r}'\Theta(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\right] \equiv\nabla\omega(\mathbf{r})\ . \tag{2.28}$$

This is consistent with $B=\nabla\times\mathbf{A}=-(e/\kappa)\rho$ as long as ρ vanishes almost everywhere, which it does in nonrelativistic quantum field theory. The magnetic field $\nabla\times\nabla\omega$ [see (1.3d)] is present at isolated points owing to the multivaluedness of the gauge function ω, which in turn is a consequence of Θ being multivalued and ρ being a sum of δ functions.

But when ρ is nonvanishing and is smoothly distributed over an extended region, (2.28) is not consistent with nonvanishing B, no matter whether or not ω is single valued.

There have been published analyses of relativistic quantum field theory, where the gauge field (2.27) is written as in (2.28).[9] However, in relativistic field theory the eigenvalues of ρ are *not* sums of δ functions; particles are *not* points, but are extended. Hence the representation (2.28) is inconsistent with nonvanishing B arising from a smooth ρ.[10] Rather, the correct expression for $\mathbf{A}$, which takes into account the contribution from the cut, is

$$\mathbf{A}=\nabla\omega-\frac{e}{\kappa}\epsilon^{ij}\int d\mathbf{z}^j\rho(\mathbf{z})\ . \tag{2.29}$$

Here the line integral of ρ is along the cut $\mathbf{z}$, which begins at $\mathbf{r}$ and passes to infinity in some arbitrary but $\mathbf{r}$-independent fashion, e.g., $\mathbf{z}=\mathbf{r}+\mathbf{f}(\tau)$, $\mathbf{f}(0)=0$, $\mathbf{f}(1)=\infty$.

Another context where the Chern-Simons gauge potential is not a pure gauge is classical field theory, whether nonrelativistic or relativistic, because there ρ is a smooth, distributed function. In the next section we shall give vivid demonstration of the influence of a nontrivial Chern-Simons vector potential on classical field-theory dynamics.

Returning now to our nonrelativistic quantum field theory, we may use (2.28) together with a redefinition of the matter field

$$\Psi(\mathbf{r})=e^{i(e/\hbar c)\omega}\Psi^0(\mathbf{r}) , \tag{2.30}$$

to reduce the covariant derivative operator (2.2) to $\mathbf{\Pi}(\mathbf{r})=e^{i(e/\hbar c)\omega}\nabla\Psi^0(\mathbf{r})$ and the Hamiltonian density (2.8b) to

$$\mathcal{H}=\frac{\hbar^2}{2m}\nabla\Psi^{0\dagger}\nabla\Psi^0-\frac{g}{2}:(\Psi^{0\dagger}\Psi^0)^2: . \tag{2.31}$$

The gauge fields also disappear from the equation of motion (2.9), and this happens in an interesting way. First, A^0 of (2.10) is rewritten as

$$\begin{aligned}A^0(\mathbf{r})&=\frac{e}{c\kappa}\int d^2\mathbf{r}'[\nabla\times G(\mathbf{r}-\mathbf{r}')]\cdot\mathbf{j}(\mathbf{r}')\\&=-\frac{e}{2\pi c\kappa}\int d^2\mathbf{r}'\nabla\Theta(\mathbf{r}-\mathbf{r}')\cdot\mathbf{j}(\mathbf{r}')\\&=-\frac{e}{2\pi c\kappa}\int d^2\mathbf{r}'\Theta(\mathbf{r}-\mathbf{r}')\nabla\cdot\mathbf{j}(\mathbf{r}')\\&=\frac{e}{2\pi c\kappa}\int d^2\mathbf{r}'\Theta(\mathbf{r}-\mathbf{r}')\partial_t\rho(\mathbf{r}')\\&=-\frac{1}{c}\partial_t\omega(\mathbf{r}) .\end{aligned} \tag{2.32}$$

We have used current conservation and freely dropped surface terms because the densities are local—for distributed densities these steps would be illegitimate. Thus, according to (2.30) and (2.32), the operator equation of motion (2.9) in the nonrelativistic quantum field theory becomes

$$i\hbar\partial_t\Psi^0(\mathbf{r})=-\frac{\hbar^2}{2m}\nabla^2\Psi^0(\mathbf{r})-g\rho(\mathbf{r})\Psi^0(\mathbf{r})-\left[i\hbar e^{-i(e/\hbar c)\omega}\partial_t e^{i(e/\hbar c)\omega}+\frac{e}{c}e^{-i(e/\hbar c)\omega}\partial_t\omega e^{i(e/\hbar c)\omega}-\frac{e^4}{2mc^2\kappa^2}\int d^2\mathbf{r}'\nabla G(\mathbf{r}-\mathbf{r}')\cdot\nabla G(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\right]\Psi^0(\mathbf{r}) . \tag{2.33}$$

The first two terms in the large parentheses do not cancel, because $\partial_t\omega$, which involves $\partial_t\rho=-\nabla\cdot\mathbf{j}$, does not commute with ω, as is seen from (2.14). When this noncommutation is taken into account, one finds that the entire expression in large parentheses vanishes—once again, the last reordering term is needed.

We conclude that the Ψ^0 fields satisfy equations of motion without gauge potentials; with $g=0$ these are free equations. However, the commutation relations are now modified:

$$\begin{aligned}\Psi^0(\mathbf{r})\Psi^0(\mathbf{r}')&=e^{-i(e/\hbar c)\omega(\mathbf{r})}\Psi(\mathbf{r})e^{-i(e/\hbar c)\omega(\mathbf{r}')}\Psi(\mathbf{r}')\\&=e^{-i(e/\hbar c)[\omega(\mathbf{r})+\omega(\mathbf{r}')]}e^{i(e^2/\hbar c\kappa)\Theta(\mathbf{r}'-\mathbf{r})}\Psi(\mathbf{r})\Psi(\mathbf{r}')\\&=e^{-i(e/\hbar c)[\omega(\mathbf{r})+\omega(\mathbf{r}')]}e^{i(e^2/\hbar c\kappa)\Theta(\mathbf{r}'-\mathbf{r})}e^{i(e/\hbar c)\omega(\mathbf{r}')}\Psi^0(\mathbf{r}')e^{i(e/\hbar c)\omega(\mathbf{r})}\Psi^0(\mathbf{r})\\&=e^{i(e^2/\hbar c\kappa)[\Theta(\mathbf{r}'-\mathbf{r})-\Theta(\mathbf{r}-\mathbf{r}')]}\Psi^0(\mathbf{r}')\Psi^0(\mathbf{r})\\&=e^{i\pi\nu}\Psi^0(\mathbf{r}')\Psi^0(\mathbf{r}) .\end{aligned} \tag{2.34}$$

[We use ν from (2.24).] Thus upon defining the wave function u_E^0 by

$$\langle\Omega|\Psi^0(\mathbf{r}_1)\Psi^0(\mathbf{r}_2)|E,2\rangle=u_E^0(\mathbf{r}_1,\mathbf{r}_2) , \tag{2.35}$$

we see that under interchange, u_E^0 acquires the phase $e^{i\pi\nu}$, and under double interchange, the phase change is $e^{i2\pi\nu}$, in agreement with (2.26).

Finally we remark that a calculation similar to (2.35) shows that the canonical commutator is unchanged:

$$[\Psi^0(\mathbf{r}_1),\Psi^{0\dagger}(\mathbf{r}_2)]=\delta(\mathbf{r}_1-\mathbf{r}_2) . \tag{2.36}$$

III. NONRELATIVISTIC CLASSICAL FIELD THEORY

A. Hamiltonian and equation of motion

We now consider Ψ to be a classical c-number field and $\Psi^\dagger$ its complex conjugate Ψ^*. The Hamiltonian is, as in (2.8),

$$H=\int d^2\mathbf{r}\,\mathcal{H} , \tag{3.1a}$$

$$\mathcal{H}=\frac{\hbar^2}{2m}(\mathbf{D}\Psi)^*\cdot(\mathbf{D}\Psi)-\frac{g}{2}(\Psi^*\Psi)^2 , \tag{3.1b}$$

but the classical equation of motion does not include the quantum reordering term—the last term in (2.9):

$$\begin{aligned}i\hbar\partial_t\Psi(t,\mathbf{r})&=\frac{\delta H}{\delta\Psi^*(t,\mathbf{r})}\\&=-\frac{\hbar^2}{2m}D^2\Psi(t,\mathbf{r})+eA^0(t,\mathbf{r})\Psi(t,\mathbf{r})\\&\quad-g\rho(t,\mathbf{r})\Psi(t,\mathbf{r}) .\end{aligned} \tag{3.2}$$

The derivative $\mathbf{D}$ is covariant, as in (2.2); the vector and scalar potentials describe Chern-Simons gauge fields that satisfy the field-current identities:

$$B=\nabla\times\mathbf{A}=-\frac{e}{\kappa}\rho , \tag{3.3a}$$

$$E^i=-\partial_i A^0-\frac{1}{c}\partial_t A^i=\frac{e}{c\kappa}\epsilon^{ij}j^j , \tag{3.3b}$$

where the matter density and current are as in (2.3) and (2.11). Explicit formulas for $\mathbf{A}$ and A^0 are as in (2.5) and (2.10), except now it is useful to leave arbitrary a gauge function:

$$\mathbf{A}(t,\mathbf{r})=\nabla\times\frac{e}{\kappa}\int d^2\mathbf{r}'G(\mathbf{r}-\mathbf{r}')\rho(t,\mathbf{r}') + \text{gauge term} , \tag{3.4a}$$

$$A^0(t,\mathbf{r})=-\nabla\times\frac{e}{c\kappa}\int d^2\mathbf{r}'G(\mathbf{r}-\mathbf{r}')\mathbf{j}(t,\mathbf{r}') + \text{gauge term} . \tag{3.4b}$$

Upon setting

$$\Psi=e^{i(e/\hbar c)\omega}\rho^{1/2} , \tag{3.5}$$

Eq. (3.1) is decomposed into real and imaginary parts. It is found that the imaginary part of (3.2) is the continuity equation

$$\partial_t\rho+\nabla\cdot\mathbf{j}=0 , \tag{3.6}$$

while the real part yields

$$\nabla^2\ln\rho=\frac{4m}{\hbar^2}(eA^0-g\rho) + \frac{2}{\hbar^2c^2}\left[e\,\mathbf{A}-\frac{\hbar c}{2}\nabla\times\ln\rho\right]\cdot\left[e\,\mathbf{A}+\frac{\hbar c}{2}\nabla\times\ln\rho\right] , \tag{3.7}$$

where the potentials are given by formulas (3.4) with additional gauge contributions coming from ω. Note that the current density takes the London form[11]

$$\mathbf{j}=-\frac{e}{mc}\rho\,\mathbf{A} . \tag{3.8}$$

From (3.4a) we see that the nontrivial part of $\mathbf{A}$ is a nonlocal functional of ρ, and therefore so is $\mathbf{j}$. Decomposing the latter into a longitudinal component $\mathbf{j}_L$, determined by $\partial_t\rho$ ($\partial_t\rho=-\nabla\cdot\mathbf{j}_L$) and a transverse component $\nabla\times j$,

$$\mathbf{j}=\mathbf{j}_L+\nabla\times j , \tag{3.9}$$

allows expressing the nontrivial part of A^0 in terms of j:

$$A^0=\frac{e}{c\kappa}j . \tag{3.10}$$

Thus the nontrivial part of A^0 also is a functional of ρ, and (3.7) is recognized to be a nonlinear and nonlocal equation for ρ at fixed time, while (3.6) determines the time dependence.

For static configurations, (3.6) requires $\mathbf{j}$ to be transverse; i.e., $\mathbf{j}_L$ vanishes and $\mathbf{j}=\nabla\times j$. Let us choose a gauge so that $\mathbf{A}$ in (3.8) is transverse. To ensure transversality of $\mathbf{j}$, we must have $\mathbf{A}\cdot\nabla\rho=0$. This suggests a local *Ansatz* for the ρ dependence of $\mathbf{A}$: $\mathbf{A}=\nabla\times a(\rho)=(\nabla\times\rho)a'(\rho)$, where a is a function of ρ. Comparison with (3.8) and (3.9) shows that $a'(\rho)=-mcj'(\rho)/e\rho$, and

$$e\,\mathbf{A}=-mc(\nabla\times\ln\rho)j'(\rho) . \tag{3.11}$$

Hence, with the local *Ansatz*, (3.7) reduces to

$$\nabla^2\ln\rho=\frac{4e^2m}{\hbar^2c\kappa}\left[j-\frac{c\kappa g}{e^2}\rho\right] + \frac{2m^2}{\hbar^2}\left[j'+\frac{\hbar}{2m}\right]\left[j'-\frac{\hbar}{2m}\right]\frac{(\nabla\rho)^2}{\rho^2} , \tag{3.12}$$

while (3.3a) and (3.11) imply

$$\nabla^2\ln\rho=-\frac{e^2}{mc\kappa}\frac{\rho}{j'}-\frac{j''}{j'}\frac{(\nabla\rho)^2}{\rho} . \tag{3.13}$$

Below we shall present solutions of these static equations.

B. Space-time symmetries

The field-theoretic dynamics admits the same space-time symmetries as the particle dynamics. These symmetries were discussed in Sec. I, but we did not there include the g-dependent magnetic interaction. However, this addition does not alter the invariances of the theory. Here we record the field-theoretic conserved quantities (generators) and the field transformation laws.

Consider first space-time translations $t\to t-a$, $\mathbf{r}\to\mathbf{r}+\mathbf{r}_0$. Under the former the field transforms as

$$\Psi(t,\mathbf{r})\to\Psi(t+a,\mathbf{r}) , \tag{3.14}$$

and the conserved generator is the Hamiltonian H of (3.1)—the spatial integral of the Hamiltonian density, which we here call T^{00}, the time-time component of a nonrelativistic energy-momentum tensor:

$$H=\int d^2\mathbf{r}\,\mathcal{H} , \tag{3.15a}$$

$$\mathcal{H}=T^{00}=\frac{\hbar^2}{2m}|\mathbf{D}\Psi|^2-\frac{g}{2}\rho^2 . \tag{3.15b}$$

The time independence of H is assured by the continuity equation satisfied, as a consequence of the equations of motion, by T^{00} and the energy flux T^{i0},

$$\partial_t T^{00}+\partial_i T^{i0}=0 , \tag{3.16}$$

the latter being given by

$$T^{i0}=-\frac{\hbar^2}{2m}[(D_t\Psi)^*D_i\Psi+(D_i\Psi)^*(D_t\Psi)] , \qquad D_t\Psi\equiv\left[\partial_t+i\frac{e}{\hbar}A^0\right]\Psi . \tag{3.17}$$

Space translations induce the field change

$$\Psi(t,\mathbf{r})\to\Psi(t,\mathbf{r}-\mathbf{r}_0) \tag{3.18}$$

generated by the momentum $\mathbf{P}$, which is an integral of the momentum density $\mathcal{P}^i$—also denoted by T^{0i}:

$$\mathbf{P}=\int d^2\mathbf{r}\,\mathcal{P} , \tag{3.19a}$$

$$\mathcal{P}^i=T^{0i}=mj^i=\frac{\hbar}{2i}[\Psi^*(D_i\Psi)-(D_i\Psi)^*\Psi] . \tag{3.19b}$$

Note that the momentum density does not coincide with the energy flux $T^{0i}\neq T^{i0}$: the energy-momentum tensor is not symmetric since the theory is not Lorentz invariant. Again, a continuity equation holds with the momentum

flux T^{ij}:

$$\partial_t T^{0i}+\partial_j T^{ji}=0 \ . \tag{3.20}$$

The momentum flux, the stress tensor T^{ij}, is given by

$$T^{ij}=\frac{\hbar^2}{2m}[(D_i\Psi)^*(D_j\Psi)+(D_j\Psi)^*(D_i\Psi) -\delta^{ij}(D_k\Psi)^*(D_k\Psi)] +\frac{\hbar^2}{4m}(\delta^{ij}\nabla^2-2\partial_i\partial_j)\rho+\delta^{ij}\mathcal{H} \ . \tag{3.21}$$

T^{ij} is symmetric—this follows from rotation invariance (see below). Also, T^{ij} has been improved so that its trace is twice the energy density:

$$\delta^{ij}T^{ij}=2\mathcal{H}=2T^{00} \ . \tag{3.22}$$

The possibility of finding such an improvement is related to the conformal symmetry (see below). [Recall that in a Lorentz-invariant theory with conformal symmetry, the relation corresponding to (3.22) contains a single factor of energy density: $\delta^{ij}T^{ij}=\mathcal{H}=T^{00}$.]

Next, we consider rotations, whereby the coordinate $\mathbf{r}$ is rotated by a rotation matrix R^{ij}:

$$r^i\rightarrow R^{ij}r^j, \quad R^{ij}R^{kj}=\delta^{ik} \ , \qquad \Psi(t,\mathbf{r})\rightarrow\Psi(t,R^{-1}\mathbf{r}) \ . \tag{3.23}$$

The density $\mathcal{J}$ for the angular momentum J,

$$J=\int d^2\mathbf{r}\,\mathcal{J} \ , \tag{3.24a}$$

$$\mathcal{J}=\mathbf{r}\times\mathcal{P}=m\mathbf{r}\times\mathbf{j} \ , \tag{3.24b}$$

satisfies a continuity equation as a consequence of T^{ij} being symmetric:

$$\partial_t\mathcal{J}+\partial_i\epsilon^{nm}x^nT^{im}=0 \ . \tag{3.25}$$

Equation (3.24) shows that the angular momentum is proportional to the magnetic dipole moment. One verifies that J generates on $\Psi(t,\mathbf{r})$ an infinitesimal rotation of the coordinate $\mathbf{r}$, supplemented by a gauge transformation $\delta\Psi=-\mathbf{r}\times\nabla\Psi-i(e/hc\kappa)Q\Psi$, $Q=e\int d^2\mathbf{r}\rho$. Hence J may be decomposed into orbital and spin parts $J=L+S$, where L generates the coordinate change, while S generates the gauge transformation. The magnitude of the spin $S=(1/4\pi c\kappa)Q^2$ coincides with what is found by other methods.[2]

Finally, the remaining conventional symmetry comprises Galileo boosts: $\mathbf{r}\rightarrow\mathbf{r}+\mathbf{V}t$, which act on the fields with a one-cocyle:

$$\Psi(t,\mathbf{r})\rightarrow\exp i(m/\hbar)(\mathbf{V}\cdot\mathbf{r}-\tfrac{1}{2}V^2t)]\Psi(t,\mathbf{r}-\mathbf{V}t) \ . \tag{3.26}$$

The generator $\mathbf{G}$ involves the electric dipole moment

$$\mathbf{G}=t\mathbf{P}-m\int d^2\mathbf{r}\,\mathbf{r}\rho \ , \tag{3.27a}$$

and is obtained from the density $\mathcal{G}$,

$$\mathcal{G}=t\mathcal{P}-m\mathbf{r}\rho \ , \tag{3.27b}$$

which satisfies a continuity equation:

$$\partial_t\mathcal{G}^i+\partial_j(tT^{ij}-r^iT^{0j})=0 \ . \tag{3.28}$$

Now we turn to the unexpected conformal symmetries.[4] Under a dilation $t\rightarrow at$, $\mathbf{r}\rightarrow\sqrt{a}\,\mathbf{r}$, the field transforms as

$$\Psi(t,\mathbf{r})\rightarrow\frac{1}{\sqrt{a}}\Psi\left[\frac{t}{a},\frac{r}{\sqrt{a}}\right] \ , \tag{3.29}$$

which is generated by D:

$$D=tH-\tfrac{1}{2}\int d^2\mathbf{r}\,\mathbf{r}\cdot\mathcal{P} \ . \tag{3.30a}$$

The corresponding density $\mathcal{D}$,

$$\mathcal{D}=t\mathcal{H}-\tfrac{1}{2}\mathbf{r}\cdot\mathcal{P} \ , \tag{3.30b}$$

satisfies the continuity equation

$$\partial_t\mathcal{D}+\partial_i(tT^{i0}-\tfrac{1}{2}r^jT^{ij})=0 \ , \tag{3.31}$$

as a consequence of the trace property of T^{ij} [see (3.22)].

A conformal redefinition $1/t\rightarrow1/t+a$, $\mathbf{r}\rightarrow\mathbf{r}/(1+at)$ is implemented on the fields by

$$\Psi(t,\mathbf{r})\rightarrow\frac{1}{1-at}e^{-imar^2/2\hbar(1-at)}\Psi\left[\frac{t}{1-at},\frac{\mathbf{r}}{1-at}\right] \ . \tag{3.32}$$

The constant of motion can be constructed from (3.30a), because dilation invariance frequently (but not always) implies conformal invariance: We write (3.30a) from (3.19b) as

$$\frac{d}{dt}(tD)=\frac{d}{dt}\left[\frac{t^2}{2}H\right]-\frac{1}{4}\int d^2\mathbf{r}(\partial_ir^2)mj^i \ .$$

An integration by parts and the current continuity equation allow presenting the above as

$$0=\frac{d}{dt}\left[\frac{t^2}{2}H-tD-\frac{m}{4}\int d^2\mathbf{r}\,r^2\rho\right] \ .$$

We conclude therefore that the following quantity, involving the electric quadrupole moment, is a constant of motion; indeed, it generates the transformation (3.32):

$$K=-t^2H+2tD+\frac{m}{2}\int d^2\mathbf{r}\,r^2\rho \ . \tag{3.33a}$$

The corresponding density $\mathcal{K}$,

$$\mathcal{K}=t^2\mathcal{H}-t\mathbf{r}\cdot\mathcal{P}+\frac{m}{2}r^2\rho \ , \tag{3.33b}$$

satisfies the continuity equation, again as a consequence of the trace property of T^{ij}:

$$\partial_t\mathcal{K}+\partial_i(t^2T^{i0}-tr^jT^{ij}+\tfrac{1}{2}r^2T^{0i})=0 \ . \tag{3.34}$$

It should be understood that the transformation laws (3.14), (3.18), (3.23), (3.26), (3.29), and (3.32) imply that if $\Psi(t,\mathbf{r})$ solves the field equation (3.2), then so also does the transformed field, given in the right-hand side of these equations. Using this fact allows constructing time-dependent solutions from time-independent ones.

C. Static self-dual systems

According to (3.2), the Hamiltonian (3.1) is stationary on static solutions. They satisfy (3.7), or if the local *Ansatz* (3.11) is made, then (3.12) and (3.13) hold. Rather than solving these equations, we take an indirect approach.

Observe the identity

$$|\mathbf{D}\Psi|^2=|(D_1\pm iD_2)\Psi|^2\pm\frac{m}{\hbar}\nabla\times\mathbf{j}\pm\frac{e}{\hbar c}B\rho \ . \tag{3.35}$$

Therefore, in view of (3.3a), the energy density $\mathcal{H}$ in (3.1b) is

$$\mathcal{H}=\frac{\hbar^2}{2m}|(D_1\pm iD_2)\Psi|^2\pm\frac{\hbar}{2}\nabla\times\mathbf{j}-\left[\frac{g}{2}\pm\frac{\hbar e^2}{2mc\kappa}\right]\rho^2 \ . \tag{3.36}$$

Consequently, with

$$g=\mp\frac{\hbar e^2}{mc\kappa} \ , \tag{3.37}$$

and sufficiently well-behaved fields so that the integral over all space of $\nabla\times\mathbf{j}$ vanishes, the energy is

$$H=\frac{\hbar^2}{2m}\int d^2\mathbf{r}|(D_1\pm iD_2)\Psi|^2 \ . \tag{3.38}$$

This is non-negative and attains its minimum, zero, when Ψ satisfies

$$D_1\Psi=\mp iD_2\Psi \ . \tag{3.39a}$$

The self-dual character of this equation is recognized when it is written as

$$\mathbf{D}\Psi=\mp i\mathbf{D}\times\Psi \ . \tag{3.39b}$$

We shall henceforth make the choice (3.37) for the strength g of the nonlinearity, equivalently for the strength of the magnetic-field–charge-density coupling.[5] As will be indicated below, this is in fact a very natural choice.

To solve (3.39), Ψ is decomposed into its phase and amplitude as in (3.5). Equation (3.39) then implies that the vector potential is given by

$$\mathbf{A}=\nabla\omega\pm\frac{\hbar c}{2e}\nabla\times\ln\rho \ . \tag{3.40}$$

From (3.3a) it now follows that away from the zeros of ρ, $\ln\rho$ satisfies the Liouville equation, all whose solutions are known:

$$\nabla^2\ln\rho=\pm2\frac{e^2}{\hbar c\kappa}\rho \ . \tag{3.41}$$

The matter density ρ must be positive and nonsingular; in particular, there can be no poles in ρ.

The Liouville equation possesses nonsingular, non-negative solutoins for ρ when the numerical constant on the right-hand side of (3.41) is negative. Hence the $\pm$ sign is chosen opposite to κ: For negative κ the upper sign is taken, for positive κ the lower; there is no sign ambiguity: $\mp e^2/\hbar c\kappa$ is always a positive, dimensionless constant $\alpha=e^2/\hbar c|\kappa|$. According to (3.37), this means that the nonlinear coupling g is always positive, $g=\hbar e^2mc|\kappa|$, and therefore according to (3.1b), the self-interaction is attractive, as in the one-dimensional, nonlinear Schrödinger equation. Nevertheless, our Hamiltonian is non-negative [see (3.38)].

From (3.40) we see that in the self-dual system the local *Ansatz* (3.11) for the vector potential charge-density dependence is satisfied with

$$j'(\rho)=\mp\frac{\hbar}{2m} \ . \tag{3.42a}$$

With this choice, the last terms in (3.12) and (3.13) vanish. Equation (3.13) then reproduces (3.41), while Eq. (3.12) also gives (3.41), provided g is taken at the value (3.37) and (3.42a) is integrated to

$$j(\rho)=\mp\frac{\hbar}{2m}\rho \ , \tag{3.42b}$$

$$\mathbf{j}=\mp\frac{\hbar}{2m}\nabla\times\rho \ . \tag{3.42c}$$

Thus the second-order equations (3.7), (3.12), and (3.13) are indeed solved by the self-dual system, which evidently provides a first integral, corresponding to zero energy. Remarkably, the remaining integration can also be performed, owing to the integrability of the Liouville equation.

D. Self-dual symmetry generators

The various symmetry generators of Sec. III B take simplified expressions in the self-dual sector. We have already remarked that the energy density is a total derivative,

$$\mathcal{H}=\pm\left[\frac{\hbar}{2}\right]\nabla\times\mathbf{j}=\left[\frac{\hbar^2}{4m}\right]\nabla^2\rho \ , \tag{3.43}$$

so that for well-behaved ρ the total energy is zero, consistent with (3.38) and (3.39).

The momentum density follows from (3.19b) and (3.42c); it too is a total derivative:

$$\mathcal{P}=\mp\frac{\hbar}{2}\nabla\times\rho \ . \tag{3.44}$$

This integrates to zero, provided ρ is sufficiently well behaved—our zero-energy solitons are at rest.

The angular momentum density is evaluated from (3.24b) and (3.42c):

$$\mathcal{J}=\pm\frac{\hbar}{2}\mathbf{r}\cdot\nabla\rho=\mp\hbar\rho\pm\frac{\hbar}{2}\nabla\cdot(\mathbf{r}\rho) \ . \tag{3.45a}$$

Thus, provided $r^2\rho$ vanishes at infinity, the angular momentum is proportional to $N=\int d^2\mathbf{r}\,\rho$ and also to the charge $Q=eN$, as well as to the flux $\Phi=-(1/\kappa)Q$:

$$J=\mp\hbar N=\mp\frac{\hbar}{e}Q=\pm\frac{\kappa\hbar}{e}\Phi \ . \tag{3.45b}$$

As remarked previously, J is also proportional to the magnetic dipole moment.

The Galileo generator (3.27a) is proportional to the electric dipole moment:

$$\mathbf{G}=-m\int d^2\mathbf{r}\,\mathbf{r}\rho\ . \tag{3.46}$$

The dilation generator (3.30a)

$$D=\frac{\hbar}{4}\int d^2\mathbf{r}\,\mathbf{r}\times\nabla\rho \tag{3.47}$$

vanishes with single-valued ρ.

Finally, the conformal generator (3.33a) is related to the electric quadrupole moment:

$$K=\frac{m}{2}\int d^2\mathbf{r}\,r^2\rho\ . \tag{3.48}$$

The integral converges provided $r^4\rho$ vanishes at infinity.

We have established that the density of energy T^{00} and of momentum T^{0i} are total divergences when evaluated on self-dual solutions [see (3.43) and (3.44)]. The same is true of the remaining components of the energy-momentum tensor, T^{i0} and T^{ij}, (3.17) and (3.21), which are given by manifestly transverse expressions:

$$T^{i0}=\left[-\frac{\hbar^2e^2}{8m^2c\kappa}\right]\epsilon^{ij}\partial_j\rho^2\ ,$$

$$T^{ij}=\left[\frac{\hbar^2}{2m}\right](\delta^{ij}\nabla^2-\partial_i\partial_j)\rho\ .$$

Thus, apart from total derivative terms, the energy-momentum tensor vanishes on a self-dual solution—a circumstance familiar from other self-dual systems.

E. Explicit, self-dual solutions

The matter density that solves (3.41) is

$$\rho(\mathbf{r})=\frac{4}{\alpha}\frac{|f'(z)|^2}{[1+|f(z)|^2]^2}\ ,\qquad z=re^{i\theta}\ , \tag{3.49}$$

where $f(z)$ is an arbitrary function, but so chosen that ρ is well behaved. A natural choice is a superposition of poles and/or zeros.

The matter density ρ vanishes at the zeros of $|f'(z)|^2$ and/or poles of $|f(z)|^2$. There, $\ln\rho$ is singular, and so is $\nabla^2\ln\rho$, which according to (3.40) contributes to the magnetic field. Nevertheless, the complete magnetic field will remain nonsingular, because ω in (3.40) can be chosen to be discontinuous, so that singularities on $\nabla\times\nabla\omega$ cancel those of $\mp(\hbar c/2e)\nabla^2\ln\rho$. However, since the modulus of Ψ is $\rho^{1/2}$ and the phase is ω, discontinuities of ω must be quantized, so that Ψ remains single valued when zeros of $\rho^{1/2}$ are encircled. Below, we show in detail how this works.

Let us examine some explicit solutions. To begin we consider the radially symmetric, positive and nonsingular solution to the Liouville equation. The radial equation is an ordinary differential equation that can be solved by quadrature. The solution depends on two parameters, n and r_0:

$$\rho(\mathbf{r})=\frac{4n^2}{\alpha r^2}\left[\left[\frac{r_0}{r}\right]^n+\left[\frac{r}{r_0}\right]^n\right]^{-2}\ . \tag{3.50}$$

Alternatively, (3.50) is obtained from (3.49) with $f(z)$ taken to be an nth-order zero or pole, $f(z)=c_nz^{\pm n}$, with $|c_n|$ related to r_0^n. Note that (3.50) is even in n, and so we take $n\geq 0$.

The matter density vanishes as $1/r^{2n+2}$ for large r and behaves as r^{2n-2} for small r. Hence ρ is regular at the origin for $n\geq 1$, and vanishes there for $n>1$, producing a singular contribution to the vector potential at $r=0$:

$$A^i=\partial_i\omega\pm\frac{\hbar c}{e}\epsilon^{ij}\frac{\hat{r}^j}{r}\left[n-1-\frac{2n}{1+(r_0/r)^{2n}}\right]$$

$$\underset{r\to 0}{\sim}\partial_i\omega\pm\frac{\hbar c}{e}\epsilon^{ij}\frac{\hat{r}^j}{r}(n-1)\ . \tag{3.51}$$

This singularity is removed when we chose $\omega=\pm(\hbar c/e)(n-1)\theta$, and so the field profile is

$$\Psi(\mathbf{r})=e^{\pm i(n-1)\theta}\frac{2n}{\sqrt{\alpha}r}\left[\left[\frac{r_0}{r}\right]^n+\left[\frac{r}{r_0}\right]^n\right]^{-1}\ . \tag{3.52}$$

We now see that n must be an integer for single-valued Ψ. Moreover, at the origin, Ψ is an analytic or antianalytic function: $\Psi(\mathbf{r})\underset{r\to 0}{\sim}(2n/\sqrt{\alpha})z^{n-1}$ or $(2n/\sqrt{\alpha})(z^*)^{n-1}$.

Integrating ρ on (3.50) over all space evaluates N,

$$N=\left[\frac{hc|\kappa|}{e^2}\right]2n\ . \tag{3.53}$$

Consequently, the configuration carries flux Φ given by an even number $2n$ of flux quanta hc/e. The Galileo generator (3.46) vanishes with this spherically symmetric solution, while the conformal generator (3.48) takes the value

$$K=\frac{\pi hc\kappa m}{e^2}\frac{r_0^2}{\sin(\pi/n)}\ , \tag{3.54}$$

which requires $n>1$.

In general, N is not an integer. But recall that quantized solitons, unlike ordinary particles, are *not* approximate eigenstates of the number operator; rather they are coherent superpositions of an undetermined number of ordinary particles, approximating eigenstates of an operator conjugate to the number operator.[7]

(In a non-Abelian theory, κ is quantized so that $2hc\kappa$ is an integer multiple of e^2.[2,12] Hence, with non-Abelian quantization for κ, N is an integer, but there is no reason for invoking this quantization in the present Abelian context.)

Even though the solution carries $2n$ units of flux, it clearly describes n solitons, all located at the origin—evidently each soliton carries two units of flux. Also, all the solitons are characterized by a common scale r_0—recall our theory is dilation invariant. More generally, the solitons can be located at different points and have different scales. Also, each can carry its own U(1) phase,

although an overall phase in the wave function is immaterial.

With the above counting, one is led to expect that an n-soliton solution depends on $4n$ parameters (two positions, one scale, one phase per soliton) and is described by (3.49) with $f(z)$ taken as

$$f(z)=\sum_{i=1}^{n}\frac{c_i}{z-z_i} . \tag{3.55a}$$

The locations of the poles z_i specify the positions, while the complex residues c_i determine the scales and phases of the solitons. In fact, one may give another formula for $f(z)$, which also leads to an n-soliton solution, but depends on two additional (real) parameters. The expression

$$f(z)=f_0+\sum_{i=1}^{n}\frac{c_i}{z-z_i} \tag{3.55b}$$

is more natural, because unlike (3.55a), it is closed under coordinate inversion $z\to z^{-1}$, which is a symmetry of the Liouville equation:

$$f(z)\to f_0+\sum_{i=1}^{n}\frac{c_i}{z^{-1}-z_i}=\tilde{f}_0+\sum_{i=1}^{n}\frac{\tilde{c}_i}{z-\tilde{z}_i} ,$$

$$\tilde{f}_0=f_0-\sum_{i=1}^{n}\frac{c_i}{z_i},\quad \tilde{c}_i=-\frac{c_i}{z_1^2},\quad \tilde{z}_i=\frac{1}{z_i} .$$

In spite of its richer structure, (3.55b) leads to a soliton profile with the same number of parameters ($4n$) as that arising from (3.55a)—the additional quantity f_0 enters the formula for ρ only to redefine the positions and residues of the poles. This is easily seen for $n=1$ by computing ρ, with $f(z)=f_0+c_1/(z-z_1)$, and finding the resulting matter density to be the same as the one with f_0 absent, but with the residue c_1 replaced by $c_1/(1+|f_0|^2)$ and the location z_1 shifted to $z_1-c_1f_0^*/(1+|f_0|^2)$. For arbitrary n, similar but more complicated redefinitions of the c_i and z_i again remove dependence on f_0.[13] Thus, regardless of which of the expressions in (3.55) is used for f, the solitons depend on $4n$ parameters, with one overall phase parameter being irrelevant.

Supplementing $f(z)$ of (3.55) by contributions involving simple or multiple zeros increases the n "number" of solitons—an effect that can be alternatively achieved by increasing the number of poles. Higher-order poles in $f(z)$ can be obtained as suitable limits of our expression (3.55).

The matter density (3.49) for the n-soliton solution constructed from (3.55b) is

$$\rho(\mathbf{r})=\frac{4}{\alpha}\frac{\left|\sum_{i=1}^{n}[c_i/(z-z_i)^2]\right|^2}{\left[1+\left|f_0+\sum_{i=1}^{n}[c_i/(z-z_i)]\right|^2\right]^2} , \tag{3.56a}$$

or in rationalized form

$$\rho(\mathbf{r})=\frac{4}{\alpha}\frac{|f'(z)V^2(z)|^2}{[|V(z)|^2+|f(z)V(z)|^2]^2} , \tag{3.56b}$$

where $f(z)$ is given by (3.55) and $V(z)=\prod_{i=1}^{n}(z-z_i)$. The denominator of (3.56b) is nonsingular and nonvanishing, but the numerator may possess zeros that give rise to singularities in $\nabla\times\ln\rho$. However, since the logarithm of the numerator is a harmonic function, it can be removed from $\nabla\times\ln\rho$ by a suitable choice of ω in (3.40), so that the vector potential is singularity free. This is achieved with

$$\omega=\pm\frac{\hbar c}{e}\arg f'(z)V^2(z) , \tag{3.57}$$

and with the upper sign the wave-function profile becomes

$$\Psi(\mathbf{r})=\frac{2}{\sqrt{\alpha}}\frac{f'(z)V^2(z)}{|V(z)|^2+|f(z)V(z)|^2} , \tag{3.58}$$

so that Ψ is an analytic function near its zeros; when the lower sign applies, $\Psi(\mathbf{r})$ is given by the complex conjugate of (3.58) and is antianalytic near its zeros.

We can determine $N=\int d^2\mathbf{r}\,\rho$ simply and universally in terms of asymptotic data by employing the Liouville equation to replace ρ by $\nabla^2\ln\rho$ and then using Gauss' law to evaluate the integral. However, before $\nabla^2\ln\rho$ is identified with ρ on the entire plane, the singularities coming from the zeros of ρ must be removed. These zeros are contained in the numerator of (3.56b)—numρ. Since the logarithm of numρ is a harmonic function, $\nabla^2\ln(\text{num}\rho)$ is zero, except at the singularities where a δ function arises. We conclude that removing singularities amounts to ignoring the numerator of (3.56b) and retaining only ρ's denominator. Thus,

$$\rho(\mathbf{r})=\frac{1}{\alpha}\nabla^2\ln d(\mathbf{r}) , \tag{3.59a}$$

$$d(\mathbf{r})=|V(z)|^2+|f(z)V(z)|^2 . \tag{3.59b}$$

For large $\mathbf{r}$, $d(\mathbf{r})$ behaves as

$$d(\mathbf{r})\to(1+|f_0|^2)r^{2n}\left[1-2\,\mathrm{Re}\frac{1}{z}\sum_{i=1}^{n}\left[z_i-\frac{c_if_0^*}{(1+|f_0|)^2}\right]\right] . \tag{3.60}$$

Thus we find, as in the rotationally symmetric special case (3.53),

$$N=\int d^2\mathbf{r}\,\rho=\frac{2\pi}{\alpha}\left[r\frac{\partial}{\partial r}\ln d\right]\Bigg|_{r=\infty}$$

$$=\left[\frac{hc|\kappa|}{e^2}\right]2n , \tag{3.61}$$

which again corresponds to an even number of flux quanta.

The Galileo generator (3.46) becomes

$$\mathbf{G}=-\frac{m}{\alpha}\int_0^{2\pi}d\theta\left[r^ir^2\frac{\partial}{\partial r}\frac{1}{r}\ln d\right]\Bigg|_{r=\infty} . \tag{3.62a}$$

The angular integral picks out the subdominant contribution to d in (3.60). We get

$$\mathbf{G}=-mN\mathbf{R} , \tag{3.62b}$$

where

$$R_x+iR_y=\frac{1}{n}\sum_{i=1}^{n}\left[z_i-\frac{c_i f_0^*}{1+|f_0|^2}\right] . \tag{3.63}$$

This is the center-of-mass coordinate (shifted by f_0).

The conformal generator (3.48) is not expressible solely in terms of asymptotic data.

We conclude this discussion of classical solutions by noting that time-dependent solutions may be obtained from the above static ones by using the symmetries of the problem. Obviously, owing to Galilean invariance, the boosted static solution solves the time-dependent field equation. From (3.26) we have

$$\Psi(t,\mathbf{r})=\exp[i(m/\hbar)(\mathbf{V}\cdot\mathbf{r}-\tfrac{1}{2}V^2t)]\Psi(\mathbf{r}-\mathbf{V}t) . \tag{3.64}$$

Less trivially obvious is the conformal transform (3.32) of the static solution:

$$\Psi(t,\mathbf{r})=\frac{1}{1-at}e^{-imar^2/2\hbar(1-at)}\Psi\left[\frac{\mathbf{r}}{1-at}\right] . \tag{3.65}$$

It remains an open question whether other time-dependent solutions can be constructed, indeed, whether Eq. (3.2), or equivalently (3.6) and (3.7), can be integrated. In this connection it is useful to note that there exists a completely integrable (2+1)-dimensional system described by the Davey-Stewartson equation, which bears some resemblance to our planar, gauged nonlinear Schrödinger equation.[14]

IV. DISCUSSION

A. Nature of the magnetic interaction

We elaborate on the nonminimal magnetic interaction, which we included in our model, and which gives rise to the cubic nonlinearity with strength g in the equation of motion (3.2). Note first that even with $g=0$, the equation possesses cubic nonlinearities, owing to the interaction with $\mathbf{A}$ and A^0, which are functionals of field bilinears. Thus our addition can be viewed as local modification of already present nonlocal terms. This is especially evident in the quantum equation of motion (2.9). As already mentioned there, the additional local kernel $g\delta(\mathbf{r}-\mathbf{r}')$ is a natural partner of the nonlocal kernel $e^4/8\pi^2mc^2\kappa^2|\mathbf{r}-\mathbf{r}'|^2$ that arises from quantum-mechanical reordering.

Not only is the form of the nonminimal interaction "natural," but also the specific value of the coupling that renders the system self-dual, $g=\mp e^2\hbar/mc\kappa$, may be understood in the following manner. Consider a two-component spinor χ and construct

$$S=\sigma^i\left[\partial_i-i\frac{e}{\hbar c}A^i\right]\chi , \tag{4.1}$$

where σ^i are two Pauli matrices $i=1,2$ satisfying $\sigma^i\sigma^j=\delta^{ij}+i\epsilon^{ij}\sigma^3$. It follows that the Hamiltonian density

$$\mathcal{H}=\frac{\hbar^2}{2m}S^\dagger S \tag{4.2a}$$

also equals, apart from a total derivative,

$$\mathcal{H}=\frac{\hbar^2}{2m}|\mathbf{D}\chi|^2-\frac{e\hbar}{2mc}\chi^\dagger\sigma^3\chi B . \tag{4.2b}$$

When χ is taken to be an eigenstate of σ^3, $\chi=\binom{\Psi}{0}$ or $\binom{0}{\Psi}$, the above reduces to (2.8c) with $g\kappa/e=\mp e\hbar/mc$, in agreement with (3.37). Thus we see that the nonminimal magnetic interaction corresponds to the two-dimensional Pauli interaction, which is also known to be supersymmetric.[15,16]

B. Relativistic generalization

Recently, there has been found a relativistic, Abelian Chern-Simons model that leads to self-dual equations for classical configurations.[17] In this model the matter degrees of freedom comprise a relativistic charged scalar field ϕ whose nonlinear self-interactions take a particular form. The matter Lagrange density is

$$\mathcal{L}_{\text{matter}}=\hbar^2|D_\mu\phi|^2-\frac{\hbar^2e^4}{c^4\kappa^2}|\phi|^2(|\phi|^2-v^2)^2 , \tag{4.3}$$

where $D_\mu\phi=\partial_\mu+i(e/\hbar c)A_\mu$ and v^2 is a positive constant.

The interaction potential, which also is the unqiue form that admits an $N=2$ supersymmetry,[18] allows symmetry-breaking ($|\phi|=v$) and symmetric ($\phi=0$) realizations of the U(1) gauge symmetry. The potential describes a theory at the critical point of a first-order transition between the two symmetry phases and, furthermore, supports solitons that are respectively topological[17,19] and non-topological.[19,20] We now show that the nonrelativistic limit of (4.3) with the symmetric realization leads to our nonlinear Schrödinger theory, with nonminimal coupling at the self-dual value (3.37).

Observe first that the quadratic term in a scalar field potential defines the mass through its coefficient, which is m^2c^2. Comparison with (4.3) shows that v^2 should be evaluated as

$$v^2=\frac{mc^3|\kappa|}{\hbar e^2} , \tag{4.4}$$

so that the Lagrange density (4.3) is

$$\mathcal{L}_{\text{matter}}=\frac{\hbar^2}{c^2}\left|\left[\partial_t+i\frac{e}{\hbar}A^0\right]\phi\right|^2-\hbar^2|\mathbf{D}\phi|^2$$
$$-m^2c^2|\phi|^2+\frac{2m\hbar e^2}{c|\kappa|}|\phi|^4-\frac{\hbar^2e^4}{c^4\kappa^2}|\phi|^6 . \tag{4.5}$$

Next, we substitute in (4.5):

$$\phi=\frac{1}{\sqrt{2m}}(e^{-i(mc^2t)/\hbar}\Psi+e^{i(mc^2t/\hbar}\overline{\Psi}^*) . \tag{4.6}$$

All terms that oscillate as $c\to\infty$ are dropped. Keeping dominant inverse powers of c leaves

$$\mathcal{L}_{\text{matter}} \to i\hbar\Psi^\dagger\left(\partial_t + i\frac{e}{\hbar}A^0\right)\Psi + i\hbar\bar{\Psi}^*\left(\partial_t - i\frac{e}{\hbar}A^0\right)\bar{\Psi}^* - \frac{\hbar^2}{2m}\left|\left(\nabla - i\frac{e}{\hbar c}\mathbf{A}\right)\Psi\right|^2 - \frac{\hbar^2}{2m}\left|\left(\nabla + i\frac{e}{\hbar c}\mathbf{A}\right)\bar{\Psi}\right|^2$$
$$+ \frac{\hbar^2 e^2}{2mc|\kappa|}(\rho^2 + 4\rho\bar{\rho} + \bar{\rho}^2) - \frac{\hbar^2 e^4}{8m^3c^4\kappa^2}(\rho^3 + 9\rho^2\bar{\rho} + 9\rho\bar{\rho}^2 + \bar{\rho}^3) \ . \tag{4.7a}$$

Here ρ is the density of particles (charge e) $\rho = \Psi^*\Psi$, while $\bar{\rho}$ correspond to antiparticles (charge $-e$), $\bar{\rho} = \bar{\Psi}^*\bar{\Psi}$. In (4.7a) particles and antiparticles are separately conserved, and so we may work in the zero antiparticle sector by setting $\bar{\Psi} = 0$. This leaves

$$\mathcal{L}_{\text{matter}} \to i\hbar\Psi^*\left(\partial_t + i\frac{e}{\hbar}A^0\right)\Psi - \frac{\hbar^2}{2m}|\mathbf{D}\Psi|^2 + \frac{\hbar e^2}{2mc|\kappa|}\rho^2 - \frac{\hbar^2 e^4}{8m^3c^4\kappa^2}\rho^3 \ . \tag{4.7b}$$

Comparison with (2.8b) and (3.37) shows that we have regained the attractive quartic (ρ^2) self-interaction with precisely the self-dual strength, while the repulsive sixth-order (ρ^3) self-coupling is $\mathcal{O}(c^{-4})$ and may be dropped in the nonrelativistic limit, leaving us with the nonrelativistic Chern-Simons–matter Lagrangian:

$$L = \frac{\kappa}{2c}\int d^2\mathbf{r}\,\partial_t\,\mathbf{A}\times\mathbf{A} + i\hbar\int d^2\mathbf{r}\,\Psi^*\partial_t\Psi - \int d^2\mathbf{r}\,A^0(\kappa B + e\rho) - \int d^2\mathbf{r}\left[\frac{\hbar^2}{2m}|\mathbf{D}\Psi|^2 - \frac{\hbar e^2}{2mc|\kappa|}\rho^2\right] \ . \tag{4.8}$$

The terms containing only gauge fields comprise the Chern-Simons Lagrangian [see (1.5b)].

To accomplish unconstrained quantization of this system, we make use of the recently publicized formalism for quantizing first-order (in time) Lagrangians.[21] It is useful to decompose the gauge potential into longitudinal and transverse components; i.e., we replace the two dynamical variables contained in the two-vector $\mathbf{A}$ by ω and B defined through

$$\mathbf{A}(\mathbf{r}) = \nabla\omega - \nabla\times\int d\mathbf{r}'G(\mathbf{r}-\mathbf{r}')B(\mathbf{r}') \ . \tag{4.9}$$

Then (4.8) becomes, apart from total derivatives,

$$L = -\frac{\kappa}{c}\int d^2\mathbf{r}\,B\dot{\omega} + i\hbar\int d^2\mathbf{r}\,\Psi^*\partial_t\Psi - \int d^2\mathbf{r}\,A^0(\kappa B + e\rho) - \int d^2\mathbf{r}\left[\frac{\hbar^2}{2m}\left|\left(\nabla - i\frac{e}{\hbar c}\nabla\omega - i\frac{e}{\hbar c}\mathbf{A}\right)\Psi\right|^2 - \frac{\hbar e^2}{2mc|\kappa|}\rho^2\right] \ . \tag{4.10a}$$

In (4.10a) the Lagrange multiplier A^0 enforces the Chern-Simons Gauss law. We solve this constraint by setting $B = -(e/\kappa)\rho$ and also redefine the phase of Ψ by $\Psi \to e^{i(e/\hbar c)\omega}\Psi$. This leaves

$$L = i\hbar\int d^2\mathbf{r}\,\Psi^*\partial_t\Psi - \int d^2\mathbf{r}\left[\frac{\hbar^2}{2m}|\mathbf{D}\Psi|^2 - \frac{\hbar e^2}{2mc|\kappa|}\rho^2\right] \ , \tag{4.10b}$$

where now the derivative $\mathbf{D}$ is covariant with respect to the gauge potential

$$\mathbf{A}(\mathbf{r}) = \nabla\times\frac{e}{\kappa}\int d^2\mathbf{r}G(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') \ . \tag{4.11}$$

In (4.10b) we recognize our self-dual Hamiltonian [compare (2.5), (2.8b), and (3.37)].

C. Dynamical role of the Chern-Simons interaction

We have already remarked in Sec. II C above that it is impossible to remove the interaction with the Chern-Simons gauge field, when the matter density ρ is an extended quantity, as in classical field theory or in relativistic quantum field theory, contrary to assertions in the literature that such an interaction can be removed.[9] The existence of regular, static soliton solutions both in the relativistic[17,19,20] and nonrelativistic[5] classical models, vividly supports our assertion. For if the gauge field could be removed by a gauge transformation, our solutions would be (gauge equivalent to) solutions that stationarize a scalar field Hamiltonian in two spatial dimensions. But according to well-known scaling arguments, such stationary configurations do not exist beyond one spatial dimension. Moreover, since in the solution $B \propto \rho$ is extended, $\mathbf{A}$ cannot be a pure gauge.

Thus it is not true that the "effect of the (Chern-Simons) terms is to transmute the statistics of the particles and to do nothing else."[22] On the contrary, the Chern-Simons term supports nonperturbative excitations, whose role in the nonrelativistic and relativistic quantum field theory still needs to be further explored.

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