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# Self-dual Chern-Simons solitons

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Recently discovered self-dual relativistic solitons in an Abelian Chern-Simons theory are discussed in detail. The model simultaneously supports topological and nontopological solitons.

## I. INTRODUCTION

In this paper we study static classical solutions to a relativistic U(1)-invariant gauge theory in three space-time dimensions, where the kinetic action for the gauge field $A_\mu$ is solely the Chern-Simons term.[1] The matter degrees of freedom comprise a self-interacting charged scalar field $\phi$ with a potential $V(\phi)$ that possesses a minimum away from the U(1)-symmetric value $\phi=0$. It was recently shown[2] that with a special choice for $V(\phi)$ the energy obeys a Bogomol'nyi-type[3] lower bound, which is achieved by fields satisfying a set of first-order "self-duality" equations. The topologically stable vortex solutions of these self-duality equations were presented. In this paper we shall examine these vortex solutions in more detail, but in addition we shall study a class of nontopological soliton solutions which are also present.[4]

In three dimensions, any U(1)-invariant renormalizable potential possessing a symmetry-breaking minimum at $|\phi|=v$ can be written in the form[5]

$$V(\phi)=\frac{\alpha e^4}{\kappa^2}(|\phi|^2-v^2)^2[|\phi|^2-\beta(|\phi|^2-v^2)] . \quad (1.1)$$

Here $e$ and $\kappa$, which are introduced in Eq. (1.1) without loss of generality for later convenience, are the gauge coupling and strength of the Chern-Simons term, respectively. (We have set $\hbar$ and $c$ to unity.) When $\beta<\frac{1}{3}$, $V$ acquires a symmetric minimum at $\phi=0$, which becomes lower than the asymmetric minimum if $\beta<0$. The theory possesses two propagating modes. About the symmetric vacuum they are degenerate scalars with mass

$$\mu=\frac{\sqrt{\alpha(1-3\beta)}e^2v^2}{|\kappa|} . \quad (1.2)$$

About the Higgs vacuum there is a scalar with mass

$$m_H=\frac{2\sqrt{\alpha}e^2v^2}{|\kappa|} , \quad (1.3)$$

and a gauge-field excitation that carries mass

$$m_A=\frac{2e^2v^2}{|\kappa|} . \quad (1.4)$$

The Bogomol'nyi limit is obtained by setting $\alpha=1$, $\beta=0$. With this choice of parameters, the scalar and gauge masses in the asymmetric vacuum are equal:

$$m_H=m_A=\frac{2e^2v^2}{|\kappa|}\equiv m , \quad (1.5)$$

while the scalar mass in the symmetric vacuum is $\mu=m/2$. In addition, the symmetric and asymmetric vacua are degenerate, so that one is at a (first-order) transition point between the symmetric and asymmetric phases. This choice of parameters also has the special property that the model is the bosonic part of a theory with an $N=2$ extended supersymmetry.[6]

In the asymmetric phase of the theory, there can be topologically stable vortex solutions; for these, the phase of the scalar $\phi$ varies around a circle at spatial infinity in such a manner that

$$\oint_{r=\infty} d\mathbf{l}\cdot\nabla\ln\phi=2\pi i n , \quad (1.6)$$

with $n$ a topological invariant, which takes only integer values. In order that the energy be finite, the covariant derivative $\mathbf{D}\phi=(\nabla-ie\mathbf{A})\phi$ must vanish asymptotically. This fixes the asymptotic behavior of $\mathbf{A}=(A^1,A^2)=(A_x,A_y)$ and implies a nonvanishing magnetic flux:

$$\Phi=\int d^2r\,\nabla\times\mathbf{A}=\int d^2r\,B=\frac{2\pi n}{e} . \quad (1.7)$$

These vortices are also charged since in the presence of the Chern-Simons term any object carrying magnetic flux must also carry electric charge:[7]

$$Q=-\kappa\Phi . \quad (1.8)$$

There are no topological invariants in the symmetric vacuum phase of the theory. However, there exist nontopological soliton solutions with nonzero flux and charge related by Eq. (1.8). These have a central region in which both $\phi$ and $B$ are nonvanishing. Their stability derives from the fact, to be derived in Sec. II, that magnetic flux is confined to regions of nonzero $\phi$. In contrast with previous examples of nontopological solitons,[8] the mass of these objects is strictly proportional to their charge, with the charge-to-mass ratio being the same as for the elementary excitations. Consequently, there is no upper limit on their charge.

The remainder of the paper is organized as follows. In Sec. II we derive the self-duality (Bogomol'nyi) equations and discuss their general properties. Section III is devoted to rotationally symmetric solutions. In Sec. IV we discuss "domain-wall" solutions with finite energy per unit length and use these to examine further properties of the rotationally invariant solutions in the limit of large flux. In Sec. V index theory methods are used to count the number of self-duality preserving deformations, from which the number of parameters entering the general solution with given flux can be determined. Concluding remarks comprise the final section.

## II. MODEL

Our model is described by the action

$$S=\int d^3x[|D_\mu\phi|^2+\tfrac{1}{4}\kappa\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma}-V(\phi)]\ , \tag{2.1}$$

where $D_\mu\phi=(\partial_\mu+ieA_\mu)\phi$, the Minkowski-space metric tensor $\eta_{\mu\nu}$ is diag$(1,-1,-1)$, and the scalar field potential is obtained by setting $\alpha=1$ and $\beta=0$ in Eq. (1.1); i.e.,

$$V(\phi)=\frac{e^4}{\kappa^2}|\phi|^2(v^2-|\phi|^2)^2\ . \tag{2.2}$$

Variation of this action yields the field equations

$$D_\mu D^\mu\phi=-\frac{\partial V}{\partial\phi^*}\ , \tag{2.3}$$

and

$$\tfrac{1}{2}\kappa\epsilon^{\alpha\beta\gamma}F_{\beta\gamma}=J^\alpha\ , \tag{2.4}$$

where the conserved matter current $J^\mu=(\rho,\mathbf{J})$ is given by

$$J_\mu=ie(\phi^*D_\mu\phi-\phi D_\mu\phi^*)\ . \tag{2.5}$$

The time component of Eq. (2.4),

$$-\kappa B=\rho\ , \tag{2.6}$$

is just the Chern-Simons version of Gauss's law and implies the relation (1.8) between magnetic flux and electric charge. It can also be used to solve for $A_0$, giving

$$A_0=\frac{\kappa}{2e^2}\frac{B}{|\phi|^2}-\frac{1}{e}\partial_0\text{Arg}(\phi)\ . \tag{2.7}$$

The energy can be found from the energy-momentum tensor

$$T_{\mu\nu}=D_\mu\phi^*D_\nu\phi+D_\mu\phi D_\nu\phi^*-g_{\mu\nu}[|D_\lambda\phi|^2-V(\phi)]\ , \tag{2.8}$$

which is obtained by varying the curved-space form of the action with respect to the metric. Integration of the time-time component yields

$$E=\int d^2r[|D_0\phi|^2+|\mathbf{D}\phi|^2+V(\phi)]$$
$$=\int d^2r\left[(\partial_0|\phi|)^2+\frac{\kappa^2B^2}{4e^2|\phi|^2}+|\mathbf{D}\phi|^2+V(\phi)\right]\ , \tag{2.9}$$

where Eq. (2.7) has been used to obtain the second equality.

The term involving $B^2/|\phi|^2$ should be noted; it forces the magnetic field to vanish whenever $\phi$ does. One consequence of this is that the magnetic flux in the vortex solutions lies in a ring, rather than being concentrated at the center as in the Landau-Ginzburg model. This term also provides crucial stabilization for the nontopological solitons of this theory.

Returning to the energy, we note that, with the potential given by Eq. (2.2), Eq. (2.9) can be rewritten as

$$E=\int d^2r\left\{|(D_1\pm iD_2)\phi|^2+\left|\frac{\kappa}{2e}\phi^{-1}B\mp\frac{e^2}{\kappa}\phi^*(v^2-|\phi|^2)\right|^2+(\partial_0|\phi|)^2\right\}$$
$$\pm ev^2\Phi+\frac{1}{2}\oint_{r=\infty}dl\cdot\mathbf{J}\ , \tag{2.10}$$

where the line integral vanishes for any finite-energy solution. Starting with this expression, we can make the usual Bogomol'nyi-type argument. For a fixed value of the flux, there is a lower bound on the energy :

$$E\geq ev^2|\Phi|\ . \tag{2.11}$$

Since static configurations that are stationary points of the energy are also stationary points of the action, the Euler-Lagrange equations of the theory will be satisfied by static configurations obeying the self-duality equations

$$D_1\phi=\mp iD_2\phi\ , \tag{2.12}$$

$$eB=\pm\frac{m^2}{2}\frac{|\phi|^2}{v^2}\left[1-\frac{|\phi|^2}{v^2}\right]\ , \tag{2.13}$$

where the upper (lower) sign corresponds to a positive (negative) value of $\Phi$; these solutions achieve the lower bound in Eq. (2.11).

These equations possess topologically stable vortex solutions for which $|\phi|\to v$ at large distances and $\Phi$ is quantized.[2] But there also exist nontopological soliton solutions for which $\phi\to0$ asymptotically. For these, the flux is not quantized, but rather is an arbitrary parameter describing the solution.[4] This parameter can be continuously varied; therefore, it is evident, since the energy $E=ev^2|\Phi|$, that these solutions are not stationary points of the energy. This is understood by recalling that the energy is stationary, provided that the field variations vanish faster than $1/r$. Such variations are sufficient to establish the Euler-Lagrange equations, and so the solutions of the self-duality equations are indeed solutions of the full field equations. The above-mentioned nonstationary variations have $\delta\mathbf{A}\sim1/r$.

The energy of a nontopological soliton with a given charge $Q$ is

$$E=ev^2|\Phi|=\frac{ev^2}{\kappa}|Q|=\mu Q\ , \tag{2.14}$$

where $\mu$ is the scalar mass in the symmetric vacuum. Thus the energy per unit charge is identical to that of the elementary excitations in the symmetric phase. This indicates that the collective, nontopological excitations are just at the threshold of stability against emission of elementary particles. Consequently, as stated above, stability does not impose an upper bound on the nontopological soliton charges. The topological solitons are of course stable for topological reasons; their flux is quantized.

Equation (2.12) implies that

$$e\bar{A}^i \equiv eA^i - \nabla_i \mathrm{Arg}(\phi) = \pm\epsilon^{ij}\nabla_j \ln|\phi| \ . \tag{2.15}$$

When substituted into Eq. (2.13), this gives

$$\nabla^2 \ln|\phi|^2 = -m^2 \frac{|\phi|^2}{v^2}\left[1-\frac{|\phi|^2}{v^2}\right] . \tag{2.16}$$

(This equation holds only away from the zeros of $\phi$; at these zeros there is an additional $\delta$-function contribution which results when one takes the curl of the gradient of the function Arg$\phi$.) Equation (2.16) does not possess known solutions, though for small $|\phi|$, when the $O(|\phi|^4)$ term may be neglected, it becomes the Liouville equation, all of whose solutions are known. Equation (2.16) is also the Euler-Lagrange equation for the energylike functional

$$\mathcal{E} = \frac{1}{2}\int d^2r \left[(\nabla \ln|\phi|^2)^2 + m^2\left[1-\frac{|\phi|^2}{v^2}\right]^2\right] . \tag{2.17}$$

The angular momentum can be obtained from the momentum density $\mathcal{P}$ via

$$J = \int d^2r\, \mathbf{r}\times\mathcal{P} = -\int d^2r[D_0\phi^* \mathbf{r}\times \mathbf{D}\phi + D_0\phi \mathbf{r}\times(\mathbf{D}\phi)^*] \ . \tag{2.18}$$

For static configurations this reduces to

$$J = \frac{\kappa}{e}\int d^2r\, B[e\mathbf{r}\times \mathbf{A} - \mathbf{r}\times\nabla\, \mathrm{Arg}(\phi)] \ . \tag{2.19}$$

Let us exclude points where $\phi=0$ from the integration surface; since the integrand is nonsingular on this set of measure zero, the result for $J$ is unaffected. Because $\nabla\times\bar{\mathbf{A}} = \nabla\times\mathbf{A}$ away from the zeros of $\phi$, we may write

$$J = \kappa\int d^2r (\nabla\times\bar{\mathbf{A}})\cdot(\mathbf{r}\times\bar{\mathbf{A}}) = \kappa\int d^2r\{\nabla\cdot[\tfrac{1}{2}\mathbf{r}(\bar{\mathbf{A}}^2) - \bar{\mathbf{A}}(\mathbf{r}\cdot\bar{\mathbf{A}})] + (\nabla\cdot\bar{\mathbf{A}})(\mathbf{r}\cdot\bar{\mathbf{A}})\} \ . \tag{2.20}$$

The final term in the last line vanishes because $\bar{\mathbf{A}}$ is transverse, as shown in Eq. (2.15). This gives

$$J = -\kappa\oint dl \times [\tfrac{1}{2}\mathbf{r}(\bar{\mathbf{A}}^2) - \bar{\mathbf{A}}(\mathbf{r}\cdot\bar{\mathbf{A}})] \ , \tag{2.21}$$

where the integral is to be taken both around a circle at spatial infinity and around infinitesimal contours surrounding the excluded points, i.e., the zeros of $\phi$.

An alternative expression for the angular momentum is obtained by substituting Eqs. (2.13) and (2.15) into the first line of Eq. (2.20), producing

$$\begin{aligned} J &= -\frac{2e^2}{\kappa}\int d^2r|\phi|(v^2-|\phi|^2)\mathbf{r}\cdot\nabla|\phi| \\ &= \frac{e^2}{\kappa}\int d^2r\{|\phi|^2(2v^2-|\phi|^2) - \nabla\cdot[\mathbf{r}|\phi|^2(v^2-\tfrac{1}{2}|\phi|^2)]\} \\ &= -\frac{e^2}{\kappa}\int d^2r\{(v^2-|\phi|^2)^2 - \tfrac{1}{2}\nabla\cdot[\mathbf{r}(v^2-|\phi|^2)^2]\} \ . \end{aligned} \tag{2.22}$$

The divergence terms in the last two equalities lead to boundary terms which vanish for $\phi(\infty)=0$ and $\phi(\infty)=v$, respectively, and so

$$J = \begin{cases} \dfrac{e^2}{\kappa}\displaystyle\int d^2r|\phi|^2(2v^2-|\phi|^2), & \phi(\infty)=0 \ , \\ -\dfrac{e^2}{\kappa}\displaystyle\int d^2r(v^2-|\phi|^2)^2, & \phi(\infty)=v \ . \end{cases} \tag{2.23}$$

## III. ROTATIONALLY SYMMETRIC SOLUTIONS

We now specialize to rotationally symmetric solutions. With the aid of a gauge transformation, any static rotationally symmetric configuration of vorticity $n$ can be brought into the form

$$\phi = vg(r)e^{in\theta} \ , \qquad eA^i = \epsilon^{ij}\frac{\hat{\mathbf{r}}^j}{r}[a(r)-n] \ , \tag{3.1}$$

with $g(r)$ real. Substitution of this ansatz into the expression (2.9) for the energy gives

$$E = 2\pi v^2\int_0^\infty dr\, r\left[(g')^2 + \left[\frac{ag}{r}\right]^2 + \left[\frac{a'}{mrg}\right]^2 + \tfrac{1}{4}m^2g^2(1-g^2)^2\right] , \tag{3.2}$$

where primes denote differentiation with respect to $r$.

The self-duality equations (2.12) and (2.13) become

$$g' = \pm\frac{ag}{r} \ , \tag{3.3}$$

and

$$\frac{a'}{r} = \pm\frac{m^2}{2}g^2(g^2-1) \ . \tag{3.4}$$

The boundary conditions at the origin follow from the requirement that the fields be nonsingular. This implies that $a(0)=n$ and that $ng(0)$ vanish. At spatial infinity, finiteness of the energy implies that $g(\infty)$ be either 0 or 1, and that $a(\infty)g(\infty)$ vanish. These requirements leave $g(0)$ undetermined if $n=0$, and $a(\infty)\equiv -\alpha$ undetermined if $g(\infty)=0$.

Equations (3.3) and (3.4) may be combined to give

$$(\ln g^2)'' + \frac{1}{r}(\ln g^2)' = -m^2g^2(1-g^2) \ . \tag{3.5}$$

This is just the restriction of Eq. (2.16) to our rotationally symmetric *Ansatz*. For small $g$, the $O(g^4)$ term may be ignored and this equation reduces to the rotationally symmetric form of Liouville's equation, whose solution can be found; it is

$$g(r)=\frac{\sqrt{8}N}{mr}\left[\left[\frac{r}{r_0}\right]^N+\left[\frac{r_0}{r}\right]^N\right]^{-1}, \tag{3.6}$$

where $r_0$ and $N$ are arbitrary constants.

With our *Ansatz*, the magnetic field is $B=-a'/(er)$, and so the magnetic flux and electric charge are

$$\Phi=-\frac{Q}{\kappa}=\frac{2\pi}{e}[a(0)-a(\infty)]=\frac{2\pi}{e}(n+\alpha) . \tag{3.7}$$

From the expression (2.21) for the angular momentum, we obtain

$$\begin{aligned} J&=\frac{\pi\kappa}{e^2}[a(\infty)^2-a(0)^2] \\ &=\frac{\pi\kappa}{e^2}(\alpha^2-n^2) \\ &=\frac{1}{4\pi\kappa}Q^2+\frac{1}{e}nQ \\ &=-\frac{1}{4\pi\kappa}Q^2-\frac{1}{e}\alpha Q . \end{aligned} \tag{3.8}$$

In examining solutions of Eqs. (3.3) and (3.4), we shall consider only those with $\Phi>0$, i.e., those corresponding to the upper choice of sign. Solutions with negative flux are related to these by the transformation $g\to g$, $a\to -a$. It is convenient to consider separately those solutions that approach the asymmetric and symmetric vacuums at spatial infinity, and those with nonzero vorticity separately from those with $n=0$. We thus have four categories.

### A. $g(\infty)=1$, $n=0$

This includes the vacuum solution $g(r)\equiv 1$, $a(r)\equiv 0$. To see that this is the only solution, note first that the boundary conditions on the gauge field are $a(0)=a(\infty)=0$. Now consider a continuous family of configurations $a(r)=\lambda f_1(r)$, $g(r)=f_2(r)$, which obeys these boundary conditions. Since the energy is clearly a monotonically increasing function of $\lambda^2$, any stationary point must have $a(r)$ identically zero. Equation (3.3) then implies that $g(r)$ must be a constant, leaving the vacuum solution as the only possible solution.

### B. $g(\infty)=1$, $n\neq 0$

These are the vortex solutions discussed in Ref. 2. They are topologically nontrivial and hence cannot be continuously deformed to the vacuum solution. The boundary conditions are $g(0)=0$, $a(0)=n$, and $a(\infty)=0$. At large distances, the fields approach their asymptotic values exponentially. Specifically,

$$\begin{aligned} g(r)&=1-\gamma K_0(mr) , \\ a(r)&=\gamma mrK_1(mr) , \end{aligned} \tag{3.9}$$

where $\gamma$ is a constant that is not determined by the behavior at infinity, but rather by the requirement of proper behavior at the origin. Near the origin a power-series solution gives

$$\begin{aligned} g(r)&=G_n(mr)^n-\frac{G_n^3}{8(n+1)^2}(mr)^{3n+2} \\ &\quad+\frac{G_n^5}{8(2n+1)^2}(mr)^{5n+2}+O((mr)^{5n+4}) , \\ a(r)&=n-\frac{G_n^2}{4(n+1)}(mr)^{2n+2}+\frac{G_n^4}{4(2n+1)}(mr)^{4n+2} \\ &\quad+O((mr)^{4n+4}) . \end{aligned} \tag{3.10}$$

[The first two terms can be obtained directly from the Liouville approximation (3.6).] The constant $G_n$ is not determined by the behavior of the fields near the origin, but is instead fixed by requiring proper behavior as $r\to\infty$. If $G_n$ is chosen too large, $g$ reaches unity at some finite value $r_1$, with $a(r_1)>0$; for all $r>r_1$, both $g'$ and $a'$ are positive, and $g$ and $a$ both grow without bound. If $G_n$ is chosen too small, $a$ becomes negative at some value $r_2$ while $g$ is still less than unity. For all $r>r_2$, both $g'$ and $a'$ are negative, and so the boundary condition $g(\infty)=1$ clearly cannot be met; instead, asymptotically $g\to 0$, while $a$ tends to a negative constant. (These solutions are discussed further in Sec. III D, below.) The value $G_n^{\rm cr}$ separating these two regimes gives the vortex solutions.

In Fig. 1 we plot $g(r)$ and $a(r)$ for the $n=1$ solution. The value of $G_n^{\rm cr}$ as a function of $n$ is plotted in Fig. 2; in this figure the solid line indicates the prediction for the large-$n$ behavior of $G_n^{\rm cr}$ which we obtain in the next section.

### C. $g(\infty)=0$, $n=0$

For this case the Higgs field approaches the symmetric minimum at large distances, and so all configurations are topologically trivial. These are the nontopological solitons. They are characterized by the value of the magnetic flux $\Phi=-2\pi a(\infty)/e=2\pi\alpha/e$, which need not be quantized. Equation (3.3) implies that at large distances

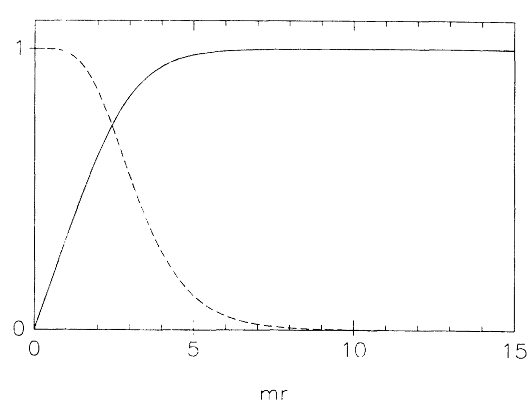


FIG. 1. Plot of $g(r)$ (solid line) and $a(r)$ (dashed line) for the $n=1$ topological vortex solution.

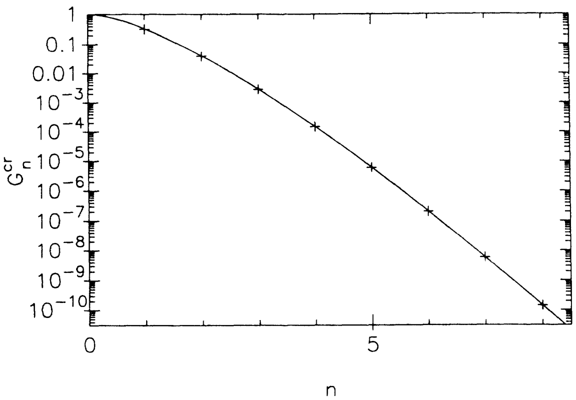


FIG. 2. Behavior of $G_n^{cr}$ as a function of $n$. The solid line corresponds to the asymptotic formula (4.18) with $b(n)=1$, i.e., $G_n^{cr}=(2n+1)^{-n}$, while the crosses represent actual data.

the Higgs field must behave as $g(r)\sim(mr)^{-\alpha}$. More precisely, we find

$$g(r)=\frac{C}{(mr)^{\alpha}}-\frac{C^3}{8(\alpha-1)^2(mr)^{3\alpha-2}}+O((mr)^{-5\alpha+4}) ,$$

$$a(r)=-\alpha+\frac{C^2}{4(\alpha-1)(mr)^{2\alpha-2}}+O((mr)^{-4\alpha+4}) . \tag{3.11}$$

Again, the first few terms in this expansion can be read off directly from the Liouville approximation.

We next consider the behavior near the origin. While $a(0)$ must vanish in order that the solution be nonsingular, $g(0)=G_0$ is not so constrained. Since the sign change $g(r)\to -g(r)$ can be implemented by a gauge transformation, we need only consider the case of positive $G_0$. To find the behavior near the origin, we attempt a power-series solution, obtaining

$$g(r)=G_0-\tfrac{1}{8}G_0^3(1-G_0^2)(mr)^2+\tfrac{1}{128}G_0^5(1-G_0^2)(2-3G_0^2)(mr)^4+O((mr)^6) ,$$

$$a(r)=-\tfrac{1}{4}G_0^2(1-G_0^2)(mr)^2+\tfrac{1}{32}G_0^{4a}(1-G_0^2)(1-2G_0^2)(mr)^4+O((mr)^6) . \tag{3.12}$$

These solutions will not in general have acceptable large distance behavior. In particular, it is clear from Eqs. (3.3) and (3.4) that if $G_0>1$, both $g$ and $a$ will be monotonically increasing functions of $r$ and, hence, that the boundary condition at $r=\infty$ cannot be met. The choices $G_0=0$ and 1 lead to the constant symmetric and asymmetric vacuum solutions, respectively. Thus the nontrivial solutions correspond to values in the range $0<G_0<1$. It is of interest to determine the correspondence between the short- and large-distance behavior of the solution, i.e., between $G_0$ and $\alpha$.

For small $G_0$ the Liouville approximation can be used near the origin. To match the boundary conditions at the origin, the constant $N$ in Eq. (3.6) should be set equal to 1, while $G_0=\sqrt{8}/(mr_0)$. This gives

$$g(r)=\frac{G_0}{1+(m^2/8)G_0^2r^2} , \tag{3.13}$$

which is in fact small, and therefore a good approximation, for all $r$. Substituting this approximation into Eq. (3.3) gives

$$a(r)=-\frac{2r^2}{r^2+8/m^2G_0^2} . \tag{3.14}$$

In the limit $r_0\to\infty$ this approximate solution should become exact, and so the solution corresponding to the limit $G_0\to 0$ will have $\alpha=2$ and flux $\Phi=4\pi/e$. By numerically integrating the field equations for nonvanishing values of $G_0$, we find that this is in fact the lower bound on the flux of these nontopological solitons. As shown in Fig. 3, the flux increases with $G_0$ and diverges as $G_0$ approaches unity; we shall derive this behavior in the next section.

An example of a nontopological soliton is shown in Fig. 4.

### D. $g(\infty)=0$, $n\neq 0$

These solutions are hybrids of the two previous cases. Their large-distance behavior is the same as that of the nontopological solitons [Eq. (3.11)]. At short distances they resemble the vortex solutions, behaving as in Eq. (3.10), but with values of $G_n$ less than the critical value $G_n^{cr}$ needed to give the vortex. They may be interpreted as nontopological solitons with vortices embedded at their origin; we shall call them nontopological vortices.

For each integer $n$ there will be a continuous set of solutions corresponding to the range $0<G_n<G_n^{cr}$. For $G_n\ll 1$, $g(r)$ is small for all $r$ and can therefore be approximated by the solution (3.6) of Liouville's equation. In this case, matching of the boundary conditions requires $N=n+1$ and $G_n=\sqrt{8}(n+1)/(mr_0)^{n+1}$. As $G_n\to 0$ ($r_0\to\infty$) the solution becomes exact; from its large-$r$ behavior we obtain the value $\alpha=a(\infty)=n+2$ and the flux $\Phi=4(n+1)\pi/e$. The flux increases with $G_n$, tending to $\infty$ as $G_n\to G_n^{cr}$. An example of this type of soliton is shown in Fig. 5.

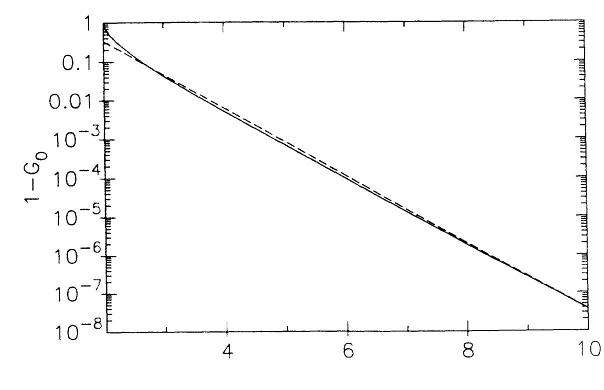


FIG. 3. Behavior of $G_0$ as a function of $\alpha$. The solid curve is actual data, while the dashed line represents the asymptotic formula (4.20) and $d(\alpha)$ set equal to 2.90.

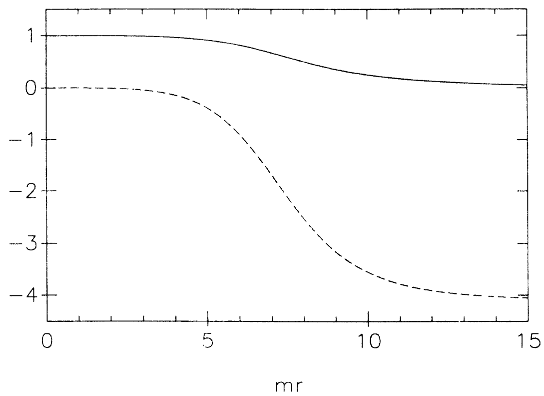


FIG. 4. Plot of $g(r)$ (solid line) and $a(r)$ (dashed line) for a nontopological soliton with $\alpha=4.10$.

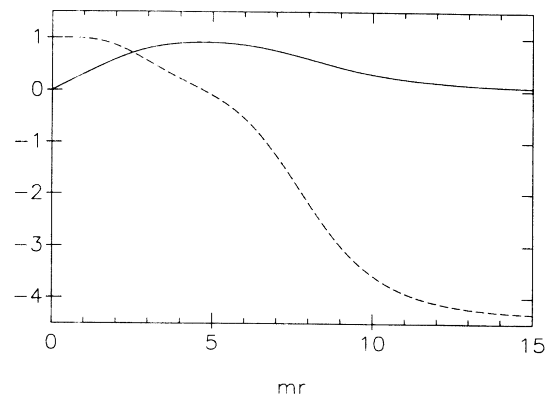


FIG. 5. Plot of $g(r)$ (solid line) and $a(r)$ (dashed line) for a nontopological vortex with $n=1$ and $\alpha=4.37$.

In conclusion, we emphasize that the limiting values of $\alpha$ are in fact bounds; the actual solutions for the nontopological solitons and vortices have $\alpha>n+2$. Note also that there is a correspondence between the limit of small $g$ and the nonrelativistic limit. Indeed, the solitons and nontopological vortices in the nonrelativistic version of our model have been found. They satisfy the Liouville equation and can be explicitly constructed.[9]

## IV. DOMAIN WALLS AND THE LARGE FLUX LIMIT

Whenever a potential possesses two disconnected, but degenerate, vacua, one expects to find a static classical solution which may be interpreted as an infinitely long boundary separating regions lying in the two vacua. Such a solution does indeed exist in the model we study in this paper; we shall refer to it as a domain wall, although it is a one-dimensional structure. In fact, because the boundary can contain an arbitrary amount of flux per unit length, there is a continuous family of domain-wall solutions.

Whereas these solutions separate two infinite regions, one can also consider the possibility of a finite domain of one vacuum completely surrounded by the other vacuum. If the radius of curvature of the domain boundary is large enough, the fields in the region of the boundary should approximate those near an infinite domain wall. In general, this boundary will not be stationary, but will instead tend to shrink. We shall find, however, that a critical amount of flux can stabilize such a boundary against collapse. These stabilized domains turn out in fact to be simply topological and nontopological solitons with large values for the flux.

To begin, let us seek a solution corresponding to an infinite domain wall parallel to the $y$ axis, with the scalar field interpolating between the asymmetric vacuum at large negative $x$ and the symmetric vacuum at large positive $x$. By an appropriate gauge transformation, $\phi=vg$ can be made real everywhere. The translational invariance of the theory then implies that $g$, $A_x$, and $A_y$ depend only on $x$. With primes denoting differentiation with respect to $x$, the energy can be written as

$$E=v^2\int d^2r\left[(g')^2+\frac{m^2}{4}g^2(1-g^2)^2+e^2(A_x^2+A_y^2)g^2+\frac{e^2}{m^2}\frac{(A_y')^2}{g^2}\right]$$
$$=v^2\int d^2r\left[\left[g'\pm\frac{m}{2}g(1-g^2)\right]^2+\left[eA_yg\mp\frac{eA_y'}{mg}\right]^2+e^2A_x^2g^2\pm\left[\frac{m}{4}(1-g^2)^2\right]'\pm\frac{e^2}{m}(A_y^2)'\right] . \tag{4.1}$$

The assumed boundary conditions on the scalar field imply $g(-\infty)=1$ and $g(\infty)=0$. To have finite energy per unit wave length, the gauge field must vanish in the asymmetric vacuum, and so $A_y(-\infty)=0$. $A_y(\infty)\equiv f$ can be chosen arbitrarily, and is equal to the magnetic flux per unit length of domain wall.

A static solution can be obtained by minimizing the energy per unit wall length, with $f$ held fixed. [Because $f$ can be varied continuously, these solutions are not stationary points of the energy; remarks similar to those following Eq. (2.13) apply.] Taking the upper signs in Eq. (4.1) for our choice of the boundary condition on $g(x)$, we find the minimum energy per unit length to be

$$\mathcal{E}=mv^2\left[\frac{1}{4}+\frac{e^2}{m^2}f^2\right] , \tag{4.2}$$

which is obtained if $A_x=0$,

$$g'=-\frac{m}{2}g(1-g^2) , \tag{4.3}$$

and

$$A_y'=mA_yg^2 . \tag{4.4}$$

Integrating these gives

$$g=(1+e^{m(x-X)})^{-1/2} , \tag{4.5}$$

$$A_y = \frac{f}{1+e^{-m(x-X)}} . \tag{4.6}$$

Here $X$ is a constant of integration determined by the position of the domain wall, which has a thickness of order $m^{-1}$. Note that the magnetic field

$$B = \frac{mf}{4}\operatorname{sech}^2 m\left[\frac{x-X}{2}\right] \tag{4.7}$$

is concentrated near $x=X$ and falls off rapidly away from the domain wall.

There is a linear momentum flow along the domain wall. The nonzero component of linear momentum density along the domain wall is

$$T^{0y} = -(D_0\phi^* D_y\phi + D_0\phi D_y\phi^*) = \frac{\kappa}{2}(A_y^2)' , \tag{4.8}$$

where Gauss's law has been used. The linear momentum density per unit length along the domain wall is then

$$\mathcal{P} = \frac{\kappa}{2} f^2 . \tag{4.9}$$

Although the structure of Eq. (4.1) is similar to that used to obtain the Bogomol'nyi bound for our solitons, Eqs. (4.3) and (4.4) are not equivalent to the self-duality equations (2.12) and (2.13). This can be easily seen by noting that the equation governing the scalar field does not involve the gauge potential, and that the solution for the scalar field does not depend on the total flux. In fact, the scalar field equation (4.3) is identical to the finite-energy soliton equation[10] in a $(1+1)$-dimensional scalar field theory with potential (2.2). The two pairs of Eqs. (2.12), (2.13) and (4.3), (4.4) are equivalent for $|f| = m/(2e)$.

We now turn to the case of a large but finite domain. Specifically, we consider a circular region of asymmetric vacuum of radius $R \gg m^{-1}$, with a magnetic flux $\Phi$ uniformly distributed along the domain boundary. Near the domain boundary, the fields should be well approximated by the infinite domain-wall solution with $f = \Phi/(2\pi R)$. Thus, for $r \approx R$, we expect

$$\begin{aligned} \phi &\approx v(1+e^{m(r-R)})^{-1/2} , \\ A^i &\approx -\epsilon^{ij}\hat{r}^j \frac{f}{1+e^{-m(r-R)}} . \end{aligned} \tag{4.10}$$

Away from the domain boundary, $\phi$ will be close to one or the other of its vacuum values, while $\mathbf{A}$ should be essentially a pure gauge.

The energy of such a configuration is concentrated near the domain boundary. From Eq. (4.2) we expect the energy to be approximately

$$E \approx 2\pi m R v^2\left[\frac{1}{4} + \left[\frac{e\Phi}{2\pi m R}\right]^2\right] . \tag{4.11}$$

The domain boundary will be stable against contraction or expansion when the energy is minimized for a given flux. Minimizing the energy as a function of the radius, we find

$$R \approx 2m^{-1}\left[\frac{e\Phi}{2\pi}\right] , \tag{4.12}$$

and

$$E \approx 2\pi v^2\left[\frac{e\Phi}{2\pi}\right] . \tag{4.13}$$

This value for the energy saturates the lower bound (2.11), indicating that the fields must be solutions of the self-duality equations. Indeed, the configuration we have described satisfies the self-duality equations near $r \approx R$ and is nothing but a nontopological soliton of large flux, $\Phi = 2\pi\alpha/e$, with $R \approx 2\alpha m^{-1}$. (Recall that for $\alpha \gg 2$, the value of $\phi$ at the center of a nontopological solition is very close to $v$.)

A topological vortex solution of flux $\Phi = 2\pi n/e$, with $n \gg 1$, can be obtained in a similar fashion by considering a large circular domain of symmetric vacuum of radius $R$ with a boundary region of width $\sim m^{-1}$. In order that the phase of $\phi$ vary uniformly with angle, we gauge transform the domain-wall solution, obtaining

$$\begin{aligned} \phi &\approx v e^{in\theta}(1+e^{m(R-r)})^{-1/2} , \\ A^i &\approx \epsilon^{ij}\hat{r}^j\left[\frac{f}{1+e^{m(r-R)}} - \frac{n}{eR}\right] , \end{aligned} \tag{4.14}$$

with $f = \Phi/(2\pi R) = n/(eR)$.

Following the same procedure as above, one finds that the minimum energy for a given flux is obtained when $R \approx 2nm^{-1}$.

This approach can be pursued further. For a rotationally symmetric solution, the magnitude of the scalar field obeys Eq. (3.5). Approximations leading to analytic solutions can be found in three regimes.

(a) When $g \ll 1$, the $g^4$ term in Eq. (3.5) can be neglected. This leads to the Liouville solution Eq. (3.6). [When $g \gg 1$, the $g^2$ term can be neglected, and Eq. (3.5) becomes the Liouville equation of opposite sign, whose solutions are unbounded and physically uninteresting.]

(b) When $1-g \ll 1$, linearization of Eq. (3.5) gives Bessel's equation, with the result that $1-g$ is a linear combination of the Bessel functions $I_0(mr)$ and $K_0(mr)$.

(c) If $(\ln g)'/r \ll m^2 g^2(1-g^2)$, the first-derivative term in Eq. (3.5) can be neglected. One integration then gives

$$g' = \mp \frac{m}{2} g[(1-g^2)^2 + k]^{1/2} , \tag{4.15}$$

where $k$ is a constant of integration. When $k=0$, this reduces to Eq. (4.3) and gives the domain-wall solutions (4.10) and (4.14).

Now consider a topological vortex solution with vorticity $n \gg 1$. Near the origin, $g \ll 1$ and the Liouville approximation can be used. At large $r$, the approximation (4.15) can be used. The boundary conditions at infinity require that $k=0$, and so $g$ can be approximated by the domain-wall solution (4.14). This solution can be extended to smaller $r$, provided that $g'/(m^2 r g^3) \approx 1/(mrg^2)$ remains small. From the previous discussion we know that transition from $g \approx 0$ to $g \approx 1$ occurs at $r \approx 2nm^{-1}$.

We therefore expect there to be a region, corresponding to values of $g$ in the range $n^{-1/2} \ll g \ll 1$, where both of these approximations should be valid. If we write $r = R - (\delta/m)$, then the domain-wall approximation (4.14) gives, at small $r$,

$$\begin{aligned} g &= (1+e^{\delta})^{-1/2} + \cdots \\ &= e^{-\delta/2} + \cdots , \end{aligned} \tag{4.16}$$

where the ellipsis in the first line indicates the corrections due to the omitted $(\ln g)'/r$ term, and the second line approximates the first for $r \ll R$. In this region the Liouville approximation gives, from Eq. (3.6) with $N$ replaced by $n+1$,

$$\begin{aligned} g &= \frac{\sqrt{8}(n+1)}{mR-\delta} \left[ \left[ \frac{mR-\delta}{mr_0} \right]^{n+1} + \left[ \frac{mr_0}{mR-\delta} \right]^{n+1} \right]^{-1} + \cdots \\ &= \sqrt{2} \left[ \frac{R}{r_0} \right]^{n+1} e^{-\delta/2} + \cdots . \end{aligned} \tag{4.17}$$

The second line is obtained by noting that the second term in the square brackets is dominant in the region of interest. Again, the ellipsis in the first line indicates corrections due to omitted terms in the differential equation, while the second line is valid in the limit of large $n$. By comparing the leading terms in the two expressions, we see that $(mr_0)^{n+1} \approx \sqrt{2}(mR)^{n+1} \approx \sqrt{2}(2n)^{n+1}$, and hence that the constant $G_n^{\rm cr}$ in the short-distance expansion (3.10) of the vortex solution must be

$$\begin{aligned} G_n^{\rm cr} &= \frac{\sqrt{8}(n+1)}{(mr_0)^{n+1}} \\ &= \frac{1}{[2n+b(n)]^n} , \end{aligned} \tag{4.18}$$

where $b(n)/n \to 0$ as $n$ tends to infinity. The nonleading behavior at large $n$ can be found by finding the corrections to the domain-wall and Liouville approximations. As shown in Fig. 2, the actual value of $G_n^{\rm cr}$ is fit rather well by Eq. (4.18) with $b(n)=1$. For $n$ ranging from 1 to 8, we find that $(G_n^{\rm cr})^{-1/n}$ is given by 3.09, 5.07, 7.05, 9.04, 11.04, 13.03, 15.02, and 17.01.

A similar procedure can be applied to the nontopological solitons with $\alpha \gg 1$. In this case one must use all three approximations: the Bessel-function solution near the origin, the Liouville solution at large distances, and the domain-wall solution in between. The constant $k$ in Eq. (4.15) does not vanish, although it tends to zero in the large-$\alpha$ limit. One finds that the constant $C$ in the large-distance expansion (3.11) is

$$C = [2\alpha + c(\alpha)]^{\alpha} , \tag{4.19}$$

while the value of the scalar field at the origin is given by

$$1 - G_0 = e^{-2\alpha + d(\alpha)} . \tag{4.20}$$

Both $c(\alpha)/\alpha$ and $d(\alpha)/\alpha$ vanish in the limit $\alpha \to \infty$. Equation (4.20) should be compared with the data shown in Fig. 3; the dashed line in that graph corresponds to setting $d(\alpha)$ equal to a constant, 2.90.

Finally, let us return to the angular momentum. In the large flux limit, the linear momentum density is nonzero only along the domain wall. With the linear momentum density (4.9) and radius (4.12) of the soliton, the angular momentum is

$$J = 2\pi R^2 \mathcal{P} = \frac{\pi\kappa}{e^2} \alpha^2 . \tag{4.21}$$

This is what we got in Eq. (3.8).

## V. MULTISOLITON SOLUTIONS AND PARAMETER COUNTING

The rotationally symmetric solutions are only special cases; the set of classical solutions is in fact much richer. For fixed values of the vorticity and flux, we expect the general solution to depend on a number of continuously variable parameters. Furthermore, these parameters may have a natural interpretation in terms of a multisoliton description of the solution.

We shall count these parameters, using methods similar to those which have been used in other self-dual systems.[11,12] The basic idea is to note that the variation of a parameter gives an infinitesimal fluctuation of the fields which preserves the self-duality equations. Such modes lie in the kernel of a matrix linear differential operator $\mathcal{D}$, from the dimension of which one can determine the number of independent parameters. This dimension is determined in two steps. One first derives an index theorem which expressed the index

$$\begin{aligned} \mathcal{I}(\mathcal{D}) &= \dim(\mathrm{kernel}\mathcal{D}) - \dim(\mathrm{kernel}\mathcal{D}^*) \\ &= \dim(\mathrm{kernel}\mathcal{D}^*\mathcal{D}) - \dim(\mathrm{kernel}\mathcal{D}\mathcal{D}^*) , \end{aligned} \tag{5.1}$$

in terms of spatial integrals involving the background fields. The second step is to prove a vanishing theorem stating that the adjoint operator $\mathcal{D}^*$ has vanishing kernel, so that the index is in fact equal to the dimension of the kernel of $\mathcal{D}$.

Although we follow rather closely the analysis[12] used to study Landau-Ginzburg vortices, we encounter two difficulties which do not arise there. First, the vanishing theorem fails whenever a certain Schrödinger equation, involving the Higgs field of the self-dual solution, has a zero-energy bound state; we believe that this occurs only for exceptional field configurations and does not affect the counting of parameters. Second, the formula we obtain for the index does not necessarily yield an integer when the background field is a nontopological soliton, whereas Eq. (5.1) clearly requires the index to be integral. The problem can be traced to the fact that the operators we deal with act on an open infinite space; they have a continuous spectrum in addition to the discrete set of eigenvalues.[13,14] For the topological vortex case, the continuum is separated by a finite gap from the zero eigenvalues and causes no problem. However, when $\phi \to 0$ at spatial infinity, the continuum extends to zero and affects the calculation of the index. Fortunately, we are able to adapt the analysis of a related problem to determine the continuum contribution.[14] Subtracting this from the result of our calculation yields an integer. (Even when the result for the index is an integer, this continuum contribution may be nonzero, although integral.) As a check of this answer, we then explicitly study the zero-mode equations in the background of a rotationally symmetric solution and show that the predicted number of zero modes emerges.

Thus, let us suppose that we are given an arbitrary self-dual solution with fields $\mathbf{A}$ and $\phi=\phi_1+i\phi_2$. Infinitesimal fluctuations which preserve self-duality satisfy

$$(D_1+iD_2)\delta\phi - ie\phi(\delta A^1+i\delta A^2)=0 , \tag{5.2}$$

and

$$e\nabla\times\delta\mathbf{A}=\frac{m^2}{v^4}(v^2-2|\phi|^2)|\phi|\delta|\phi| . \tag{5.3}$$

Many of these are simply gauge transformations; they are of no interest and can be eliminated by imposing a gauge condition. A particularly convenient choice is the background gauge condition

$$e\nabla\cdot\delta\mathbf{A}+\frac{i}{2}(\phi^*\delta\phi-\phi\delta\phi^*)=0 , \tag{5.4}$$

which is equivalent to requiring that the fluctuation be orthogonal to all gauge transformations whose gauge parameter vanishes at spatial infinity. When $|\phi|\to v$, those gauge transformations whose parameter does not vanish asymptotically are also excluded, since the corresponding $\delta\phi$ does not vanish at spatial infinity. If instead $\phi$ approaches the symmetric vacuum, there is one surviving gauge mode with asymptotically nonvanishing parameter which is not excluded by Eq. (5.4). This mode, which is analogous to the global gauge modes of the multi-instanton solutions and the global U(1) mode of the multimonopole solution,[11] must be explicitly subtracted at the end of the calculation.

These equations can be summarized by a single matrix equation

$$0=\mathcal{D}\eta , \tag{5.5}$$

where

$$\mathcal{D}=\begin{bmatrix} \nabla_1+eA^2 & -\nabla_2+eA^1 & \phi_2 & \phi_1 \\ \nabla_2-eA^1 & \nabla_1+eA^2 & -\phi_1 & \phi_2 \\ \phi_1 U & \phi_2 U & \nabla_2 & -\nabla_1 \\ \phi_2 & -\phi_1 & \nabla_1 & \nabla_2 \end{bmatrix} , \tag{5.6}$$

$U=2(m^2/v^4)(v^2-2|\phi|^2)$, and

$$\eta=\begin{bmatrix} \delta\phi_1 \\ \delta\phi_2 \\ e\delta A^1 \\ e\delta A^2 \end{bmatrix} . \tag{5.7}$$

In order to evaluate the index of $\mathcal{D}$, let us define

$$\hat{\mathcal{I}}(\mathcal{D})=\lim_{M^2\to\infty}\left[\mathrm{Tr}\left[\frac{M^2}{\mathcal{D}^*\mathcal{D}+M^2}\right]-\mathrm{Tr}\left[\frac{M^2}{\mathcal{D}\mathcal{D}^*+M^2}\right]\right] , \tag{5.8}$$

where Tr denote a functional trace. Now observe that if $\eta$ is an eigenfunction of $\mathcal{D}^*\mathcal{D}$ with nonzero eigenvalue, then $\mathcal{D}\eta$ is an eigenfunction of $\mathcal{D}\mathcal{D}^*$ with the same eigenvalue. By evaluating Eq. (5.8) in a basis corresponding to the eigenfunctions of these operators, it would seem that the contribution from nonzero eigenvalues would cancel between the two terms, and that $\hat{\mathcal{I}}(\mathcal{D})$ would be in fact just the index we seek.

If there were only a discrete spectrum, there would be no difficulty in carrying out this one-to-one cancellation between eigenvalues. However, when there is a continuum spectrum extending to zero, there can also be a contribution from the lower end of the continuum. To obtain the number of normalizable zero modes, this continuum contribution must be subtracted from Eq. (5.8).

We now evaluate $\hat{\mathcal{I}}(\mathcal{D})$. A straightforward calculation shows that

$$\begin{aligned}\mathcal{D}^*\mathcal{D}&=-\nabla^2 I-L_1 , \\ \mathcal{D}\mathcal{D}^*&=-\nabla^2 I-L_2 ,\end{aligned} \tag{5.9}$$

where $L_1$ and $L_2$ are first-order differential operators that satisfy

$$\mathrm{tr}(L_1-L_2)=4B , \tag{5.10}$$

with tr denoting the matrix trace. The two terms in Eq. (5.8) can each be expanded about $M^2(-\nabla^2+M^2)^{-1}$. The leading terms in the two expansions are identical and cancel, while the third and higher terms vanish in the $M^2\to\infty$ limit. This leaves only the second terms, which give

$$\begin{aligned}\hat{\mathcal{I}}(\mathcal{D})&=\lim_{M^2\to\infty}\int d^2r\,\mathrm{tr}(L_1-L_2)M^2\langle x|(-\nabla^2+M^2)^{-2}|x\rangle \\ &=\frac{\Phi}{\pi} \\ &=2(n+\alpha) .\end{aligned} \tag{5.11}$$

For the vortex solutions $\alpha=0$, there is no continuum contribution, and our result is indeed an integer. On the other hand, for the solutions which approach the symmetric vacuum at large distances, $\hat{\mathcal{I}}(\mathcal{D})$ need not be an integer and thus clearly contains a continuum contribution which must be calculated and subtracted.

Because this contribution arises from the zero-frequency end of the continuum, it is sensitive only to the large-distance behavior of $\mathcal{D}$ and $\mathcal{D}^*$. Omitting all terms which fall faster than $1/r^2$ makes $\mathcal{D}$ block diagonal, allowing us to divide the eigenfunctions into those with only upper components nonzero and those with only lower components nonzero. From an examination of $\mathcal{D}$, it is obvious that the latter make no contribution and can be ignored. The former eigenfunctions correspond to the solutions of $(D_1+iD_2)\delta\phi=0$. By noting that $\hat{\mathcal{D}}\equiv D_1+iD_2$ and its adjoint $\hat{\mathcal{D}}^*\equiv -D_1+iD_2$ are just the off-diagonal elements of the Dirac operator $i(\sigma_2 D_1+\sigma_1 D_2)$, one can easily show that the calculation of the index of $\hat{\mathcal{D}}$ is essentially equivalent to the problem of counting the fermion zero modes in the presence of the gauge field **A** (more precisely, the difference between the number of positive- and negative-chirality modes). The continuum contribution to this calculation was studied some time ago. In our notation, the result[14] is that the continuum contribution to the fermion problem is equal to the fractional part of $\alpha$ if $\alpha$ is nonintegral, and unity if $\alpha$ is an integer. Although $\psi$ and $i\psi$ are not counted separately in the fermion problem, the corresponding modes in the parameter counting problem must be treated as linearly independent; because of this, we must multiply by 2. Doing so, and then subtracting from $\hat{\mathcal{I}}(\mathcal{D})$, we obtain

$$\mathcal{I}(\mathcal{D})=\begin{cases}2n, & |\phi(\infty)|=v\\ 2n+2\hat{\alpha}, & \phi(\infty)=0\ ,\end{cases} \tag{5.12}$$

where $\hat{\alpha}$ is defined to be the greatest integer less than $\alpha$.

We now examine the kernel of $\mathcal{D}^*$. Any solution of $0=\mathcal{D}^*\psi$ must also be a solution of

$$0=\mathcal{O}\mathcal{D}^*\psi\ , \tag{5.13}$$

for any operator $\mathcal{O}$. In particular, let us choose

$$\mathcal{O}=\begin{pmatrix}\phi_2 & -\phi_1 & \nabla_1 & \nabla_2\\ \phi_1 & \phi_2 & -\nabla_2 & \nabla_1\\ 0 & 0 & \phi_2 & \phi_1\\ 0 & 0 & -\phi_1 & \phi_2\end{pmatrix}. \tag{5.14}$$

[It is easy to see that there are no nonzero solutions to $\mathcal{O}\eta=0$, and so all solutions to Eq. (5.13) do in fact lie in the kernel of $\mathcal{D}^*$.] Using the equations satisfied by the unperturbed solution, we find that Eq. (5.13) becomes

$$0=(-\nabla^2+|\phi|^2)\psi_4\ , \tag{5.15}$$

$$0=\left[-\nabla^2+\frac{m^2}{v^4}(2|\phi|^2-v^2)|\phi|^2\right]\psi_3\ , \tag{5.16}$$

$$0=|\phi|^2\psi_1+(\phi_1\nabla_1-\phi_2\nabla_2)\psi_3-(\phi_1\nabla_2-\phi_2\nabla_1)\psi_4\ , \tag{5.17}$$

$$0=|\phi|^2\psi_2-(\phi_1\nabla_2+\phi_2\nabla_1)\psi_3+(\phi_1\nabla_1-\phi_2\nabla_2)\psi_4\ . \tag{5.18}$$

The first of these implies that for any square-integrable solution $\psi_4$ must vanish. The last two equations then determine $\psi_1$ and $\psi_2$ in terms of $\psi_3$. We are left with the second equation for $\psi_3$, which tells us that $\mathcal{D}^*$ has a nonzero kernel only if that Schrödinger equation has a normalizable eigenstate with zero energy. We would not in general expect this to be the case; therefore, $\mathcal{I}(\mathcal{D})$ indeed counts the zero modes of $\mathcal{D}$.

As a check of these formal manipulations, let us explicitly count the zero modes about an arbitrary rotationally symmetric solution described by functions $g(r)$ and $a(r)$ according to the *Ansatz* (3.1). From Eq. (5.2) we find that

$$e\delta A^i-\nabla_i\delta\,\mathrm{Arg}(\phi)=\epsilon^{ij}\frac{\nabla_j\delta|\phi|}{|\phi|}\ . \tag{5.19}$$

Substituting this into the background gauge condition (5.4) then gives

$$0=(-\nabla^2+|\phi|^2)\delta\,\mathrm{Arg}(\phi)\ . \tag{5.20}$$

This has singular solutions, which can be accepted if the singularity in $\delta\,\mathrm{Arg}(\phi)$ occurs at a zero of $\phi$. By matching the singularities of these with the singularities we shall find for the right-hand side of Eq. (5.19), a nonsingular $\delta\mathbf{A}$ can be obtained. In addition to these, there is a also a solution with $\delta\,\mathrm{Arg}(\phi)$ asymptotically constant if $\phi=0$ at spatial infinity. This solution corresponds to the single gauge mode allowed by the background gauge condition.

Thus, once $\delta|\phi|$ is found, $\delta\mathbf{A}$ and $\delta\,\mathrm{Arg}(\phi)$ can be determined. Let us write

$$|\phi|=g(r)[1+h(r,\theta)]\ . \tag{5.21}$$

By expanding the differential equation (2.16) obeyed by the magnitude of the scalar field, we obtain

$$0=-\nabla^2 h+m^2(2g^4-g^2)h\ , \tag{5.22}$$

which is identical to Eq. (5.16). When studying the kernel of $\mathcal{D}^*$, we were only interested in normalizable solutions of this equation, which in general we expect to be absent. Here we must consider a broader class of solutions, since a non-normalizable solution for $h$ may give a normalizable $\delta|\phi|$. If

$$h(r,\theta)=h_0(r)+\sum_{J=1}^{\infty}[h_J^{(1)}(r)\cos(J\theta)+h_J^{(2)}(r)\sin(J\theta)]\ , \tag{5.23}$$

then the $h_J^{(i)}$ obey

$$0=\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}h_J^{(i)}\right]-\left[\frac{J^2}{r^2}+m^2g^2(2g^2-1)\right]h_J^{(i)}\ . \tag{5.24}$$

Near the origin, the solutions of this equation behave as

$$h_J^{(i)}\approx\begin{cases}P_J^{(i)}r^J+Q_J^{(i)}r^{-J}, & J\neq 0\ ,\\ P_J+Q_J\ln r, & J=0\ ,\end{cases} \tag{5.25}$$

provided that $g(0)$ is finite. This is also the behavior as $r\to\infty$ if $g(\infty)=0$, while if $g(\infty)=1$

$$h_J^{(i)}\approx P_J^{(i)}I_J(mr)+Q_J^{(i)}K_J(mr)\ . \tag{5.26}$$

Of course, the $P$ and $Q$ coefficients are all different in the various asymptotic regimes.

Not all of these solutions are acceptable. Near the origin, $\delta|\phi|-gh\sim r^n h$ must certainly be finite. This implies that $Q_J^{(i)}$ must vanish unless $J \le n$, and furthermore that $Q_0 = 0$ if $n = 0$. We constrain the solutions at spatial infinity by requiring square integrability, so that the terms in the energy which are quadratic in the fluctuations are finite. Thus we require that $\delta|\phi| = gh \sim r^{-\alpha}h$ tend to zero faster than $1/r$. For the vortex solutions, where $g(\infty)=1$, this means that the $Q_J^{(i)}$ must all vanish, while for the solutions which approach the symmetric vacuum, $Q_J^{(i)}$ can be nonzero only if $J < \alpha - 1$.

If for a given choice of $J$ and $\alpha$ both solutions are well behaved at both $r=0$ and $\infty$, then there are clearly two linearly dependent acceptable solutions of Eq. (5.24). If both are well behaved at the origin, but only one at $r=\infty$ (or conversely), then there is one and only one acceptable solution. This leaves only the case in which both $J > n$ and $J \ge \alpha - 1$, where there is only one well-behaved solution at each limit. In general, we would not expect to be able to match these solutions to obtain a globally acceptable solution. If they can be matched, the result is a normalizable solution of Eq. (5.22) and, thus, of Eq. (5.16). Thus, these extra solutions exist if and only if $\mathcal{D}^*$ has a nonvanishing kernel.

Once we have a solution for $\delta|\phi|$, Eqs. (5.19) and (5.20) must be used to determine $\delta\mathbf{A}$ and the phase of $\delta\phi$. When $J \neq 0$, the singularity in $\nabla h$ can be canceled by choosing a singular solution of Eq. (5.20), so that the final $\delta\phi$ and $\delta\mathbf{A}$ are nonsingular. This cancellation cannot be carried out for the $J=0$ modes, which correspond to deformations of the original solution that change the values of $n$ and $\alpha$. The former type of deformation is unexpected, since $n$ should be quantized. Indeed, although this mode gives a nonsingular solution of Eq. (5.22), it cannot be extended to a solution of the full set of equations (5.5), since $\delta\mathbf{A}$ has a $1/r$ singularity at the origin which cannot be removed by a gauge transformation. The mode corresponding to a change in $\alpha$ is nonsingular near the origin, while at large distances $\delta\mathbf{A}$ falls as $1/r$. This behavior is consistent with finite energy, and this mode is evidently counted by $\mathcal{I}(\mathcal{D})$. Nevertheless, since a change in $\alpha$ changes the energy, this mode does not correspond to a parameter. Subtracting both this mode and the gauge mode, alluded to above, which is present when $\phi(\infty)=0$, we find that the number of parameters is

$$\text{number of parameters} = \begin{cases} 2n, & |\phi(\infty)| = v\ , \\ 2n + 2\hat{\alpha} - 2, & \phi(\infty) = 0\ . \end{cases} \tag{5.27}$$

The result for the upper case, $|\phi(\infty)| = v$, is the same as for the case of Landau-Ginzburg vortices.[12] For that case the parameters specify the positions in the plane of $n$ independent vortices.[15] We expect that the $2n$ parameters in our case similarly specify the positions of $n$ independent topological vortices.

To seek a multisoliton interpretation where $\phi$ asymptotically approaches the symmetric vacuum, we first consider the case $n=0$, $\alpha \neq 0$. One might expect solutions consisting of some number $N$ "lumps," each resembling a spherically symmetric nontopological soliton, surrounded by a region of approximate vacuum. For each of these lumps there would be two position parameters $x_i$ and $y_i$, an $\alpha_i$ specifying its flux and a U(1) phase $\theta_i$. These $4N$ parameters should be reduced by 1 because a simultaneous shift of all the $\theta_i$ is simply a global gauge transformation and should again be reduced by 1 because the sum of the $\alpha_i$ is constrained to be equal to $\alpha$. Since each of the $\alpha_i$ must be greater than 2, $N$ must be less than $\alpha/2$, so that for $2k < \alpha \le 2k+2$, the generic solution should consist of $N=k$ lumps. The enumeration above would then give $4k-2$ parameters. This agrees with Eq. (5.27) for $2k < \alpha \le 2k+1$, but falls short by 2 for $2k+1 < \alpha \le 2k+2$. We do not have an understanding of these extra two parameters. The $2n$ additional parameters when $n$ is also nonzero presumably can be interpreted as the positions of $n$ vortices superimposed on a multinontopological soliton solution.

## VI. CONCLUSION

In this paper we have described a novel type of self-dual system. It occurs at the transition point between the asymmetric and symmetric phases of the theory and has soliton solutions appropriate to each. In the former case there are topologically stable vortices, carrying a quantized charge, while in the latter case the solutions are nontopological in nature and (at least classically) have a continuously variable charge. The existence of these solutions even when the gauge-field kinetic energy contains no Maxwell term vividly demonstrates the effectiveness of the Chern-Simons interaction. Without stabilization by gauge-field dynamical effects, scalar fields cannot support static, finite-energy excitations. Thus, contrary to occasional assertions in the literature,[16] the Chern-Simons interaction does more than merely change statistics.

While some aspects of the classical theory, such as a complete description of multisoliton solutions and their behavior away from the self-dual point, remain to be clarified, perhaps the most interesting open questions are associated with the quantum theory. Upon quantization, the classical solutions give rise to quantum particle states, in a fashion familiar from previous studies.[17] The properties of these states, including their statistics and the quantization of charge, await further investigation.


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