Statistics transmutation in Maxwell-Chern-Simons theories

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We study (2+1)-dimensional Abelian gauge theories in which the gauge field has both Maxwell and Chern-Simons topological terms as the action and is coupled to a generic matter field. It is shown that the statistics of the matter field is transmuted into an anyonic one when a suitable basis for the field operators is chosen. The redefined matter-field operator with transmuted statistics has a local intrinsic flux.

I. INTRODUCTION

The observation that spin and statistics may transmute into an exotic one due to topological terms[1,2] in (2+1)-dimensional space-time has attracted much attention recently. It has been argued that the statistics transmutation has an important experimental consequence in the physics of high-T_c superconductivity.[3]

Wilczek[4] considered the possibility of exotic statistics appearing in 2+1 dimensions. The object obeying the unusual statistics is called an *anyon.* Wilczek and Zee[5] studied the CP^1 model [O(3) nonlinear σ model] with the *Hopf term*. The topological soliton contained in this theory, discovered first by Belavin and Polyakov[6] some time ago, is anyonic and becomes fermionic in a special case. The theory of the anyon was examined further by several authors.[7,8]

Dzyaloshinskii, Polyakov, and Wiegmann[9,10] studied the CP^1 model with the Chern-Simons term.[1] This theory may be regarded as a low-energy effective theory of the model of Wilczek and Zee.[5] They demonstrated the change of statistics, so-called Bose-Fermi transmutation, in this effective theory. The statistics transmutation can be shown in a more elegant fashion in the path-integral method.[10] These inspiring works[9,10] have suggested a possible connection of the gauge theory with the Chern-Simons term to high-T_c superconductivity and have led to extensive study of the statistics transmutation in 2+1 dimensions.[11-19]

It was suggested in a series of papers[9,10] that the charged fermions coupled to the U(1) Chern-Simons term will be transmuted to bosons and the Bose condensation leads to a superconducting state. Inspired by these works, we investigated the Chern-Simons CP^1 model coupled to charged fermions.[17] We constructed the generalized Hamiltonian formalism of this theory and quantized the system by using the canonical method and the path-integral method. The Gauss-law constraint in this theory can be solved by expressing the gauge potential in terms of a multivalued function. This vector potential can be absorbed in the matter field by passing to a new operator basis. The canonical (anti)commutation relations of the matter-field variables become anyonic in this new basis. In particular, for a special value of a parameter that is the coefficient of the Chern-Simons term, commutators are changed to anticommutators and vice versa. Thus the transmutation of the statistics is understood in terms of the canonical commutator algebra. The quantum-mechanical version was also considered and we found some interesting aspects.[18]

A similar result was obtained in a general way for the U(1) Chern-Simons theory. We were able to show that the statistics transmutation occurs in the U(1) Chern-Simons theory coupled to a *generic* matter field.[19] We applied the symplectic geometrical method[20] to construct the generalized Hamiltonian formalism in this theory.

The Chern-Simons CP^1 model (coupled to fermions) is believed to be obtained from some statistical model as an effective theory.[9,10,13,14] The Chern-Simons term (which is the first-derivative term) is expected to appear as the leading contribution in the long-wavelength limit of this model. As the next to leading order, there can arise the usual Maxwell term of gauge field (which is the second-derivative term). The canonical structures of the theories with and without the Maxwell term are vastly different. It is an interesting question whether the statistics transmutation due to the Chern-Simons term will disappear or it will be unaffected by adding the Maxwell term.[21]

In this paper we extend our previous approach to the case where *both* the Maxwell term and the Chern-Simons topological term are present in the Lagrangian and the gauge field is coupled to a generic matter field by minimal interactions. We call such theories an Abelian topological massive gauge theory[1] or Maxwell-Chern-Simons theory. We construct the generalized Hamiltonian formalism of this theory using the symplectic geometric method.[20] We are able to solve the Gauss-law constraint explicitly by dividing the gauge field into the transverse and longitudinal components. We show that the statistics of each matter field is transmuted to the anyonic one when a suitable operator basis is chosen. The equal-time (anti)commutation relations between the matter fields are changed into the anyonic ones. The longitudinal part of the gauge field is absorbed in the matter fields and the statistics of the matter is transmuted. Different from the case where the gauge field has only the Chern-Simons term as the action, the dynamical degree of the freedom of the gauge field is remained. The equal-time commutation relations between the gauge field and the matter fields also are changed in a nontrivial way, from which we show that the intrinsic local flux is attached to the transmuted matter fields.

This paper is organized as follows. We construct the generalized Hamiltonian formalism following the symplectic geometric method. The system is quantized canonically in Sec. III. In Sec. IV, a new basis of the field operators is introduced, and it is shown that the system can have the anyonic statistics. We devote Sec. V to conclusions.

II. GENERALIZED HAMILTONIAN FORMALISM

The Lagrangian density of the Maxwell-Chern-Simons theory coupled to a generic matter field φ is given by

$$\mathcal{L}=-\frac{k}{4}F_{\mu\nu}F^{\mu\nu}+\frac{\mu}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + A_\mu J^\mu+\mathcal{L}^{\text{matter}}(\varphi, A_\mu) \ . \tag{1}$$

Here the second term in the right-hand side of Eq. (1) is the Chern-Simons term, multiplied by a parameter μ. The parameter k is introduced in the first term in order to trace the effect of the Maxwell term. We assume that the theory is gauge invariant and the interactions are the minimal gauge coupling, and we do not need to specify the matter Lagrangian $\mathcal{L}^{\text{matter}}$ further.

We wish to construct the generalized Hamiltonian formalism of the theory described by the Lagrangian (1). This can be made in a most transparent way by using the symplectic geometric method recently advocated by Faddeev and Jackiw.[20] In the present case, this method allows us to solve all the constraints in terms of dynamical variables and to cast the Lagrangian in the standard canonical form.

First we rewrite Eq. (1) in the first-order formalism treating A_μ and $F_{\mu\nu}$ as independent variables:

$$\mathcal{L}=-\frac{k}{2}(\partial_\mu A_\nu-\partial_\nu A_\mu-\tfrac{1}{2}F_{\mu\nu})F^{\mu\nu}+\frac{\mu}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + A_\mu J^\mu+\mathcal{L}^{\text{matter}} \ . \tag{2}$$

For the canonical method, it is convenient to use the two-dimensional vector notation. Equation (2) now becomes

$$\mathcal{L}=kE_i\dot{A}^i-\frac{\mu}{2}\epsilon^{ij}A_i\dot{A}_j+\frac{k}{2}E_iE^i-\frac{k}{2}B^2+A_iJ^i + A_0(k\partial_iE^i+\mu\epsilon^{ij}\partial_iA_j+J^0)+\mathcal{L}^{\text{matter}} \ , \tag{3}$$

where the overdot indicates the time derivative. We have defined the electric field E^i and the magnetic field B by $E^i=F^{i0}$ and $B=-\frac{1}{2}\epsilon^{ij}F_{ij}$, respectively $(i,j=1,2)$. Here we regard A^i and E^i as independent variables and B is expressed in terms of A^i.

The first and second terms in the right-hand side of Eq. (3) contain first time-derivative terms. We define new variables

$$\Pi^j\equiv kE^j-\frac{\mu}{2}\epsilon^{ij}A_i$$

and eliminate E^j to cast two terms in Eq. (3) into the standard canonical term. Then Eq. (3) is rewritten as

$$\mathcal{L}=\Pi^j\dot{A}_j+\frac{1}{2k}\left[\Pi^j+\frac{\mu}{2}\epsilon^{ij}A_i\right]\left[\Pi_j+\frac{\mu}{2}\epsilon_{kj}A^k\right] -\frac{k}{2}B^2+A_iJ^i + A_0\left[\partial_j\Pi^j+\frac{\mu}{2}\epsilon^{ij}\partial_iA_j+J^0\right]+\mathcal{L}^{\text{matter}} \ . \tag{4}$$

Π^j and A^j are the phase-space variables that are conjugate to each other. Notice that the system is accompanied with the constraint

$$\partial_j\Pi^j+\frac{\mu}{2}\epsilon^{ij}\partial_iA_j+J^0=0 \ . \tag{5}$$

This is the Gauss-law constraint reflecting the gauge invariance of the system.

Our next problem is to solve the Gauss-law constraint Eq. (5). We separate each of the field variables A^j and Π^j into the transverse and longitudinal parts as $A^j=A_T^j+A_L^j$ and $E^j=E_T^j+E_L^j$. So Eq. (5) becomes

$$\partial_j\Pi_L^j=-\left[\frac{\mu}{2}\epsilon^{ij}\partial_iA_j^T+J^0\right] \ .$$

The longitudinal component Π_L^j can be expressed as $\Pi_L^j=\partial^j\lambda$ introducing a scalar function λ. Then we get

$$\partial_j\partial^j\lambda=-\left[\frac{\mu}{2}\epsilon^{ij}\partial_iA_j^T+J^0\right] \ ,$$

which is the Poisson equation in two dimensions. The solution is

$$\lambda(\mathbf{x})=\frac{1}{4\pi}\int d^2y(\ln|\mathbf{x}-\mathbf{y}|^2+\text{const}) \times\left[\frac{\mu}{2}\epsilon^{ij}\partial_i^yA_j^T(\mathbf{y})+J^0(\mathbf{y})\right] \ .$$

We obtain the solution of the Gauss-law constraint (5):

$$\Pi_L^j(\mathbf{x})=\frac{1}{2\pi}\int d^2y\frac{(x-y)^j}{|\mathbf{x}-\mathbf{y}|^2}\left[\frac{\mu}{2}\epsilon^{lm}\partial_l^yA_m^T+J^0(\mathbf{y})\right] \ . \tag{6a}$$

Further, the solution (6a) is formally rewritten in the form of a total derivative by using a *multivalued* function. We introduce the angle variable $\Omega(\mathbf{x}-\mathbf{y})$ between the vector $\mathbf{x}-\mathbf{y}$ and the first axis in the two-dimensional space, that is,

$$\tan\Omega(\mathbf{x}-\mathbf{y})=\frac{x^2-y^2}{x^1-y^1} \ .$$

By noting the relation

$$\partial_x^j\Omega(\mathbf{x}-\mathbf{y})=-e^{jk}\frac{(x-y)_k}{|\mathbf{x}-\mathbf{y}|^2} \ ,$$

we can represent Eq. (6a) as

$$\Pi_L^j(\mathbf{x})=\frac{1}{2\pi}\int d^2y\, e^{jk}\partial_k^x\Omega(\mathbf{x}-\mathbf{y})\left[\frac{\mu}{2}\epsilon^{lm}\partial_l^y A_m^T+J^0(\mathbf{y})\right] . \tag{6b}$$

As an important property of $\Omega(\mathbf{x}-\mathbf{y})$, we should notice the relation

$$-\epsilon^{ij}\partial_i^x\partial_j^x\Omega(\mathbf{x}-\mathbf{y})=2\pi\delta(\mathbf{x}-\mathbf{y}) , \tag{7}$$

which indicates that we should consider $\epsilon^{ij}\partial_i^x\partial_j^x\Omega(\mathbf{x}-\mathbf{y})$ as a distribution function reflecting the multivalued nature of $\Omega(\mathbf{x}-\mathbf{y})$. Keep in mind that Π_L^j is expressed in terms of A_j^T and J^0 and is not an independent field variable any more.

After the separation into the transverse and longitudinal parts, the Lagrangian Eq. (4) is written as

$$\begin{aligned}\mathcal{L}=&\Pi_T^j\dot{A}_j^L+\pi_L^j\dot{A}_j^L\\&+\frac{1}{2k}\left[\Pi_T^i+\frac{\mu}{2}\epsilon^{ij}A_i^T+\Pi_L^i+\frac{\mu}{2}\epsilon^{ij}A_i^L\right]\\&\times\left[\Pi_j^T+\frac{\mu}{2}\epsilon_{kj}A_T^k+\Pi_j^L+\frac{\mu}{2}\epsilon_{kj}A_L^k\right]\\&-\frac{k}{2}B^2+(A_i^T+A_i^L)J^i+\mathcal{L}^{\text{matter}} ,\end{aligned} \tag{8}$$

where $B=-\epsilon^{ij}\partial_i A_j^T$. The second term in the right-hand side of Eq. (8) is not the canonical form since Π_L^j is given as Eq. (6). We must reconstruct the standard canonical term by changing variables and defining suitable new variables.

We try to eliminate A_j^L by the phase change of the matter fields. Now we are assuming the minimal gauge coupling so that the interaction term appears in the form of $(\partial_i-ieA_i^L)\varphi$, where e is the gauge coupling constant. Under the change as $\varphi\to e^{i\Xi}\varphi$, the term is transformed as

$$(\partial_i-ieA_i^L)\varphi\to e^{i\Xi}(\partial_i+i\partial_i\Xi-ieA_i^L)\varphi .$$

In order to eliminate A_i^L, the condition

$$\partial_i\Xi=eA_i^L \tag{9}$$

should be satisfied. We solve Eq. (9) using the same technique as we have solved the Gauss-law constraint (5). The solution is

$$\Xi(\mathbf{x})=-\frac{1}{4\pi}\int d^2y(\ln|\mathbf{x}-\mathbf{y}|^2+\text{const})e\partial_y^i A_i^L(\mathbf{y}) . \tag{10}$$

Now A_L^i disappears from the interaction term. On the other hand, this phase change of the matter field affects the canonical term of the matter field like $w\dot{\varphi}$, where w is a canonically conjugate variable of φ. The term is transformed as $w\dot{\varphi}\to w\dot{\varphi}+i\dot{\Xi}w\varphi$. The additional term is rewritten using Eqs. (5) and (9) as

$$\begin{aligned}i\dot{\Xi}w\varphi&=-\frac{1}{e}\dot{\Xi}J^0\\&=\frac{1}{e}\dot{\Xi}\left[\partial_i\Pi_L^i+\frac{\mu}{2}\epsilon^{ij}\partial_i A_j^T\right]\\&=-\frac{1}{e}\partial_i\dot{\Xi}\left[\Pi_L^i+\frac{\mu}{2}\epsilon^{ij}A_j^T\right]\\&=-\dot{A}_i^L\left[\Pi_L^i+\frac{\mu}{2}\epsilon^{ij}A_j^T\right] ,\end{aligned} \tag{11}$$

where the charge-density operator is given as $J^0=-iew\varphi$. We notice that the term $\Pi_L^j\dot{A}_j^L$ in Eq. (11) is canceled by the second term in the right-hand side of Eq. (8). Finally we obtain

$$\begin{aligned}\mathcal{L}=&\left[\Pi_T^j+\frac{\mu}{2}\epsilon^{ij}A_i^L\right]\dot{A}_j^T\\&+\frac{1}{2k}\left[\Pi_T^i+\frac{\mu}{2}\epsilon^{ij}A_i^T+\Pi_L^i+\frac{\mu}{2}\epsilon^{ij}A_i^L\right]\\&\times\left[\Pi_j^T+\frac{\mu}{2}\epsilon_{kj}A_T^k+\Pi_j^L+\frac{\mu}{2}\epsilon_{kj}A_L^k\right]\\&-\frac{k}{2}B^2+A_i^TJ^i+\mathcal{L}^{\text{matter}} .\end{aligned} \tag{12}$$

Π_i^L has disappeared from terms containing the time derivative. But A_i^L remains and the form of the standard canonical term has not been realized. To accomplish the construction of the standard term, we define a new field variable P_T^j as

$$\Pi_T^j+\frac{\mu}{2}\epsilon^{ij}A_i^L\equiv P_T^j .$$

Then the Lagrangian density becomes

$$\begin{aligned}\mathcal{L}=&P_T^j\dot{A}_j^T\\&+\frac{1}{2k}\left[P_T^i+\frac{\mu}{2}\epsilon^{ij}A_i^T+\Pi_L^i\right]\left[P_j^T+\frac{\mu}{2}\epsilon_{kj}A_T^k+\Pi_j^L\right]\\&-\frac{k}{2}B^2+A_j^TJ^j+\mathcal{L}^{\text{matter}} .\end{aligned} \tag{13}$$

We have completed the construction of the standard canonical term. Remember that B is represented by A_T^i, and Π_L^j is given by using A_T^i and J^0 as Eq. (6). In the sector of gauge field, all of the constraints are solved and the system is described by the true dynamical variables. The matter sector denoted by $\mathcal{L}^{\text{matter}}$ might have some constraints. If so, we repeat the above prescription of the symplectic geometric method or we may rely on the Dirac method[22] if it is still hard to solve the constraints. In any case, it is possible to obtain a canonical form of the matter fields.

III. CANONICAL QUANTIZATION

Based on the Lagrangian density (13), we quantize the theory by imposing equal-time (anti)commutation relations as

$$[A_i^T(\mathbf{x}),P_j^T(\mathbf{y})]_-=i\hbar\delta_{ij}^T(\mathbf{x}-\mathbf{y}) \; ; \tag{14}$$

$$\begin{aligned}&[\varphi(\mathbf{x}),\varphi(\mathbf{y})]_\pm=(\ ) ,\\ &[\omega(\mathbf{x}),\omega(\mathbf{y})]_\pm=(\ ) ,\\ &[\varphi(\mathbf{x}),\omega(\mathbf{y})]_\pm=(\ ) .\end{aligned} \tag{15}$$

$\delta_{ij}^T(\mathbf{x}-\mathbf{y})$ is the transverse delta function which satisfies

$$\partial_x^i\delta_{ij}^T(\mathbf{x}-\mathbf{y})=\partial_y^j\delta_{ij}^T(\mathbf{x}-\mathbf{y})=0 \; .$$

The empty parentheses represent terms that arise from $\mathcal{L}^{\text{matter}}$ and we do not need to specify them for the present purpose. The index $-$ $(+)$ means the commutator (anticommutator). The Hamiltonian density of the system is

$$\mathcal{H}=\frac{-1}{2k}\left[P_T^j+\frac{\mu}{2}\epsilon^{ij}A_i^T+\Pi_L^j\right]\left[P_j^T+\frac{\mu}{2}\epsilon_{kj}A_T^k+\Pi_j^L\right]+\frac{k}{2}B^2-A_j^TJ^j+\mathcal{H}^{\text{matter}} \; . \tag{16}$$

The equal-time (anti)commutation relations, Eqs. (14) and (15), and the Hamiltonian density, Eq. (16), describe the quantum theory on the true phase space consistently.

It should be noticed that we arrived at the quantized theories without specifying the gauge-fixing condition. This is the superior feature of the algorithm of the symplectic geometrical method.[20] We have solved the constraint explicitly in the construction of the generalized Hamiltonian formalism. The remaining variables are the true phase-space variables. So we have not needed the gauge-fixing condition. The result obtained in this paper is independent of the gauge choice.

IV. STATISTICS TRANSMUTATION

Now we present the statistics transmutation of the system. In the Hamiltonian density Eq. (16), Π_j^L contains the multivalued angle variable as shown in Eq. (6b). We wish to eliminate Π_j^L by transforming the *operator* field variables. First, we introduce a new variable X_i^T as

$$\frac{\mu}{2}\epsilon^{ij}A_i^T+\Pi_L^j\equiv\mu\epsilon^{ij}X_i^T \; . \tag{17}$$

The transversality of X_i^T is kept consistently in the definition of Eq. (17). Using the Gauss-law constraint Eq. (5), we can rewrite Eq. (17) to

$$\mu\epsilon^{ij}\partial_i(A_j^T-X_j^T)=-J^0 \; .$$

If we introduce a scalar function ϕ by $A_j^T-X_j^T=\epsilon_{jk}\partial^k\phi$ noticing that $A_j^T-X_j^T$ is transversal, we have $\partial_i\partial^i\phi=(1/\mu)J^0$, which can be solved as before. The solution is

$$\phi(\mathbf{x})=-\frac{1}{4\pi}\int d^2y(\ln|\mathbf{x}-\mathbf{y}|^2+\text{const})\frac{1}{\mu}J^0(\mathbf{y})$$

and we obtain

$$\begin{aligned}A_j^T(\mathbf{x})&=X_j^T(\mathbf{x})-\frac{1}{2\pi}\int d^2y\frac{\epsilon_{jk}(x-y)^k}{|\mathbf{x}-\mathbf{y}|^2}\frac{1}{\mu}J^0(\mathbf{y})\\&=X_j^T(\mathbf{x})+\frac{1}{e}\partial_j^x\Theta(\mathbf{x}) \; ,\end{aligned} \tag{18a}$$

$$\Theta(\mathbf{x})=\frac{e}{2\pi\mu}\int d^2y\;\Omega(\mathbf{x}-\mathbf{y})J^0(\mathbf{y}) \; , \tag{18b}$$

where $\Omega(\mathbf{x}-\mathbf{y})$ is defined in Sec. II. On the other hand, $B=-\epsilon^{ij}\partial_i A_j^T$ is rewritten as

$$B=-\epsilon^{ij}\partial_iX_j^T+\frac{1}{\mu}J^0(\mathbf{x})$$

by using Eqs. (7) and (17). So, we obtain the Hamiltonian density

$$\mathcal{H}=\frac{-1}{2k}(P_T^j+\mu\epsilon^{ij}X_i^T)(P_j^T+\mu\epsilon_{kj}X_T^k)+\frac{k}{2}B^2-\left[X_j^T+\frac{1}{e}\partial_j^x\Theta(\mathbf{x})\right]J^j+\mathcal{H}^{\text{matter}} \; . \tag{19}$$

The multivalued function still remains in the interaction term of Eq. (19). Let us perform the phase transformation of the matter fields as

$$\varphi\equiv e^{i\Theta}\hat{\varphi}, \quad \omega\equiv\hat{\omega}e^{-i\Theta} \; , \tag{20}$$

where the variables with the hat are newly defined. The interaction term has the form

$$\left[\partial_j-ie\left[X_j^T+\frac{1}{e}\partial_j^x\Theta(\mathbf{x})\right]\right]\varphi \; .$$

Under the definition Eq. (20), the term is rewritten as $e^{i\Theta}(\partial_j-ieX_j^T)\hat{\varphi}$ so that the multivalued function is eliminated. (The phase factor $e^{i\Theta}$ is canceled by $e^{-i\Theta}$ from w.) Finally we obtain the Hamiltonian density

$$\mathcal{H}=\frac{-1}{2k}(P_T^j+\mu\epsilon^{ij}X_i^T)(P_j^T+\mu\epsilon_{kj}X_T^k)+\frac{k}{2}B^2-X_j^TJ^j+\mathcal{H}^{\text{matter}} \; . \tag{21}$$

The multivalued function has been eliminated completely.

How about the canonical equal-time (anti)commutation relations? The transformations Eqs. (18) and (20) include the quantum operator J^0 and the multivalued function Ω so that they may change the commutation relations. In fact, we show that the commutation relations are modified in a nontrivial way.

Before that, we derive important formulas for later convenience. We are now considering the gauge-invariant theories so that the commutation relations

$$[\varphi(\mathbf{x}),J^0(\mathbf{y})]_-=-e\varphi(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) \; , \tag{22a}$$

$$[\omega(\mathbf{x}),J^0(\mathbf{y})]_-=e\omega(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) \tag{22b}$$

hold. [Equation (22) and the current-conservation law lead to the Ward-Takahashi identity.] From Eqs. (18b) and (22), we obtain

$$[\varphi(\mathbf{x}),\Theta(\mathbf{y})]_-=-\frac{e^2}{2\pi\mu}\Omega(\mathbf{y}-\mathbf{x})\varphi(\mathbf{x}) \; , \tag{23a}$$

$$[\omega(\mathbf{x}),\Theta(\mathbf{y})]_-=\frac{e^2}{2\pi\mu}\Omega(\mathbf{y}-\mathbf{x})\omega(\mathbf{x}) \; . \tag{23b}$$

We should notice that Eqs. (22) and (23) are guaranteed by the gauge invariance of the theories. Equation (23) gives us important relations as

$$e^{-i\Theta(\mathbf{x})}\varphi(\mathbf{y})e^{i\Theta(\mathbf{x})}=\exp\left[-i\frac{e^2}{2\pi\mu}\Omega(\mathbf{x}-\mathbf{y})\right]\varphi(\mathbf{y}) , \tag{24a}$$

$$e^{-i\Theta(\mathbf{x})}\omega(\mathbf{y})e^{i\Theta(\mathbf{x})}=\exp\left[i\frac{e^2}{2\pi\mu}\Omega(\mathbf{x}-\mathbf{y})\right]\omega(\mathbf{y}) , \tag{24b}$$

which are very useful later.

Now we derive the equal-time (anti)commutation relations in the transformed system, which is described by the variables X_i^T, P_i^T, $\hat{\varphi}$, and $\hat{\omega}$. The commutation relations between the gauge-field variables are

$$\begin{aligned}&[X_i^T(\mathbf{x}),X_j^T(\mathbf{y})]_-=[P_i^T(\mathbf{x}),P_j^T(\mathbf{y})]_-=0 ,\\&[X_i^T(\mathbf{x}),P_j^T(\mathbf{y})]_-=i\hbar\delta_{ij}^T(\mathbf{x}-\mathbf{y}) ,\end{aligned} \tag{24c}$$

which are the usual ones. Between the gauge-field variables and the matter-field variables, we obtain commutation relations as

$$[\hat{\varphi}(\mathbf{x}),P_j^T(\mathbf{y})]_-=[\hat{\omega}(\mathbf{x}),P_j^T(\mathbf{y})]_-=0 , \tag{25a}$$

$$[\hat{\varphi}(\mathbf{x}),X_i^T(\mathbf{y})]_-=\frac{e}{2\pi\mu}\partial_i^y\Omega(\mathbf{y}-\mathbf{x})\hat{\varphi}(\mathbf{x}) , \tag{25b}$$

$$[\hat{\omega}(\mathbf{x}),X_i^T(\mathbf{y})]_-=-\frac{e}{2\pi\mu}\partial_i^y\Omega(\mathbf{y}-\mathbf{x})\hat{\omega}(\mathbf{x}) , \tag{25c}$$

where Eqs. (25b) and (25c) are nontrivial. The commutation relations between the matter field variables become

$$\hat{\varphi}(\mathbf{x})\hat{\varphi}(\mathbf{y})\mp\exp\left[-i\frac{e^2}{2\pi\mu}\Delta\Omega(\mathbf{x}-\mathbf{y})\right]\hat{\varphi}(\mathbf{y})\hat{\varphi}(\mathbf{x})=(\ ) , \tag{26a}$$

$$\hat{\omega}(\mathbf{x})\hat{\omega}(\mathbf{y})\mp\exp\left[-i\frac{e^2}{2\pi\mu}\Delta\Omega(\mathbf{x}-\mathbf{y})\right]\hat{\omega}(\mathbf{y})\hat{\omega}(\mathbf{x})=(\ ) , \tag{26b}$$

$$\hat{\varphi}(\mathbf{x})\hat{\omega}(\mathbf{y})\mp\exp\left[i\frac{e^2}{2\pi\mu}\Delta\Omega(\mathbf{x}-\mathbf{y})\right]\hat{\omega}(\mathbf{y})\hat{\varphi}(\mathbf{x})=(\ ) , \tag{26c}$$

where the difference of the angle variables is defined as

$$\Delta\Omega(\mathbf{x}-\mathbf{y})=\Omega(\mathbf{x}-\mathbf{y})-\Omega(\mathbf{y}-\mathbf{x}) .$$

We call Eqs. (24)–(26) *anyonic commutation relations.*

We find some marvelous features of the transformed system from the anyonic commutation relations derived above. In Eq. (26), there appears the novel phase factor under interchanging the order of the product of the matter variables. The $\Delta\Omega(\mathbf{x}-\mathbf{y})$ is the angle between the antiparallel two vectors so that we have $\Delta\Omega(\mathbf{x}-\mathbf{y})=(2n+1)\pi$ (n is an integer). The phase factor becomes

$$\exp\left[\mp i\frac{e^2}{2\mu}(2n+1)\right] .$$

Then Eq. (26) gives us the following two cases for typical values of the parameters.

(i) When $e^2/2\mu=2m\pi$, we obtain

$$\begin{aligned}&\hat{\varphi}(\mathbf{x})\hat{\varphi}(\mathbf{y})\mp\hat{\varphi}(\mathbf{y})\hat{\varphi}(\mathbf{x})=(\ ) ,\\&\hat{\omega}(\mathbf{x})\hat{\omega}(\mathbf{y})\mp\hat{\omega}(\mathbf{y})\hat{\omega}(\mathbf{x})=(\ ) ,\\&\hat{\varphi}(\mathbf{x})\hat{\omega}(\mathbf{y})\mp\hat{\omega}(\mathbf{y})\hat{\varphi}(\mathbf{x})=(\ ) .\end{aligned} \tag{27}$$

(ii) When $e^2/2\mu=(2m+1)\pi$, we obtain

$$\begin{aligned}&\hat{\varphi}(\mathbf{x})\hat{\varphi}(\mathbf{y})\mp(-1)\hat{\varphi}(\mathbf{y})\hat{\varphi}(\mathbf{x})=(\ ) ,\\&\hat{\omega}(\mathbf{x})\hat{\omega}(\mathbf{y})\mp(-1)\hat{\omega}(\mathbf{y})\hat{\omega}(\mathbf{x})=(\ ) ,\\&\hat{\varphi}(\mathbf{x})\hat{\omega}(\mathbf{y})\mp(-1)\hat{\omega}(\mathbf{y})\hat{\varphi}(\mathbf{x})=(\ ) ,\end{aligned} \tag{28}$$

where m is integer.

In case (i), the (anti)commutation relations of the transformed variables Eq. (27) agree with the original ones Eq. (15). The case (ii) is specific where the commutators (anticommutators) of the original variables are changed to the anticommutators (commutators) of the transformed variables. This is just the Bose-Fermi transmutation. In the other region of the value of the parameters, the interpolating statistics between the Bose and Fermi statistics is realized. Such an exotic case corresponds to the *anyon* statistics.

The commutation relations between the gauge field and the matter fields also give us an interesting interpretation. We can find that the transformed matter variables have an intrinsic magnetic flux. In applying $-\epsilon^{ki}\partial_k$ to Eqs. (25b) and (25c), we have

$$[\hat{\varphi}(\mathbf{x}),\hat{B}(\mathbf{y})]_-=\frac{e}{\mu}\delta(\mathbf{y}-\mathbf{x})\hat{\varphi}(\mathbf{x}) , \tag{29a}$$

$$[\hat{\omega}(\mathbf{x}),\hat{B}(\mathbf{y})]_-=-\frac{e}{\mu}\delta(\mathbf{y}-\mathbf{x})\hat{\omega}(\mathbf{x}) , \tag{29b}$$

where we have defined the magnetic field operator in the anyonized system as $\hat{B}(\mathbf{y})=-\epsilon^{ki}\partial_k X_i^T$ and used the relation Eq. (7). Equation (29) gives us

$$\begin{aligned}&[\hat{\varphi}(\mathbf{x}),\hat{\Phi}]_-=\frac{e}{\mu}\hat{\varphi}(\mathbf{x}) ,\\&[\hat{\omega}(\mathbf{x}),\hat{\Phi}]_-=-\frac{e}{\mu}\hat{\omega}(\mathbf{x}) ,\end{aligned}$$

where $\hat{\Phi}=-\int d^2y\,\epsilon^{ij}\partial_i X_j^T$. Equation (29) means that the anyonized field variables $\hat{\varphi}(\mathbf{x})$ and $\hat{\omega}(\mathbf{x})$ have local magnetic flux $-(e/\mu)\delta(\mathbf{y}-\mathbf{x})$ and $(e/\mu)\delta(\mathbf{y}-\mathbf{x})$, respectively. We can say that the phase factor appeared in the anyonic commutation relations between the matter fields are induced by a *local* Aharonov-Bohm effect that is due to these local magnetic flux attached to the anyonized matter field variables.

V. CONCLUSIONS

In conclusion, we have studied $(2+1)$-dimensional U(1) gauge theories, which have both the Maxwell term and the Chern-Simons topological term as the action of the gauge field and are coupled to the generic matter field. By using the symplectic geometric method, we have constructed the generalized Hamiltonian formalism and have quantized the system. The Gauss-law constraint has been solved and the longitudinal part of the vector potential has been represented by the matter variables using the multivalued function. We have changed the basis of the field operators so as the multivalued function has been eliminated. Then the commutation relation of the transformed system has become anyonic. Thus the anyonization has been accomplished. A transverse part of the gauge field has remained dynamically. From the commutation relations between the gauge field and the matter fields, it has been shown that the anyonic matter fields have the intrinsic flux. Thus the appearance of the exotic statistics has its origin in the local Aharonov-Bohm effect due to these flux attached to the anyonized matter fields.

In our previous paper,[19] we showed that the Chern-Simons U(1) gauge theories, which have only the Chern-Simons term as the action of the gauge field and include the generic matter field, can be anyonized. In this paper, we have shown that the Maxwell-Chern-Simons theories, in which the gauge field has both the Maxwell and Chern-Simons terms as the action and is coupled to the generic matter field, can be anyonized. The Chern-Simons term is the first-derivative term and Maxwell term is the second-derivative term, so that the canonical structure of these theories is vastly different. After the careful analyses, we found that in both theories, the statistics transmutation can occur. Intuitively saying, the mechanism of the statistics transmutation is that the matter field can absorb the effect of the Chern-Simons term by changing the basic of the field operators. This is the essence of the statistics transmutation. More generally, we can anyonize almost all U(1) gauge theories in $2+1$ dimensions by taking the effect of the Chern-Simons term in and out matter fields. Theories with the Chern-Simons term in the action can be transformed to theories without the Chern-Simons term and vice versa.

In the Maxwell-Chern-Simons theories discussed here, the dynamical degree of freedom of the gauge field remains. It is interesting to study a dynamics of the theories, in particular by using anyonic field operator basis. The description of the anyonic model by using the anyonic field operator basis is important because it will make possible the analysis of the model beyond the mean-field approximation. There are some open questions. How can we construct a Fock representation of the anyonic field operator? Is the anyonic state realized in asymptotic physical states? The anyonic state may be confined. What kind of a bound state appears? These analyses progress now and will appear elsewhere separately.

In the practical application of quantum field theories containing the Chern-Simons term to high-T_c superconductivity, we might consider the theories with the Maxwell term in addition to the Chern-Simons term. It is plausible that the fundamental statistical model may induce the second-derivative term as the next to the leading contribution in the long-wavelength limit. So, it is important that we have shown the statistics transmutation can occur in the Maxwell-Chern-Simons theories. We hope that the general argument presented here will encourage investigations based on the more concrete models.

Finally, it seems that at present we encounter a new type of quantum field theory, which is the anyonic quantum field theory. We need more deep studies in order to understand what it is.

Note added. After completing this work, I was informed of closely related work.[23] I would like to thank Professor G. W. Semenoff for sending me their paper and giving me encouraging comments on my work.

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