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# Squeezed quantum states of relic gravitons and primordial density fluctuations

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The close relationship between the theory of particle creation in external fields and the theory of quantum-mechanical squeezed states is clarified. It is shown that relic gravitons (and other primordial perturbations), created from zero-point quantum fluctuations in the course of cosmological evolution, should now be in strongly squeezed states. The statistical properties of the stochastic collection of relic gravitational waves are investigated. Some other examples of particle creation, and in particular Hawking's process of black-hole evaporation, are considered in the context of the theory of squeezed states.

## I. INTRODUCTION

Relic gravitons can be created from zero-point quantum fluctuations of the gravitational field in the course of cosmological expansion.[1] They may provide extremely valuable information on the physical conditions in the very early Universe (see Ref. 2 for recent reviews). Until recently, only quasiclassical characteristics of relic gravitational waves, such as their spectral energy density, have been discussed. The primary concern of the present paper is a very important new feature of the created particles. It will be shown that they must exist now in specific quantum states known in quantum optics and measurement theory[3] as squeezed quantum states (for a review of squeezed states, see, for example, Ref. 4). Generally speaking, one can say that the variable gravitational field of the cosmological evolution is a "phase-sensitive amplifier which squeezed the vacuum."

Usually, it is said that relic gravitational waves should form a stochastic collection of waves with randomly distributed amplitudes and phases. The important feature of this stochastic background of relic gravitational waves, attributed to the phenomenon of squeezing, is that the variances in the amplitude distribution are very large, while the variances in the phase distribution are practically equal to zero. As a result, one must now deal with a collection of standing waves rather than with a collection of traveling waves. The discovered property applies equally well to the states produced from zero-point quantum fluctuations of other fields, say, a massive scalar field. It is believed that the primordial density fluctuations, which have led to the observed large-scale structure in the Universe, may have been produced by the same process of amplifying ("squeezing") the zero-point quantum fluctuations of some scalar field.

The squeezed vacuum states under discussion are the many-particle quantum states. The mean number of the given particles in this state is much larger than 1. From this point of view the resulting field can be called classical or, better to say, macroscopical. However, the statistical properties of this field differ greatly from those corresponding to the coherent quantum state (in a sense, most classical of all possible quantum states) with the same mean number of particles. From this point of view the produced field is highly nonclassical. The actual statistical properties of the produced fluctuations can be revealed observationally. In the case of gravitational waves, this may allow us to distinguish the relic stochastic background from other sources of stochastic waves, such as the huge number of binary stars which independently emit overlapping gravitational waves.

Production of relic gravitons and primordial density fluctuations is covered by the theory known as particle creation in external fields.[5] A seemingly unrelated subject is the quantum-mechanical theory of squeezed states and, in particular, quantum optics. However, it turns out that these two areas of research are intimately related. Their mathematical formalism and physical concepts are very similar. Yet there is a difference in final results. It is known how much experimental effort is required to achieve a modest squeezing in the case of light generated under laboratory conditions. In contrast with this, the squeezed relic gravitational waves are produced, in a sense, for free and with a much greater amount of squeezing. Unfortunately, the electromagnetic waves cannot be squeezed in the course of cosmological expansion in a similar way since they do not interact with the external gravitational field in the same manner as the gravitational waves do.

The theory of squeezed quantum states is well developed. One of the motivations of the present paper is to show that every case in the classification of the squeezed states has a counterpart in the theory of particle creation. In the case of gravitons (Sec. III) we encounter one- and two-mode squeezed states with the same squeezing parameters. For completeness we will also consider here a pair of scalar fields which yield the most general squeezed states (Appendix B). The similarity between particle creation and squeezing (Sec. II) goes even further. It applies not only to the time-variable gravitational fields, such as cosmological expansion, but also to such geometries as black holes. From this perspective we will briefly consider (Appendix C) Hawking's well-known process of black-hole evaporation.[6]

The paper is organized in such a way that the detailed discussion is present only for the gravity-wave equations in the Friedmann-Robertson-Walker (FRW) cosmological background. All the subtleties of the theory are illustrated for this case. However, the theory can be easily generalized to other cases, some of which are explicitly spelled out below.

## II. PARTICLE CREATION VERSUS SQUEEZING

We begin from the simplest FRW metric

$$ds^2=l_P^2\sigma^2a^2(\eta)[d\eta^2-(dx^1)^2-(dx^2)^2-(dx^3)^2]\ , \quad (1)$$

where $a(\eta)$ is the dimensionless scale factor. For easier comparison with the quantum-cosmological treatment of the spatially finite geometries, we have introduced into Eq. (1) the Planck length $l_P$ and the dimensionless normalizing constant $\sigma$, $\sigma^2=4\pi/3v$, $v=\int d^3x$.[7] The volume $v$ can be finite even for the metric (1) due to a nontrivial topology. Each polarization component $h(\eta,\mathbf{x})$ of the classical gravity-wave field satisfies the curved-space-time D'Alambert equation

$$\Box h(\eta,\mathbf{x})=0\ . \quad (2)$$

The general solution to Eq. (2) can be presented as the sum over the independent $h_{\mathbf{n}}$-mode functions, $\mathbf{n}=(n^1,n^2,n^3)$:

$$h(\eta,\mathbf{x})=\sum_{\mathbf{n}} h_{\mathbf{n}}\ ,$$
$$h_{\mathbf{n}}=\frac{1}{a}\mu_{\mathbf{n}}(\eta)U_{\mathbf{n}}(\mathbf{x})\ . \quad (3)$$

(We will make this presentation more accurate later on.) For the time-dependent functions $\mu_{\mathbf{n}}(\eta)$, one obtains from Eq. (2) the second-order differential equation[1] (index $\mathbf{n}$ is omitted)

$$\mu''+[n^2-V(\eta)]\mu=0\ , \quad (4)$$

where $V(\eta)\equiv a''/a$, $n^2=(n^1)^2+(n^2)^2+(n^3)^2$, and a prime denotes $d/d\eta$. Let us first analyze Eq. (4) which determines the temporal dependence of $h(\eta,\mathbf{x})$.

### A. Temporal dependence

The function $h(\eta,\mathbf{x})$ must be real, but the functions $\mu(\eta)$ and $U(\mathbf{x})$ may be complex. The general complex solution to Eq. (4) has the form

$$\mu=a\xi+b^\dagger\xi^*\ , \quad (5a)$$

where $\xi,\xi^*$ are the complex-conjugated linearly independent base functions, and $a,b^\dagger$ are arbitrary complex constants. (For the time being the dagger means the complex conjugation, but later on, for the operator-valued coefficients, the dagger will mean the Hermitian conjugation.) The same general solution can be decomposed over other base functions $\chi$ and $\chi^*$:

$$\mu=c\chi+d^\dagger\chi^*\ . \quad (5b)$$

Since (5a) and (5b) represent the same solution, their coefficients are related:

$$a=uc+vd^\dagger\ ,$$
$$b^\dagger=wc+zd^\dagger\ , \quad (6)$$

where the complex numbers $u$, $v$, $w$, and $z$ satisfy the conditions

$$u=z^*,\quad v=w^*\ ,$$
$$u^*u-v^*v=z^*z-w^*w=\frac{W(\chi,\chi^*)}{W(\xi,\xi^*)}\ , \quad (7)$$

and $W(f,g)\equiv f'g-fg'$ is the Wronski determinant. Relations (6) and (7) can be verified by considering (5a) and (5b) at arbitrary point $\eta$ and by joining $\mu(\eta)$ and $\mu'(\eta)$ at that point continuously.

Below we will use the normalized basic solutions, such that

$$W(\xi,\xi^*)=W(\chi,\chi^*)=-i\ .$$

This allows us to introduce the new parameters $r$, $\varphi$, and $\theta$:

$$u=e^{-i\theta}\cosh r,\quad v=-e^{-i(\theta-2\varphi)}\sinh r\ ,$$
$$w=-e^{i(\theta-2\varphi)}\sinh r,\quad z=e^{i\theta}\cosh r\ , \quad (8)$$

where $r$, $\varphi$, and $\theta$ are real numbers, and $r\geq 0$.

Until now we were considering the classical solutions to the classical Eq. (4). In the quantum theory the same equation governs the operator-valued function $\mu(\eta)$. The complex coefficients $a$, $b^\dagger$, $c$, and $d^\dagger$ become the operators satisfying the standard commutation relations

$$[a,a^\dagger]=[b,b^\dagger]=[c,c^\dagger]=[d,d^\dagger]=1\ ,$$

with all other commutators equal to zero.

Let the potential $V(\eta)$ vanish asymptotically for $\eta\rightarrow-\infty$ (in region) and for $\eta\rightarrow+\infty$ (out region). The basic solutions $\xi\equiv\xi_{\rm in}$ and $\chi\equiv\xi_{\rm out}$, valid for all $\eta$, can be chosen in such a way that

$$\xi_{\rm in}(\eta)\rightarrow\frac{1}{\sqrt{2n}}e^{-in\eta}\quad\text{for }\eta\rightarrow-\infty\ ,$$

and

$$\xi_{\rm out}(\eta)\rightarrow\frac{1}{\sqrt{2n}}e^{-in\eta}\quad\text{for }\eta\rightarrow+\infty\ .$$

Because of this choice (see, for example, Ref. 5) the operators $(a,b)$ and $(a^\dagger,b^\dagger)$ can be interpreted as the annihilation and creation operators for in particles $a$ and $b$. Similarly, the operators $(c,d)$ and $(c^\dagger,d^\dagger)$ can be interpreted as the annihilation and creation operators for out particles $c$ and $d$. Since the positive- and negative-frequency solutions in in and out regions are defined with respect to the same time parameter $\eta$, the in and out particles are indistinguishable in their physical properties (not in their total particle numbers, of course); that is, $c$

particles are the same as $a$ particles, and $d$ particles are the same as $b$ particles.

The operator-valued relations (6) are called the Bogoliubov transformations. They are the primary concept in the theory of particle creation. On the other hand, by taking into account Eq. (8), one can rewrite Eq. (6):

$$a=R^{\dagger}S^{\dagger}cSR \; , \quad b^{\dagger}=R^{\dagger}S^{\dagger}d^{\dagger}SR \; , \tag{9}$$

where $S$ and $R$ are the unitary operators

$$S(r,\varphi)=\exp[r(e^{-2i\varphi}cd-e^{2i\varphi}c^{\dagger}d^{\dagger})] \; ,$$

$$R(\theta_1,\theta_2)=\exp(-i\theta_1c^{\dagger}c-i\theta_2d^{\dagger}d), \quad \theta_1=\theta_2=\theta \; .$$

In the theory of squeezed quantum states, the operator $S(r,\varphi)$ is called the two-mode squeeze operator and the operator $R(\theta_1,\theta_2)$ is called the two-mode rotation operator.[4]

Let us consider a quantum state $|\phi_{\rm in}\rangle$ and a function of operator arguments $F(a,a^{\dagger},b,b^{\dagger})$. One can find the mean value of $F$ and show, with the help of (9), that

$$\langle\phi_{\rm in}|F(a,a^{\dagger},b,b^{\dagger})|\phi_{\rm in}\rangle=\langle\phi_{\rm in}|R^{\dagger}S^{\dagger}F(c,c^{\dagger},d,d^{\dagger})SR\,|\phi_{\rm in}\rangle$$
$$=\langle\phi_{\rm out}|F(c,c^{\dagger},d,d^{\dagger})|\phi_{\rm out}\rangle \; ,$$

where

$$|\phi_{\rm out}\rangle=S(r,\varphi)R(\theta_1,\theta_2)|\phi_{\rm in}\rangle \; . \tag{10}$$

Since the in particles are indistinguishable from the out particles, one can view Eq. (10) as a result of transforming the initial quantum state into the final quantum state in the process of evolution from in region to out region. In particular, the initial vacuum state, $|0,0\rangle$ for $a$ and $b$ particles, transforms into a two-mode squeezed state $|SS\rangle_2$:

$$|SS\rangle_2=S(r,\varphi)|0,0\rangle \; .$$

Equation (10) can also be obtained from the Schrödinger equation in the following way. The equations of motion (4) can be derived from the Lagrange function

$$L=\mu^{*\prime}\mu'-\frac{a'}{a}(\mu'\mu^*+\mu^{*\prime}\mu)+\left[\left(\frac{a'}{a}\right)^2-n^2\right]\mu^*\mu \; .$$

The associated Hamiltonian is

$$H=p^*p+\frac{a'}{2a}(\mu p+p\mu+\mu^*p^*+p^*\mu^*)+n^2\mu^*\mu \; ,$$

where $p\equiv\partial L/\partial\mu'$. In a standard manner, $H$ can be presented in terms of creation and annihilation operators. The solution to the Schrödinger equation

$$i\frac{d|\phi\rangle}{d\eta}=H|\phi\rangle \; ,$$

with the Hamiltonian derived above, is given by formula (10).

The notion of the two-mode squeeze operator has appeared because we were using the complex solutions to Eq. (4), $\mu^*\neq\mu$. If we confine ourselves, from the very beginning, to the real solutions $\mu^*=\mu$, the notion of the one-mode squeeze operator appears. In this case one deals with only one sort of particles, since $a=b$ and $c=d$. Instead of (9), one will have

$$a=R_1^{\dagger}S_1^{\dagger}cS_1R_1 \; ,$$

where

$$S_1(r,\varphi)=\exp\left[\frac{r}{2}[e^{-2i\varphi}c^2-e^{2i\varphi}(c^{\dagger})^2]\right]$$

is the one-mode squeeze operator, and

$$R_1(\theta)=\exp(-i\theta c^{\dagger}c)$$

is the one-mode rotation operator.[4] Further on, instead of (10), one will have

$$|\phi_{\rm out}\rangle=S_1(r,\varphi)R_1(\theta)|\phi_{\rm in}\rangle \; , \tag{11}$$

so that the initial vacuum state $|0\rangle$ transforms into a one-mode squeezed state $|SS\rangle_1$:

$$|SS\rangle_1=S_1(r,\varphi)|0\rangle \; .$$

Equation (11) can also be derived as a solution to the Schrödinger equation. In the case of the real $\mu$ field, the equations of motion (4) follow from the Lagrange function

$$L=\frac{1}{2}\left[(\mu')^2-\frac{a'}{a}(\mu'\mu+\mu\mu')+\left(\frac{a'}{a}\right)^2\mu^2-n^2\mu^2\right] \; .$$

The associated Hamiltonian

$$H=\frac{1}{2}\left[p^2+\frac{a'}{a}(\mu p+p\mu)+n^2\mu^2\right] \tag{12}$$

defines the Schrödinger equation and its solution (11).

The actual transition from the complex solutions of Eq. (4) to the real ones (marked by index $r$) can be performed as follows. One takes two complex-conjugated solutions $\mu_1=a_1\xi+b_1^{\dagger}\xi^*$ and $\mu_2=a_2\xi+b_2^{\dagger}\xi^*$, where $\mu_1^*=\mu_2$; that is, $a_2=b_1$, $b_2^{\dagger}=a_1^{\dagger}$. Then one constructs two real solutions $\mu_1^r=a_1^r\xi+(b_1^{\dagger})^r\xi^*$ and $\mu_2^r=a_2^r\xi+(b_2^{\dagger})^r\xi^*$ according to the rule

$$\mu_1^r=\frac{1}{\sqrt{2}}(\mu_1+\mu_2), \quad \mu_2^r=\frac{i}{\sqrt{2}}(\mu_1-\mu_2) \; . \tag{13}$$

This transformation generates the transformation between the annihilation and creation operators associated with the complex and real solutions:

$$a_1^r=\frac{1}{\sqrt{2}}(a_1+a_2), \quad a_2^r=\frac{i}{\sqrt{2}}(a_1-a_2) \; ,$$
$$(b_1^{\dagger})^r=\frac{1}{\sqrt{2}}(b_1^{\dagger}+b_2^{\dagger}), \quad (b_2^{\dagger})^r=\frac{i}{\sqrt{2}}(b_1^{\dagger}-b_2^{\dagger}) \; . \tag{14}$$

By using (14) one can see that the operator $S(r,\varphi)$ factors into the product of two identical operators $S_1(r,\varphi)$, and as a consequence, formula (11) follows from (10). From the standpoint of squeezed states theory, transformation (14) is generated by the mixing operator $T$.[4,8]

The physical meaning of the complex and real solutions and their associated operators will be better seen when the spatial dependence of $h(\eta,\mathbf{x})$ is taken into account. It will be shown that the complex solutions correspond to the decomposition of the field over traveling waves, whereas the real solution are over standing waves. Both are equally good.

### B. Spatial dependence

It follows from Eq. (2) that every spatial function $U_{\mathbf{n}}(\mathbf{x})$ satisfies the Laplace equation

$$\Delta U_{\mathbf{n}}+n^2U_{\mathbf{n}}=0 \ . \tag{15}$$

One can use either complex solutions to this equation, such as $U_{\mathbf{n},1}\sim e^{i\mathbf{n}\cdot\mathbf{x}}$, $U_{\mathbf{n},2}\sim e^{-i\mathbf{n}\cdot\mathbf{x}}$, or real solutions, such as $U^r_{\mathbf{n},1}\sim\cos\mathbf{n}\cdot\mathbf{x}$, $U^r_{\mathbf{n},2}\sim\sin\mathbf{n}\cdot\mathbf{x}$. (We will mark two linearly independent components of $U_{\mathbf{n}}$ by index $J$: $U_{\mathbf{n},J}$, $J=1,2$.) The most general representation for every $h_{\mathbf{n}}$ component is

$$h_{\mathbf{n}}=\frac{1}{a}(\mu_{\mathbf{n},1}U_{\mathbf{n},1}+\mu_{\mathbf{n},2}U_{\mathbf{n},2}) \ . \tag{16}$$

Since $h_{\mathbf{n}}$ must be real, one has $U^*_{\mathbf{n},2}=U_{\mathbf{n},1}$ and $\mu^*_{\mathbf{n},2}=\mu_{\mathbf{n},1}$. Another possibility is to have all the functions real, so that

$$h_{\mathbf{n}}=\frac{1}{a}(\mu^r_{\mathbf{n},1}U^r_{\mathbf{n},1}+\mu^r_{\mathbf{n},2}U^r_{\mathbf{n},2}) \ . \tag{17}$$

We will start from the complex solutions to Eq. (15). One can choose a complete set of the normalized functions

$$U_{\mathbf{n},1}=Ke^{i\mathbf{n}\cdot\mathbf{x}}, \quad U_{\mathbf{n},2}=Ke^{-i\mathbf{n}\cdot\mathbf{x}} \ ,$$

where the constant $K$ is equal to $K=(2v)^{-1/2}$ for the finite volume $v$ and $K=(2\pi)^{-1/2}$ for the infinite volume. If the temporal dependence in an asymptotic region is chosen as

$$\mu_1=\frac{1}{\sqrt{2n}}(a_1e^{-in\eta}+b_1^\dagger e^{in\eta}) \ ,$$

$$\mu_2=\frac{1}{\sqrt{2n}}(a_2e^{-in\eta}+b_2^\dagger e^{in\eta}), \quad \mu_2^*=\mu_1 \ ,$$

then $a_1=b_2$ is the annihilation operator for a particle with the momentum $\mathbf{n}$, $b_2^\dagger=a_1^\dagger$ is the creation operator for a particle with the same momentum $\mathbf{n}$, and $a_2=b_1$ and $b_1^\dagger=a_2^\dagger$ are annihilation and creation operators for a particle with the opposite momentum $-\mathbf{n}$. In terms of classical waves, Eq. (16) presents a decomposition over traveling waves.

The general solution to Eq. (2) can be written in the form

$$h(\eta,\mathbf{x})=\frac{1}{a(\eta)}{\sum_{\mathbf{n}}}'\sum_{J=1}^{2}\mu_{\mathbf{n},J}(\eta)U_{\mathbf{n},J}(\mathbf{x}) \ , \tag{18}$$

where $\sum'_{\mathbf{n}}$ means that the summation (integration) extends over half of the space of wave vectors $\mathbf{n}$, for instance, over a region $\mathbf{n}\cdot\mathbf{x}\geq 0$, in order not to count the components twice.

The transition to the real solutions is performed by the transformation

$$U^r_{\mathbf{n},1}=\frac{1}{\sqrt{2}}(U_{\mathbf{n},1}+U_{\mathbf{n},2}) \ ,$$

$$U^r_{\mathbf{n},2}=\frac{i}{\sqrt{2}}(U_{\mathbf{n},2}-U_{\mathbf{n},1}) \ ,$$

along with transformation (13). In this case one has, for every $\mathbf{n}$,

$$\mu_1^r=a_1^r\xi+(a_1^\dagger)^r\xi^*, \quad \mu_2^r=a_2^r+(a_2^\dagger)^r\xi^* \ ,$$

$$U_1^r=K\sqrt{2}\cos\mathbf{n}\cdot\mathbf{x}, \quad U_2^r=K\sqrt{2}\sin\mathbf{n}\cdot\mathbf{x} \ .$$

Formula (17) corresponds to the decomposition of the field over standing waves.

Thus one encounters a two-mode operator or a pair of one-mode squeeze operators, depending on whether one associates the particles with a pair of traveling waves or a pair of standing waves. Now we will consider in detail the graviton production in an expanding universe.

## III. SQUEEZED QUANTUM STATES OF RELIC GRAVITONS

It was shown in Sec. II that squeezed quantum states arise inescapably whenever particle creation occurs. The actual values of the squeeze parameters depend on a concrete model.

### A. Squeezing in an inflationary model

We will consider a simple cosmological model which includes the inflationary ($i$), radiation-dominated ($e$), and matter-dominated ($m$) epochs. The scale factor $a(\eta)$ has the following $\eta$-time dependence:

$$a_i=-(\kappa\eta)^{-1}, \quad \eta_b\leq\eta\leq\eta_1, \quad \eta<0 \ ,$$

$$a_e=+(\kappa\eta_1^2)^{-1}(\eta-2\eta_1), \quad \eta_1\leq\eta\leq\eta_2 \ ,$$

$$a_m=[4\kappa\eta_1^2(\eta_2-2\eta_1)]^{-1}(\eta+\eta_2-4\eta_1)^2, \quad \eta\geq\eta_2 \ ,$$

where $\kappa$ determines the Hubble constant at the inflationary stage. It is assumed that the total duration of expansion from some $\eta_b$ until the present time $\eta_0$ is sufficiently long, so that the scale of order of Planck's length has increased up to, at least, the present-day Hubble radius.

Without any loss of generality we will use the presentation (17) so that, for every $(\mathbf{n},J)$ mode, the initial vacuum state will go over into a one-mode squeezed state. This transition is governed by the Schrödinger equation

$$i\frac{d\phi}{d\eta}=H\phi \ , \tag{19}$$

where $\phi$ stands for every $(\mathbf{n},J)$-mode wave function and $H$ is the Hamiltonian (12).

It is known that the squeezed-state wave functions are always Gaussian, so that one can seek a solution to Eq. (19) in the form

$$\phi(\eta,\mu)=c(\eta)e^{-B(\eta)\mu^2} \ . \tag{20}$$

[In order to avoid confusion we want to emphasize that in our earlier paper[9] we were using the coordinate

$y=a^{-1}\mu$ and the function $B(\eta)$, which was equal to the factor $a^2$ times the $B(\eta)$ used here.] By substituting (20) into (19) one can show that the function $B(\eta)$ satisfies the equation $B'=-2(a'/a)B-2iB^2+(i/2)n^2$ and can be expressed in terms of solutions to Eq. (4) according to the relation

$$B(\eta)=-\frac{i}{2}\frac{(\mu/a)'}{\mu/a} .$$

The initial vacuum state corresponds to the wave function (20) with $B=n/2$. This means that one will have to choose a solution $\mu(\eta)$ that satisfies the relation $(\mu/a)'=in(\mu/a)$ initially, for $\eta\to-\infty$ or $\eta=\eta_b$.

The function $B(\eta)$ determines completely the squeeze parameter $r(\eta)$ and the squeeze angle $\varphi(\eta)$ through the relation[4]

$$B(\eta)=\frac{n}{2}\frac{\cosh r+e^{2i\varphi}\sinh r}{\cosh r-e^{2i\varphi}\sinh r} . \tag{21}$$

[One should note that $r$ and $\varphi$ are functions of time here, whereas they are constants as discussed in Sec. II. This is because in Sec. II we were interested just in the asymptotic regions $\eta\to\pm\infty$, where the positive- and negative-frequency solutions were defined. Here we are interested in the gradual evolution of the wave function that corresponds to defining the positive- and negative-frequency solutions at every current $\eta$. As a result, the parameters $r$ and $\varphi$ enter the general formulas as functions of time. However, $r(\eta)$ and $\varphi(\eta)$ approach some definite and unambiguous values in asymptotic regions $\eta\to\pm\infty$, which we are interested in. In particular, for waves much shorter than the present-day Hubble radius, the squeeze parameter $r$ has practically reached its asymptotic constant value.] In order to calculate $r(\eta)$ and $\varphi(\eta)$ explicitly, one has to use the solution $\mu(\eta)=\gamma^{(1)}\xi+\gamma^{(2)}\xi^*$ with the appropriate initial conditions. The exact solution for the base functions $\xi(\eta)$ and $\xi^*(\eta)$ has the form[9]

$$\xi=\left[1-\frac{iq}{n(\eta+\theta)}\right]e^{-in(\eta+\theta)} ,$$

at all three stages $(i)$, $(e)$, and $(m)$, where, at the $i$ stage, $q_i=1$ and $\theta_i=0$, at the $e$ stage, $q_e=0$ and $\theta_e=-2n_1$, and, at the $m$ stage, $q_m=1$ and $\theta_m=\eta_2-4\eta_1$. The squeeze parameter $r(\eta_0)$ was explicitly calculated in Ref. 9 for a reasonable cosmological model. This parameter depends on $n$; that is, it varies with the frequency. It was shown that $r$ increases from $r\approx1$ and up to $r\approx10^2$ for waves with the present-day frequencies ranging from $\nu\approx10^8$ Hz up to $\nu\approx10^{-16}$ Hz. These are the waves which were amplified at the $i$ stage only. For waves which were additionally amplified at the $m$ stage, that is, for waves with frequencies $\nu\approx10^{-16}$–$10^{-18}$ Hz, $r$ is further increased and reaches $r\approx1.2\times10^2$ at $\nu\approx10^{-18}$ Hz.

The easiest way to find the squeeze angle $\varphi(\eta)$ is just to compare $B(\eta)$ from Eq. (21) with the asymptotic behavior of $B(\eta)$ (for $r\to\infty$) (Ref. 9):

$$B\approx\frac{n}{2}\frac{1-\delta_m e^{-2in(\eta+\eta_2-4\eta_1)}}{1+\delta_m e^{-2in(\eta+\eta_2-4\eta_1)}} ,$$

where $\delta_m\approx-e^{-2in(\eta_2-2\eta_1)}$ for $\nu\approx10^8$–$10^{-16}$ Hz and $\delta_m\approx1$ for $\nu\approx10^{-16}$–$10^{-18}$ Hz. For these two frequency intervals one obtains $\varphi\approx-n\eta-2n(\eta_2-3\eta_1)$ and $\varphi\approx-n\eta-n(\eta_2-4\eta_1)$. In both cases one has

$$\varphi\approx-n\eta+\varphi_0 , \tag{22}$$

where $\varphi_0=$const and the first term, $\sim n\eta$, just reflects the free evolution of the squeezed state. The constant $\varphi_0$ is the same for both $J=1,2$, and for all unit vectors $\mathbf{n}/n$.

Since the mean number of particles in every mode is determined by the relation $\langle N\rangle=\sinh^2 r$, one can see that $\langle N\rangle\gg1$ for all modes with frequencies lower than $\nu\approx10^8$ Hz. It means that we will be dealing with the many-particle states, and in this sense the generated field is classical (macroscopical). However, its statistical properties are very unusual, as we will now see.

### B. Amplitude and phase fluctuations

Our next goal is to relate the rigorous quantum-mechanical treatment developed above with classical description based on the notion of the classical waves with randomly distributed amplitudes and phases. In the high-frequency limit $n\eta\gg1$, every classical mode function $\mu_{\mathbf{n},J}(\eta)$ can be presented in the form

$$\mu=A\sin(-n\eta+\phi) , \tag{23}$$

where $A$ and $\phi$ are randomly distributed numbers. Their actual mean values and variances depend on a specific quantum state.

In order to relate the quantum and classical descriptions, it is convenient to use the Wigner function. This is a function of the phase-space variables $(\mu,p)$ and time $\eta$ and is defined as[10]

$$P_w(\eta,\mu,p)=\frac{1}{\pi}\int_{-\infty}^{\infty}\phi^*(\mu+\epsilon)\phi(\mu-\epsilon)e^{2ip\epsilon}d\epsilon . \tag{24}$$

For the quantum-mechanical problem at hand [the variable mass and elasticity oscillator governed by the Hamiltonian (12)], $P_w(\eta,\mu,p)$ satisfies the Liouville equation

$$\frac{\partial P_w}{\partial\eta}=-\left[p+\frac{a'}{a}\mu\right]\frac{\partial P_w}{\partial\mu}+\left[n^2\mu+\frac{a'}{a}p\right]\frac{\partial P_w}{\partial p} .$$

The Wigner function $P_w$ can be interpreted as the probability distribution for $\mu$ and $p$.

It is instructive to first recall the form of the function $P_w$ for a coherent state:

$$P_w=\left[\frac{n}{\pi}\right]^{1/2}\exp\left\{-n\left[\mu-\left[\frac{2}{n}\right]^{1/2}\mathrm{Re}\alpha\right]^2\right\}\times\frac{1}{\sqrt{\pi n}}\exp[-(p-\sqrt{2n}\ \mathrm{Im}\alpha)^2n^{-1}] ,$$

where $\alpha$ is a complex number and $\langle N\rangle=\alpha^*\alpha$. It is clearly seen that $\mu$ and $p$ obey the Gaussian distribution law. The variances of the (dimensionless) $\mu n^{1/2}$ and $pn^{-1/2}$ are equal and their product is minimal. The same is true for the variances of the amplitude and phase distributions. The quantum "noise" of a coherent state can be visualized as a circle. In contrast with that, the quantum "noise" of a squeezed state can be described by an ellipse. It is precisely the phase variances that will be severely "squeezed" in our case under consideration.

Let us substitute the wave function (20) into Eq. (24). One will get (see also Ref. 11)

$$P_w(\eta,\mu,p)=\phi^*(\eta,\mu)\phi(\eta,\mu)F(\eta,\mu,p) ,$$

where

$$F(\eta,\mu,p)=(2\pi\,\mathrm{Re}B)^{-1/2} \times\exp[-(2\,\mathrm{Re}B)^{-1}(2\,\mathrm{Im}B\mu+p)^2] .$$

We are interested in the limit $r\to\infty$. From (21) we see that $\mathrm{Re}B\to 0$, $\mathrm{Im}B\to i(n/2)\cot\varphi$ in this limit. Hence one obtains the remarkable δ-function-like behavior for $F$:

$$F\to\delta(2\,\mathrm{Im}B\mu+p)\approx\delta(n\mu\cot\varphi+\mu') . \quad (25)$$

Taking into account (22) and (23) when analyzing the argument of the δ function, one arrives at the conclusion that the distribution of the random variable $\phi$ is highly "squeezed" in the δ-function manner near the values

$$\phi=\varphi_0+\pi l,\quad l=0,\pm 1,\ldots . \quad (26)$$

The same conclusion can be reached by direct computation of the mean values associated with the phase operator $\hat{\phi}$ defined as[12]

$$e^{i\hat{\phi}}=(a^\dagger a+1)^{-1/2}a,\quad e^{-i\hat{\phi}}=a^\dagger(a^\dagger a+1)^{-1/2} .$$

The mean values should be computed with respect to the one-mode squeezed state with the parameters $r$ and $\varphi$:

$$|SS\rangle_1=S_1(r,\varphi)|0\rangle =(\cosh r)^{-1/2}\sum_{m=0}^{\infty}\frac{\sqrt{(2m)!}}{m!}\left[-\frac{1}{2}\tanh r\right]^m \times e^{2im\varphi}|2m\rangle ,$$

where $|m\rangle$ are the particle-number eigenstates (the Fock basis).

One can show that

$$\langle(\cos\hat{\phi})^{2k+1}\rangle=\langle(\sin\hat{\phi})^{2k+1}\rangle=0,\quad k=0,1,\ldots,$$

$$\langle(\cos\hat{\phi})^{2k}\rangle\xrightarrow[r\to\infty]{}(\sin\varphi)^{2k} ,$$

$$\langle(\sin\hat{\phi})^{2k}\rangle\xrightarrow[r\to\infty]{}(\cos\varphi)^{2k} .$$

These numbers define the distribution of the random variable $\phi$, which shows up in the classical expression $\mu=A\cos\phi$. Combining these formulas leads us again to (23) and (26).

In a similar way the distribution of the operator amplitude $\hat{A}=(\mu^2+n^{-2}p^2)^{1/2}$ can be computed. One can show that

$$\langle(\hat{A})^{2k+1}\rangle=0 ,$$

$$\langle(\hat{A})^{2k}\rangle=\left[\frac{1+4n^{-2}(\mathrm{Im}B)^2}{2\,\mathrm{Re}B}\right]^k\frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})},\quad (r\to\infty).$$

The mean value of $A$ is zero. The variance of $A$ can be easily related to the mean number of particles $\langle N\rangle$ and to the characteristic gravity-wave amplitude $h$ for a given mode.

In concluding this section we would like to comment on the δ-function behavior of formula (25). One can see that for a fixed time $\eta$ the Wigner function is sharply peaked near a line in the phase space $(\mu,p)$. The factor $\phi^*(\mu)\phi(\mu)$ gives the probability distribution for the ponts on this line. Every point generates a classical trajectory when time evolution is taken into account. The set of these trajectories forms a two-dimensional surface in the three-dimensional space $(\mu,p,\eta)$ (in this context, see also Ref. 13). This behavior is totally different from a coherent-state case where a "pencil" of trajectories is singled out.

### C. Relic gravitons: A stochastic collection of standing waves

We have studied the time-dependent components $\mu^r_{\mathbf{n},J}(\eta)$ [Eq. (23)]. Now we can construct the $h_{\mathbf{n}}$ modes [Eq. (17)] and sum them to produce the total field $h(\eta,\mathbf{x})$. Let us first take the sum over all the modes with the same number $n, n^2=\mathbf{n}\cdot\mathbf{n}$:

$$h_n(\eta,\mathbf{x})=\frac{1}{a}\sum_{\mathbf{n}}{}'\sum_J\mu^r_{\mathbf{n},J}(\eta)U^r_{\mathbf{n},J}(\mathbf{x}) .$$

Since for a given $n$ all the phases $\phi_{\mathbf{n},J}$ obey the same condition (26), one can get the time-dependent function from under the summation symbol and write

$$h_n(\eta,\mathbf{x})=\frac{1}{a}\sin(-n\eta+\varphi_0)\sum_{\mathbf{n}}{}'\sum_J A^r_{\mathbf{n},J}U^r_{\mathbf{n},J}(\mathbf{x}) . \quad (27)$$

[This could not have been done if the statistical properties of $\phi_{\mathbf{n},J}$ did not obey (26).] The function $h_n(\eta,\mathbf{x})$ describes a stochastic field in the form of a standing wave. A characteristic feature of Eq. (27) is that $h_n(\eta,\mathbf{x})$ vanishes at some moments of time separated by a half of the period. The spatial pattern is determined by the random coefficients $A^r_{\mathbf{n},J}$ described above. It is worth noting that this pattern does not depend on the kind of spatial base functions $U(\mathbf{x})$ chosen. If one changes to another set of base functions, $U\to\bar{U}$, the random coefficients also change, $A\to\bar{A}$, but their statistical properties remain the same.

The total field $h(\eta,\mathbf{x})$ is obtained by summing over the components (27). Of course, the total field loses the property of vanishing at some moments of time since the various $\sin(-n\eta+\varphi_0)$ factors are shifted with respect to each other in their arguments. However, this shift is not random, but well prescribed, at least for a simple cosmological model considered above. For instance, if at some moment of time $\eta=\eta_0$ the component $h_{n_1}(\eta,\mathbf{x})$ vanishes, then the same is true for all other components $h_{n_2}(\eta,\mathbf{x})$, where $n_2=n_1(1+k/l)$ and $k/l$ is an arbitrary rational number. Perhaps this property can be somehow used in a specific strategy for the observational discrimination of this stochastic gravity-wave field from others.

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## APPENDIX A: CLASSICAL WAVE AMPLIFICATION AND THE PHENOMENON OF SQUEEZING

Classical gravity waves, obeying Eq. (4), amplify in the course of their evolution.[1] Let the gravity-wave field have the form of traveling waves in the in and out regions:

$$h_{\text{in}}=\frac{1}{a_{-\infty}}\frac{1}{\sqrt{2n}}(ae^{-in\eta}+b^{\dagger}e^{in\eta})e^{i\mathbf{n}\cdot\mathbf{x}} ,$$

$$h_{\text{out}}=\frac{1}{a_{+\infty}}\frac{1}{\sqrt{2n}}(ce^{-in\eta}+d^{\dagger}e^{in\eta})e^{i\mathbf{n}\cdot\mathbf{x}} ,$$

where $a_{-\infty}$ and $a_{+\infty}$ are scale factors in the in and out regions, and $\mathbf{n}\cdot\mathbf{x}>0$. Suppose that there is only one wave traveling to the right in the in region; i.e., $b^{\dagger}=b=0$. Thus, in the out region, there will be an amplified wave traveling in the same direction and the "reflected" wave traveling to the left. The amplification coefficient $Z$ is defined as the ratio of the actual amplitude of the transmitted wave to the amplitude the wave would have if it behaved adiabatically,[1] that is,

$$Z_a=\frac{c^{\dagger}c}{a^{\dagger}a},\quad b^{\dagger}b=0 .$$

From relations (6)–(8) one can get

$$Z_a=\cosh^2 r=1+\langle N_a\rangle ,$$

where $r$ is the squeeze parameter of the two-mode squeezed state arising in the quantum treatment, and $\langle N_a\rangle$ is the mean number of the created $a$ particles. A similar conclusion holds for the waves traveling initially to the left, i.e., for $b$ particles:

$$Z_b=\cosh^2 r=1+\langle N_b\rangle,\quad \langle N_b\rangle=\langle N_a\rangle .$$

Thus the classical amplification coefficient determines the squeeze parameter of the second-quantized theory.

The classical equation (4) can also be viewed as the Schrödinger equation for the wave function $\mu(\eta)$, which describes a "particle" having energy $n^2$ and moving in the presence of the external "potential barrier" $V(\eta)$.[1] From this point of view the transition from in to out region has the meaning of the tunneling through the barrier. The tunneling coefficient $D$ can be defined as $D_a=c^{\dagger}c/a^{\dagger}a$, $d^{\dagger}d=0$ for the right-traveling waves in the out region and $D_b=d^{\dagger}d/b^{\dagger}b$, $c^{\dagger}c=0$ for the left-traveling waves in the out region. From Eqs. (6)–(8) one derives, for both cases,

$$D_a=D_b=\frac{1}{\cosh^2 r} .$$

In the limit $r\to\infty$, one has $D\sim e^{-2r}$. Thus, again, the classical equation (4), being interpreted as the Schrödinger equation, sheds some new light on the physical meaning of the squeeze parameter $r$.

## APPENDIX B: PRODUCING THE MOST GENERAL SQUEEZED STATE (AN EXAMPLE)

The most general squeezed state is defined as[4]

$$|SS\rangle=S_1^{(1)}(r_1,\varphi_1)S_1^{(2)}(r_2,\varphi_2)S(r,\varphi)|0\rangle . \tag{B1}$$

We will give an example of the quantum-cosmological problem where this state appears naturally.

Let us consider two coupled massive scalar fields $\phi_1(t),M_1$ and $\phi_2(t),M_2$ at the FRW cosmological background metric (1) with the scale factor $a(t)$. The Lagrangian of the system is

$$L=\int d^3x\sqrt{-g}\left[-\frac{R}{16\pi G}+\frac{1}{2}(\dot\phi_1^2-M_1^2\phi_1^2+\dot\phi_2^2-M_2^2\phi_2^2)-12\kappa R\phi_1\phi_2+T\right] ,$$

where $R$ is the four-curvature scalar, $\kappa$ is the coupling constant, and $T$ is the term describing some matter source which governs the FRW evolution. We assume that this term has a simple form $\int d^3x\sqrt{-g}\,T=\frac{1}{2}V(a)$.

By assuming $l_P=1$ and introducing the dimensionless variables

$$\mu_1(t)=a^{-3/2}v^{1/2}\sigma\phi_1,\quad \mu_2(t)=a^{+3/2}v^{1/2}\sigma\phi_2 ,$$

$$m_1=\sigma^{+1}M_1,\quad m_2=\sigma^{+1}M_2 ,$$

we can reduce $L$ to the form

$$L=-\tfrac{1}{2}a\dot a^2+\tfrac{1}{2}V(a)+l_1+l_2+l_\kappa , \tag{B2}$$

where

$$l_1=\frac{1}{2}\dot\mu_1^2-\frac{1}{2}m_1^2\mu_1^2-\frac{3}{2}\frac{\dot a}{a}\dot\mu_1\mu_1+\frac{9}{8}\left[\frac{\dot a}{a}\right]^2\mu_1^2 ,$$

$l_2$ is obtained from $l_1$ by the replacement $\mu_1\to\mu_2$, $m_1\to m_2$, and

$$l_\kappa=\kappa\left[\left[\frac{\dot a}{a}\right]^2\mu_1\mu_2-\frac{1}{2}\frac{\dot a}{a}(\mu_1\mu_2)^{\cdot}\right] .$$

From the Lagrangian (B2) one can derive the equation of motion for $\mu_1(t)$,

$$\ddot\mu_1+\left[m_1^2-\frac{3}{2}\frac{\ddot a}{a}-\frac{3}{4}\frac{\dot a^2}{a^2}\right]\mu_1-\frac{\kappa}{2}\left[\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}\right]\mu_2=0 ,$$

and the equation of motion for $\mu_2(t)$, which can be obtained from the previous one by the replacement $\mu_1\leftrightarrow\mu_2$, $m_1\leftrightarrow m_2$. These two coupled equations of motion generalize the single Eq. (4).

In a standard way one derives the Hamiltonian of the system:

$$H=-\frac{P^2}{2M}-\frac{1}{2}V(a)+\frac{1}{2}(p_1^2+m_1^2\mu_1^2+p_2^2+m_2^2\mu_2^2) ,$$

where

$$M=a+\frac{\kappa}{a^2}\left[\mu_1\mu_2+\frac{\kappa}{4}(\mu_1^2+\mu_2^2)\right],$$

$$P=p_a+\frac{p_1}{2a}(3\mu_1+\kappa\mu_2)+\frac{p_2}{2a}(3\mu_2+\kappa\mu_1).$$

$P_a$, $p_1$, and $p_2$ are the momenta canonically conjugated to the coordinates $a$, $\mu_1$, and $\mu_2$.

We are interested in quantizing the lowest (homogeneous) mode of the scalar fields $\mu_1$ and $\mu_2$, not the gravitational field $a(t)$ itself. We will also neglect the "back action" of the scalar fields to the background geometry. For these reasons we expand $P^2/2M$ in a series and retain only the term $a^{-1}p_a^2$ [which governs, together with $V(a)$, the quasiclassical behavior of the background geometry] and the terms quadratic in the scalar field variables $\mu$ and $p$. Then the Hamiltonian $H_m$ for the scalar field components acquires the form

$$H_m=\tfrac{1}{2}(p_1^2+m_1^2\mu_1^2+p_2^2+m_2^2\mu_2^2)-\frac{p_a}{2a^2}[3(\mu_1p_1+\mu_2p_2)+\kappa(\mu_1p_2+\mu_2p_1)]+\frac{\kappa p_a^2}{2a^4}\left[\mu_1\mu_2+\frac{\kappa}{4}(\mu_1^2+\mu_2^2)\right],$$

where $p_a\approx -a\dot{a}$. One can check that the equations of motion for $\mu_1$ and $\mu_2$ derived above also follow from $H_m$. Finally, we introduce the creation and annihilation operators

$$\mu_1=\frac{1}{\sqrt{2m_1}}(b_1+b_1^\dagger),\quad p_1=i\left[\frac{m_1}{2}\right]^{1/2}(b_1^\dagger-b_1),$$

and similarly for $b_2^\dagger$ and $b_2$. This allows us to write $H_m$ in the standard form

$$H_m=\omega_1(t)b_1^\dagger b_1+\omega_2(t)b_2^\dagger b_2+f_1(t)b_1^2+f_2(t)b_2^2+g_1(t)b_1b_2+g_2(t)b_1b_2^\dagger+\text{H.c.}, \qquad \text{(B3)}$$

where

$$\omega_1(t)=\frac{1}{2}\left[m_1+\frac{\kappa^2p_a^2}{8a^4m_1}\right],$$

$$f_1(t)=\frac{\kappa^2p_a^2}{16a^4m_1}+i\frac{3p_a}{4a^2},$$

$$g_1(t)=\frac{\kappa p_a^2}{4a^4\sqrt{m_1m_2}}+i\frac{\kappa p_a(m_1+m_2)}{4a^4\sqrt{m_1m_2}},$$

and $\omega_2(t)$, $f_2(t)$, and $g_2(t)$ are obtained by $m_1\leftrightarrow m_2$.

The Hamiltonian (B3) shows clearly that the quantum states in the in and out regions are related by the general transformation[4]

$$|\phi_{\text{out}}\rangle=S_1^{(1)}(r_1,\varphi_1)S_1^{(2)}(r_2,\varphi_2)S(r,\varphi)\times T(q,\chi)R(\theta_1,\theta_2)|\phi_{\text{in}}\rangle.$$

If the initial state $|\phi_{\text{in}}\rangle$ is the vacuum state, this formula yields Eq. (B1) for the most general squeezed state. All the squeeze parameters entering Eq. (B1) are determined by the parameters of the system and by its time evolution. Since they all are different, the product $S_1^{(1)}(r_1,\varphi_1)S_1^{(2)}(r_2,\varphi_2)$ cannot be reduced to a two-mode squeeze operator, so that the model presented here is more general than the one considered in Secs. II and III.

## APPENDIX C: SQUEEZED QUANTUM STATES AND BLACK-HOLE EVAPORATION

An important peculiarity of the black-hole gravitational field, as opposed to the cosmological solutions considered above, is the black-hole event horizon. A remote observer can only register the particles defined outside the horizon. Since the particles defined inside the horizon are unobservable, one should average over them. This procedure leads to a density matrix.

Let us consider a system consisting of two sorts of particles: 1 and 2. Let the state of the system be a two-mode squeezed state:

$$|\phi\rangle=S(r,\varphi)|O_1\rangle|O_2\rangle. \qquad \text{(C1)}$$

In the case of a quantized field this assumption applies to every mode **n** of that field. If one is interested in the mean value of the operator $Q_1$, which refers to the 1-particles only, then one gets

$$\langle\phi|Q_1|\phi\rangle=\text{tr}(\rho Q_1),$$

where the density matrix $\rho$, in the case (C1) under consideration, has the form

$$\rho=(1-\tanh^2r)\sum_{m=0}^{\infty}(\tanh^2r)^m|m_1\rangle\langle m_1|. \qquad \text{(C2)}$$

For every mode **n** this density matrix appears as a "thermal" density matrix if one defines the temperature $T_n$ as[14]

$$e^{-n/T_n}=\tanh^2r_n. \qquad \text{(C3)}$$

(It is interesting to note the relation

$$R_n=e^{-n/T_n},$$

where $R_n$ is the "above-barrier" reflection coefficient $R=1-D$; see Appendix A.) In general, $T_n$ could have been different for different modes **n**, but for the black-hole geometry (as well as for some other geometries with horizons), all $T_n$ are the same. Now we will see this in more detail.

The quantized scalar field $\varphi$ can be represented in the form[15,16]

$$\varphi=\int d\omega(a_\omega^{(1)}f_\omega^{(1)}+a_\omega^{(3)}f_\omega^{(3)}+a_\omega^{(4)}f_\omega^{(4)}+\text{H.c.}),$$

where the base functions $f_\omega^{(1)}$, $f_\omega^{(3)}$, and $f_\omega^{(4)}$ satisfy the conditions

$$f^{(1)}_\omega \approx \begin{cases} e^{i\omega v} & \text{on } \mathcal{J}^-, \\ 0 & \text{on } \mathcal{H}^-, \end{cases}$$

$$f^{(3)}_\omega \approx \begin{cases} 0 & \text{on } \mathcal{J}^-, \\ e^{i\omega u_+} & \text{on } \mathcal{H}^-, \end{cases}$$

$$f^{(4)}_\omega \approx \begin{cases} 0 & \text{on } \mathcal{J}^-, \\ e^{i\omega u_+} & \text{on } \mathcal{H}^-. \end{cases}$$

Here the following conventions have been used:

$$v = t + r + 2M\ln(r/2M - 1), \quad U = \mp 4Me^{-u/4M},$$

$$u_+ = \begin{cases} -K^{-1}\ln(-U) & \text{for } U<0, \\ -K^{-1}\ln U + i\pi K^{-1} & \text{for } U>0. \end{cases}$$

$K=(4GM)^{-1}$ is the surface gravity and $(t,r)$ the Schwarzschild coordinates. The in-vacuum state is defined by the requirements

$$a^{(1)}_\omega|0_-\rangle = a^{(3)}_\omega|0_-\rangle = a^{(4)}_\omega|0_-\rangle = 0 .$$

The same scalar field $\varphi$ can be expanded over another set of the base functions:

$$\varphi = \int d\omega(g_\omega w_\omega + h_\omega y_\omega + j_\omega z_\omega + \text{H.c.}) ,$$

where

$$w_\omega \approx \begin{cases} 0 & \text{on } \mathcal{J}^-, \\ 0 & \text{on } \mathcal{H}^-, \quad U<0, \\ e^{-i\omega u_+} & \text{on } \mathcal{H}^-, \quad U>0, \end{cases}$$

$$y_\omega \approx \begin{cases} 0 & \text{on } \mathcal{J}^-, \\ e^{i\omega u_+} & \text{on } \mathcal{H}^-, \quad U<0, \\ 0 & \text{on } \mathcal{H}^-, \quad U>0, \end{cases}$$

$$z_\omega \approx \begin{cases} e^{i\omega v} & \text{on } \mathcal{J}^-, \\ 0 & \text{on } \mathcal{H}^-. \end{cases}$$

The out-vacuum state is defined by the requirements

$$g_\omega|0_+\rangle = h_\omega|0_+\rangle = j_\omega|0_+\rangle = 0 .$$

It has been shown[15] that the in and out operators are related by the Bogoliubov transformation:

$$a^{(1)}_\omega = j_\omega, \quad a^{(3)}_\omega = (1-x)^{-1/2}(h_\omega - x^{1/2}g^+_\omega) ,$$

$$a^{(4)}_\omega = (1-x)^{-1/2}(g_\omega - x^{1/2}h^+_\omega) ,$$

where $x=\exp(-8\pi GM\omega)$. We would like to emphasize a consequence of this transformation: The connection between the states $|0_-\rangle$ and $|0_+\rangle$ involves a two-mode squeeze operator:

$$|0_-\rangle \equiv |0_{f^{(1)}}\rangle|0_{f^{(3)}}\rangle|0_{f^{(4)}}\rangle$$
$$= |0_z\rangle S(r,\pi)|0_\omega\rangle|0_y\rangle , \quad \text{(C4)}$$

where $\tanh^2 r = x$. This is the fact which should have been expected according to Eqs. (6)–(10).

A remote observer has access to the $y$ particles only. One should average over the unobservable $\omega$ particles. According to Eqs. (C4), (C1), and (C2), this leads to the density matrix $\rho$. The striking property of the black-hole geometry is that $\tanh^2 r$ factor has the universal dependence of $\omega$ for all the modes $\omega$:

$$\tanh^2 r = \exp(-8\pi GM\omega) . \quad \text{(C5)}$$

As a result, one obtains from Eqs. (C5) and (C3) the universal Hawking temperature $T_H = (8\pi GM)^{-1}$.

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