Particle creation and entropy generation during a tunneling between spacetimes

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This paper explores the evolution of a quantum matter field on a gravitational instanton, investigating in particular the fact that this evolution results oftentimes in an end state which is "nearly thermal." The key features leading to this systematic evolution are the facts (a) that evolution on an instanton entails a systematic suppression of any initial excitations and (b) that an initial vacuum evolves oftentimes to an end state which, albeit pure, is characterized by a nearly thermal expectation value for the number N_k of particles in each mode $\mathbf{k}$, the temperature being determined by the duration (in imaginary time) of the instanton. Both these facts are a direct consequence of the nonunitary evolution implicit in an imaginary-time Tomonaga-Schwinger equation. The "nearly thermal" character of the end state may also be manifest by the expectation value of other functions $f(N_k)$, but significant discrepancies arise when considering mode-mode correlations or other properties of the field that probe the "phase information" complementary to the "number information," a fact interpreted within the context of work by Hu and Kandrup on information-theoretic measures of entropy for a quantum field. These results are established rigorously for the special case of homogeneous tunneling, using a straightforward analogue of techniques developed by Parker and by Zel'dovich and Starobinsky, and are motivated in more general settings, allowing in particular for topology-changing instantons, using the functional integral approach of, e.g., Lavrelashvili, Rubakov, and Tinyakov.

I. INTRODUCTION

Over the past several years, much interest has focused on the possible role of gravitational instantons in the early Universe. Practical interest in such instantons has included both the bodacious idea that the Universe may have "tunneled into being" as a quantum fluctuation,[1] and the more tame issue of understanding tunneling between various "vacua."[2] From a more pedagogical point of view, there has also been interest in the question of whether changes in the topology of space could perhaps be mediated via instantons,[3] it being by now well understood that topology change in the context of Lorentzian gravity leads to serious difficulties.[4]

Given this interest, it is natural to ask whether it be possible to make sense of quantum field theory for test fields propagating on such instantons. Is it, e.g., possible to proceed with as much (or as little) confidence in formulating such a theory as is the case for ordinary Lorentzian field theory in a curved, dynamical spacetime? One knows, of course, from the pioneering work of Parker[5] and of Zel'dovich and Starobinsky[6] how a time-dependent gravitational field can induce the net creation or destruction of particles in the context of field theory in a Lorentzian spacetime. And one would like to understand how to address similar processes for fields defined on a Euclidean instanton.

These are questions of practical, as well as conceptual, importance. Indeed, one might try to argue[7] that the basic particle content of the Universe was in fact generated primordially as the Universe "tunneled into being" (although there will also be subsequent, albeit less robust, particle creation and/or destruction as the Universe expands in a Lorentzian manner).

The basic objective of this paper is to investigate the qualitative features of particle creation on a gravitational instanton, so as to extract the salient underlying physics. Specifically, it will be seen that particle creation on a Euclidean instanton is rather similar to, but yet fundamentally different from, particle creation in a Lorentzian spacetime.

In achieving this improved understanding, the primary focus will *not* be on the development of new computational tools. Most of the necessary ingredients have already been developed by Rubakov and his co-workers, both in the language of nonunitary Bogoliubov transformations[8] and in the more general setting of a functional integral approach,[9] this latter approach being further generalizable to the consideration of particle creation on manifolds with "handles."[10] Rather, the primary focus is on understanding what it is that these formal tools are trying to say.

From a purely technical viewpoint, the crucial differences between Lorentzian and Euclidean space(time)s are largely a reflection of the fact that what were irrelevant phase factors in a Lorentzian field theory are converted into highly nontrivial exponential factors in an analogous Euclidean setting. This in turn is a direct consequence of the fact that the imaginary-time Tomonaga-Schwinger equation defining the Euclidean field theory implies a *nonunitary evolution* for the quantum fields.

An added twist in this regard is provided by the fact that, in this cosmological setting, the correct form of the imaginary-time Tomonaga-Schwinger equation is *not* completely obvious. If this equation be derived from the Wheeler-DeWitt equation, e.g., in a Born-Oppenheimer approximation, it comes equipped with a fundamental sign ambiguity which reflects, at some level, a choice of "ingoing" or "outgoing" instantons.[11] A choice of either sign leads inexorably to a preferred "arrow of time," but these two arrows are fundamentally opposite one another.

The fact that propagation on an instanton leads to a preferred "arrow of time" might immediately suggest connections with the problem of entropy generation. The question of entropy generation on an instanton leads, however, to its own set of nontrivial issues, these basically reflecting the fact that the evolution is nonunitary. In a Lorentzian spacetime, the value of the "entropy" $S(t)\equiv -\mathrm{Tr}\rho\ln\rho$ constructed from the density matrix ρ of an isolated Hamiltonian system on a $t=$const surface is necessarily conserved. On a Euclidean instanton, this is no longer true.

The basic aim of this paper is to consider particle creation on a Euclidean instanton in the closest possible analogy with particle creation in a Lorentzian spacetime, so as to see precisely where and why differences arise. In particular, it will be seen that the choice of an appropriate sign in the imaginary time Tomonaga-Schwinger equation implies an evolution entailing a systematic suppression of any nontrivial initial conditions for the quantum fields. It will, moreover, be seen that, in many cases, this systematic evolution results in a final state which, in terms of its particle content, is "nearly thermal" in a physically and mathematically well-defined sense. The "temperature" associated with this "nearly thermal" distribution is determined by the duration in imaginary cosmic time of the instanton. Its presence traces to a term which was, in a Lorentzian setting, a phase factor of unit modulus, but which, in a Euclidean manner, becomes an exponential "Boltzmann factor."

As hinted already, this analysis could be effected in either of two ways: namely, in the language of Bogoliubov transformations between "initial" and "final" states or in terms of a functional integral. The latter alternative is more general in that it does not require the assumption of a product topology $\Sigma\times R$ and, as such, makes sense even in the presence of changes in topology. The former alternative has, however, the important advantages of being much simpler and more easily understood in terms of ordinary quantum-mechanical models. For this reason, all but the last section of this paper will be formulated completely in the language of nonunitary Bogoliubov transformations.

Section II discusses more carefully, and in some detail, precisely why, both physically and mathematically, the problem of particle creation on an instanton is fundamentally different from particle creation in a Lorentzian spacetime. Section III then illustrates these conceptual points explicitly by considering the evolution of a free field on two different sorts of instantons: namely, the Euclidean analogues of the Lorentzian spacetimes first studied by Parker[5] and by Zel'dovich and Starobinsky[6] and the more general case of "time-homologous" instantons which are static up to an overall conformal factor Ω depending only on imaginary time. Section IV endeavors to interpret the results of Sec. III in terms of appropriate measures of "field entropy." Finally, Sec. V concludes by indicating which of the results derived in Secs. III and IV are generic and which seem instead to reflect the special symmetries of the concrete models considered therein.

II. PARTICLE CREATION ON EUCLIDEAN, RATHER THAN LORENTZIAN, SPACES

One can of course write down an imaginary time Tomonaga-Schwinger equation "by fiat," and use it to define a field theory on a Euclidean space, just as one can write down a real-time Tomonaga-Schwinger equation and then use it to define a field theory in a Lorentzian spacetime. However, it is more illuminating to understand both these equations as arising (at least formally) from a more complete description which takes as its starting point the Wheeler-DeWitt equation (cf. Ref. 12), i.e., the quantized Hamiltonian constraint of classical general relativity.

This can be done, at least to lowest orders, in a Born-Oppenheimer approximation. Unfortunately, however, the extension of this analysis to higher orders, with the incorporation a self-consistent back reaction, has not yet been achieved. The program advocated by Halliwell,[13] as further developed, e.g., by Padmanabhan,[14] has been shown by Brout and Venturi[15] to be unsatisfactory (because of a Berry phase) except for special minisuperspace models with trivial homotopy. Be this, however, as it may, one *can* (neglecting such problems as regularization) easily "derive" Tomonaga-Schwinger equations by assuming a wave function Ψ which is "dominated" in its gravitational sector by some vacuum solution to the Einstein equation.

Following the approach introduced by Gerlach[16] and extended by Lapchinsky and Rubakov,[17] start with the Wheeler-DeWitt equation, which takes the form

$$\left[-m_P^{-2}G_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}}-m_P^2 h^{1/2}(R^{(3)}-2\Lambda)+\mathcal{H}_M\right]\Psi[h_{ij},\Phi]=0\ . \quad (1)$$

Here Λ, $R^{(3)}$, and G_{ijkl} denote, respectively, a possible cosmological constant, the three-dimensional scalar curvature associated with the three-metric h_{ij}, and the superspace metric,[12] and $\mathcal{H}_M$ represents the matter Hamiltonian associated with some matter field(s) Φ. The explicit presence of the Planck mass m_P manifests the obvious fact that general relativity is a dimensionful theory, and that gravitational physics above and below the Planck scale may be expected to be fundamentally different.

In general, this Wheeler-DeWitt equation clearly treats the gravitational and matter sectors on an equal footing. If, however, one is interested in energy scales much below the Planck mass m_P, one can effectively "freeze-out" the gravitational degrees of freedom. Specifically, suppose that the solution for Ψ can be written in the form

$$\Psi=\left[\det\left[\frac{\delta^2 S}{\delta h_{ij}\delta h_{kl}}\right]\right]^{1/2}\exp(\pm i m_P^2 S)\chi[h_{ij},\Phi] \ . \quad (2)$$

One then discovers that, to order m_P^2, S will satisfy an equation

$$G_{ijkl}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{kl}}-h^{1/2}(R^{(3)}-2\Lambda)=0 \ , \quad (3)$$

which is nothing other than the Hamilton-Jacobi equation of vacuum classical Einstein gravity. This implies (modulo a good deal of technical work) that S may be identified as the action associated with a classical vacuum spacetime, and that the new wave function χ may be interpreted as referring to matter evolving in that classical spacetime. Indeed, to next lowest order, namely, $O(m_P^0)$, one sees that χ satisfies

$$2iG_{ijkl}\frac{\delta S}{\delta h_{ij}}\frac{\delta \chi}{\delta h_{kl}}=\mp\mathcal{H}_M\chi \ . \quad (4)$$

By interpreting the left-hand side of this equation as defining a many-fingered time, this may then be viewed as a Tomonaga-Schwinger equation of the form

$$i\frac{\delta\chi}{\delta\tau(x)}=\pm\mathcal{H}_M(x)\chi \ . \quad (5)$$

The sign ambiguity in either Eq. (4) or (5) traces back to the two possible choices $\exp(\pm i m_P^2 S)$ in the semiclassical ansatz (2), these essentially reflecting, in the language of Teitelboim,[11] the choice of "ingoing" or "outgoing" solutions. At the lowest order, semiclassical level of the Hamilton-Jacobi equation, which is quadratic in S, this choice is immaterial; but, to next order, where the matter is treated quantum mechanically, the choice does in fact leave an imprint. However, in this Lorentzian setting that imprint remains irrelevant in computing physical amplitudes: it is clear that solutions χ to the two equations with opposite signs will differ only by overall phases of unit modulus.

One can equally well look for solutions to the Wheeler-DeWitt equation which are dominated in the gravitational sector by a vacuum solution to the Euclidean Einstein equation.[10] Indeed, the ansatz

$$\Psi=\left[\det\left[\frac{\delta^2|S_E|}{\delta h_{ij}\delta h_{kl}}\right]\right]^{1/2}\exp(\mp m_P^2|S_E|)\chi[h_{ij},\Phi] \quad (6)$$

leads to lowest order to the Euclidean Hamilton-Jacobi equation

$$G_{ijkl}\frac{\delta|S_E|}{\delta h_{ij}}\frac{\delta|S_E|}{\delta h_{kl}}+h^{1/2}(R^{(3)}-2\Lambda)=0 \ , \quad (7)$$

so that $|S_E|$ may be interpreted as the action associated with a vacuum solution to the classical Euclidean field equation. To next lowest order, one then concludes that the matter wave function χ satisfies

$$2G_{ijkl}\frac{\delta|S_E|}{\delta h_{ij}}\frac{\delta\chi}{\delta h_{kl}}=\mp\mathcal{H}_M\chi \ , \quad (8)$$

which, in analogy with Eq. (4), can be interpreted as an imaginary-time Tomonaga-Schwinger equation

$$\frac{\delta\chi}{\delta\tau(x)}=\pm\mathcal{H}_M(x)\chi \ . \quad (9)$$

Once again the Tomonaga-Schwinger equation manifest a sign ambiguity, but in this case the ambiguity actually has important physical content. Because one is dealing with an imaginary-time *diffusion* equation, what were phases in a Lorentzian mode have been converted to real exponential factors which, no longer being of unit modulus, *do* impact on physical amplitudes.

More abstractly, the obvious point, both physically and mathematically, is that the evolution is no longer unitary. There is no notion of probability conservation. This implies in particular that the "ordinary" textbook entropy $S(t)=-\mathrm{Tr}\rho\ln\rho$, associated with a density matrix ρ constructed from some matter wave functions χ, is no longer guaranteed to be conserved. For an evolution described by a real-time Tomonaga-Schwinger equation, unitarity implies that, for any f, the value of the quantity $\mathrm{Tr}f(\rho)$ must remain constant as the field evolves. But, for the imaginary-time Tomonaga-Schwinger equation, this is no longer true.

Another important point is that the choice of *either* sign in Eq. (6) or (9) necessarily leads to a preferred "arrow of time." At this purely formal level, one is not constrained to make one choice as opposed to the other, but one must recognize that either choice, once made, necessarily leads to such an arrow. Diffusion, or antidiffusion, equations, unlike Schrödinger equations, clearly have a preferred direction of time.

This is well illustrated by considering the simpler (nonfunctional) imaginary time equations

$$\frac{\partial\chi}{\partial t}=\pm H\chi \ , \quad (10)$$

which, for a time-independent H, admit solutions $\propto\exp(\pm Ht)$. The obvious point then is that, if H is real, one infers either a systematic suppression or a systematic amplification of any initial excitations above the minimum energy $E_{\min}$.

If, in analogy with ordinary quantum-mechanical tunneling, one chooses the minus sign in Eq. (9), one infers a systematic tendency for a suppression of any initial excitations as the system evolves. This means that, if one starts from an initial "vacuum," one will in fact get a net creation of particles, but that, if one starts instead from a highly excited state, one would anticipate a systematic decrease in the average particle number.

The physical picture is very different from the case of evolution in a Lorentzian space where, for free fields in "time-homologous" spacetimes, one can actually identify special classes of initial configurations which, irrespective of their particle content, necessarily evolve in such a fashion as to increase the average particle number.[18] As emphasized in Sec. IV, the special role of "random phase" initial conditions in Lorentzian spacetimes reflects directly the unitary evolution that is encapsulated in Eq. (5) but absent in Eq. (9). In a Lorentzian setting the question of whether one gets a net creation or destruction of quanta depends only on the initial phase coherence or decoherence of the field. Factors of $\exp(\pm iHt)$ do not lead to any tendency for a net creation or destruction of particles because of "energetic" considerations. In a Euclidean setting one is confronted instead with factors $\exp(\pm Ht)$ which *do* have nontrivial "energetic" implications.

The analysis in the remainder of this paper is predicated upon the assumption that, for "physical" reasons, one has decided to work with an imaginary time equation

$$\frac{\delta\chi}{\delta\tau(x)} = -\mathcal{H}_M(x)\chi \; , \tag{11}$$

where τ is constrained to *increase* in value between the initial and final boundaries of the instanton.

This requirement reflects the choice in Eq. (6) of an "outgoing" instanton, and, in this sense, may be interpreted as implying semiclassically a certain sort of boundary condition for the Wheeler-DeWitt equation. Aside from the intuitive tunneling interpretation which this choice facilitates, it will be seen that this choice is satisfying in that it leads to interesting (and particularly "physical?") results for particle creation on an instanton.

One especially interesting fact about the imaginary time Tomonaga-Schwinger equation (11) is that it tends oftentimes to yield a final state in which the distribution of particle number or energy is rather "thermal." This fact, to be demonstrated carefully in Sec. III, may be understood intuitively as a reflection of the factor $\propto \exp(-Ht)$ which arises if, in Eq. (10), a minus sign is chosen: With that choice, an initial ground-state wave function will in fact evolve to a final state in which the relative amplitudes for various energies are characterized by exponential "thermal" factors. If, alternatively, the wave function is not initially in its ground state, this will no longer be exactly true, but the minus sign will nevertheless tend to "damp" the initial excitations so that the system will still tend to evolve *toward,* albeit not *to,* the same final state.

The quantum evolution described by a Tomonaga-Schwinger equation with a plus, rather than minus, sign is completely different in character, representing in some sense a "time-reverse" of the sort of evolution implicit in Eq. (11). Such an equation implies that any initial excitations will be systematically enhanced, rather than suppressed, and, moreover, that the final state should tend to be dominated by high-, rather than low-energy particles. That this should be the case follows trivially from an examination of solutions to the equation $\partial\chi/\partial t = +Ht$, and has also been examined in a minisuperspace setting by Rubakov,[1] who, in the language of this paper (cf. the discussion in Ref. 7), investigated the particle creation associated with a Universe "tunneling into being" via an "ingoing," rather than "outgoing," instanton. Not surprisingly, this alternative choice of a positive sign imaginary time Tomonaga-Schwinger equation leads to a prediction of "catastrophic"[1] particle creation, which seems rather unphysical. As illustrated in Sec. III, the choice of a minus sign leads oftentimes instead to the "more physical" prediction of a "nearly thermal" distribution of quanta.

It should, however, be emphasized that, even though the final distribution of average numbers may be "nearly thermal," the final state is in general most definitely *not* really thermal. Even though the evolution implicit in either imaginary time Tomonaga-Schwinger equation is nonunitary, it still entails a mapping from pure states to pure states. It is therefore impossible that an initial vacuum, or any other initial pure state, be mapped into a final thermal state, which is of course mixed. Indeed, that the state evolved from an initial vacuum is nonthermal is easily seen when one considers the expectation values of operators that probe correlations among the modes of the field or, more generally, the phase information that is complementary to particle number.[19] That this is true should be obvious intuitively, but will be better illustrated in Sec. III, which considers two special classes of instantons, where, for free fields, the modes decouple and one can study the evolution problem trivially using Bogoliubov transformation techniques.

The only way in which to generate a mixed state for the field(s) is (a) to introduce a "coarse-graining" that entails a neglect of some information about the field(s) or (b) to allow for the inevitable existence of correlations between the "nearly classical" gravitational sector and the "fully quantum" matter sector. This latter alternative is an obvious one to consider, but its correct implementation clearly requires a satisfactory understanding of how to incorporate a self-consistent back reaction of the matter fields on the gravitational sector.

III. PARTICLE CREATION DURING COSMOLOGICAL TUNNELING

The objective now is to study particle creation for two simple classes of models, treating the Euclidean and Lorentzian cases on an essentially equal footing. In so doing, it should be recognized that one is constrained to work in the Schrödinger picture. The fact that the case of Lorentzian evolution is unitary means that, in that setting, the Heisenberg and Schrödinger pictures are equivalent. This, however, is no longer true in a Euclidean setting. (This is manifest below in the fact that the Bogoliubov transformation relating the definition of "particle" at two different times is nonunitary.) Here one is constrained to work directly with the imaginary time Tomonaga-Schwinger equation, which is of course a functional Schrödinger equation.

Consider first the evolution of a real, massive, minimally coupled free scalar field Φ on a statically bounded space(time) with a metric

$$ds^2 = \pm dt^2 + \Omega^2(t)\delta_{ab}dx^a dx^b \; ,$$
$$\Omega \to \Omega_1, \Omega_2 \;\; \text{for} \; t \to t_1, \;\; t \to t_2 \; . \tag{12}$$

Here, as in subsequent equations of this section, the upper sign refers to the case of a Euclidean space, whereas the lower sign refers to the case of a Lorentzian spacetime. The requirement that the space(time) be statically bounded, so that Ω tends to constant values at early and late times, ensures that, in these limits, one can introduce a meaningful Fock-space representation. This requirement is particularly reasonable in the context of gravitational instantons, where the Euclidean solution is presumably supposed to be matched onto Lorentzian solutions at initial and final zero extrinsic curvature boundaries. This analysis closely parallels Rubakov's[8] discussion of particle creation, but differs in that one is expanding in complex plane waves rather than real modes. This choice can obviously have no impact on the basic physics, but, as will be evidenced below, it *does* lead to interesting differences in interpretation.

Suppose now that the scalar field Φ satisfies a field equation $\nabla_\mu \nabla^\mu \Phi - m^2 = 0$, which translates in components to

$$\pm \Omega^{-3} \frac{\partial}{\partial t}\left[\Omega^3 \frac{\partial}{\partial t}\Phi\right] + \Omega^{-2}\nabla^2\Phi - m^2\Phi = 0 , \tag{13}$$

where ∇^2 denotes the flat-space three-dimensional Laplacian. Because the $t=\text{const}$ slices are flat, one can implement the usual plane-wave decomposition

$$\Phi = (2\pi)^{-3/2}\int d^3k [A_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} f_k(\tau) + A_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} g_k(\tau)] , \tag{14}$$

where, in terms of a time $d\tau \equiv \Omega^{-3}dt$, the temporal mode functions f_k and g_k satisfy

$$\frac{d^2\Psi_k}{d\tau^2} \mp \omega_k^2 \Psi_k = 0 , \tag{15}$$

with

$$\omega_k = \Omega^3\left[\frac{k^2}{\Omega^2} + m^2\right]^{1/2} . \tag{16}$$

Near the initial boundary, one wishes of course to choose f_k and g_k as corresponding, respectively, to positive- and negative-frequency solutions, be this with respect to either real or imaginary time. Thus, e.g., in a Euclidean setting one demands that

$$g_k(\tau) \propto \{\exp[\omega_k(\tau_1)(\tau-\tau_1)]\} . \tag{17}$$

More precisely, the boundary conditions on f_k and g_k reduce in both settings to

$$g_k(\tau_1) = f_k(\tau_1) = [2\omega_k(\tau_1)]^{-1/2} ,$$

whereas the conditions on their time derivatives vary. In a Lorentzian setting one must demand that

$$\partial_\tau g_k(\tau_1) = -\partial_\tau f_k(\tau_1) = i[\omega_k(\tau_1)/2]^{1/2} ,$$

whereas, in a Euclidean setting,

$$\partial_\tau g_k(\tau_1) = -\partial_\tau f_k(\tau_1) = [\omega_k(\tau_1)/2]^{1/2} . \tag{18}$$

With these boundary conditions, $A_k^\dagger$ and A_k can be interpreted as creation and annihilation operators in the initial Fock space.

The subsequent dynamics at $\tau > \tau_1$ leads to a mixing of the temporal modes functions g_k and f_k, so that, at later times, the initial positive- (negative-) frequency functions contain admixtures of negative (positive) frequency. Thus, e.g., near time τ_2 the initial positive-frequency solution (17) will have evolved to a linear combination

$$g_k(\tau) \propto \alpha_k\{\exp[\omega_k(\tau_2)(\tau-\tau_1)]\} + \beta_k\{\exp[-\omega_k(\tau_2-\tau_1)]\} \tag{19}$$

in terms of two Bogoliubov coefficients α_k and β_k. More generally, in the Lorentzian case one can write

$$g_k(\tau_2) = [2\omega_k(\tau_2)]^{-1/2}(\alpha_k e^{\Delta_k} + \beta_k e^{-\Delta_k}) ,$$
$$\partial_\tau g_k(\tau_2) = i[\omega_k(\tau_2)/2]^{-1/2}(\alpha_k e^{\Delta_k} - \beta_k e^{-\Delta_k}) ,$$

and

$$f_k(\tau_2) = [2\omega_k(\tau_2)]^{-1/2}(\mu_k e^{-\Delta_k} + \nu_k e^{\Delta_k}) ,$$
$$\partial_\tau f_k(\tau_2) = -i[\omega_k(\tau_2)/2]^{1/2}(\mu_k e^{-\Delta_k} - \nu_k e^{\Delta_k}) ,$$

whereas, in the Euclidean case, one has instead

$$g_k(\tau_2) = [2\omega_k(\tau_2)]^{-1/2}(\alpha_k e^{\Delta_k} + \beta_k e^{-\Delta_k}) ,$$
$$\partial_\tau g_k(\tau_2) = [\omega_k(\tau_2)/2]^{1/2}(\alpha_k e^{\Delta_k} - \beta_k e^{-\Delta_k}) ,$$

and

$$\begin{aligned} f_k(\tau_2) &= [2\omega_k(\tau_2)]^{-1/2}(\mu_k e^{-\Delta_k} + \nu_k e^{\Delta_k}) ,\\ \partial_\tau f_k &= -[\omega_k(\tau_2)/2]^{1/2}(\mu_k e^{-\Delta_k} - \nu_k e^{\Delta_k}) . \end{aligned} \tag{20}$$

All of the mixing of the positive and negative frequencies is now encoded in the four Bogoliubov coefficients α_k, β_k, μ_k, and ν_k, and the additional quantity Δ_k. In the Lorentzian case, this

$$\Delta_k \equiv 2i\omega_k(\tau_2)(\tau_2-\tau_1) \tag{21}$$

is purely imaginary, whereas, in the Euclidean case,

$$\Delta_k \equiv 2\omega_k(\tau_2)(\tau_2-\tau_1) \tag{22}$$

is purely real.

At this stage, two fundamental differences between the Lorentzian and Euclidean cases should be clear. First of all, it is evident that, in the Lorentzian case, f_k and g_k are complex conjugates, so that $\mu_k = \alpha_k^*$ and $\nu_k = \beta_k^*$. This means that there are only two Bogoliubov coefficients entering into the analysis for each $\mathbf{k}$. In the Euclidean case this is no longer true. There is no symmetry to reduce the number of coefficients from four to two. It is, however, true in both settings that there *is* a single Wronskian condition $\alpha_k\mu_k - \beta_k\nu_k = 1$ relating these coefficients, a condition which reduces, in the Lorentzian case to the well-known $|\alpha_k|^2 - |\beta_k|^2 = 1$. (This Wronskian condition ensures in both Lorentzian and Euclidean cases that the formal commutation relations are preserved for $\tau > \tau_1$.)

The other, even more crucial, difference between the Euclidean and Lorentzian cases lies in the fact that, in the Euclidean case, Δ_k is *not* just a phase. In the Lorentzian case, the modulus $|\exp(\pm\Delta_k)| \equiv 1$, so that $\exp(\pm\Delta_k)$ can simply be absorbed into α_k and β_k as an additional phase. Indeed, for this reason it is customary among workers on quantum field theory in curved spaces to routinely ignore this factor, although it has been included, e.g., by Brown and Carson[20] in the study of parametric amplification for a quantum oscillator.

Given these Bogoliubov coefficients, it is straightforward, albeit tedious, to construct an "evolution operator" enabling one to map an initial wave function $\chi(\tau_1)$ to a final $\chi(\tau_2)$. Because the modes decouple, all that one need do is focus on a single pair $\pm\mathbf{k}$ and then determine the rule mapping an initial eigenstate $|n,m)$ with n quanta in the mode $\mathbf{k}$ and m in $-\mathbf{k}$ to the final state to which it evolves. This calculation may be effected by paralleling Rubakov's[8] analysis, only allowing for complex, rather than real, eigenfunctions. What one finds is that this final state can be written in the form $\zeta|n,m\rangle$, where $|n,m\rangle$ denotes a final eigenstate with n and m particles in the modes $\pm\mathbf{k}$ and

$$\zeta \equiv \zeta_1\zeta_2\zeta_3 = :\exp\left[\sum_k [D_k a_k^\dagger a_{-k}^\dagger + F_k(a_k^\dagger a_{-k} + a_{-k}^\dagger a_k) + G_k a_k a_{-k}]\right]: , \tag{23}$$

with coefficients

$$D_k = \frac{\beta_k}{\alpha_k}e^{-2\Delta_k} , \quad F_k = \frac{1}{\alpha_k}e^{-\Delta_k} - 1 ,$$

and

$$G_k = \frac{\nu_k}{\alpha_k} . \tag{24}$$

Here $a_k^\dagger$ and a_k denote, respectively, creation and annihilation operators as defined on the final Fock space, and colons indicate normal ordering.

It is easy to see that in the Euclidean case, Eq. (23) implies a systematic tendency of nontrivial excitations to be damped away. A precise formulation of the precise sense in which this is true will, however, be deferred to the second, more general, class of instantons considered below.

Given that, in the Euclidean setting, initial excitations tend to be damped, it is natural to investigate the ultimate fate of an initial vacuum state. This is easy enough to do, in both the Lorentzian and Euclidean cases. Because ζ_2 and ζ_3 act trivially on the state $|0\rangle$, one obtains immediately an unnormalized wave function

$$|\text{out}\rangle = \zeta|0\rangle = \zeta_1|0\rangle = \exp\left[\sum_k D_k a_k^\dagger a_{-k}^\dagger\right]|0\rangle . \tag{25}$$

One then computes trivially that $\langle\text{out}|\text{out}\rangle = (1-|D_k|^2)^{-1}$, so that the probability distribution $\mathcal{P}$ for finding n quanta in mode $+\mathbf{k}$ and m quanta in $-\mathbf{k}$ takes the form

$$\mathcal{P}(n,m) = \frac{|D_k|^{2n}}{1-|D_k|^2}\delta_{nm} , \tag{26}$$

where δ_{nm} denotes a Kronecker delta. The presence of this delta emphasizes the crucial fact that, as is evident already in Eq. (23), particles are always created in pairs with opposite "momenta" $\mathbf{k}$ and $-\mathbf{k}$.

From this probability distribution one determines immediately that, at late times, the average (unnormalized) number $\langle N_k(\tau_2)\rangle_V \equiv \langle a_k^\dagger a_k\rangle_V$ generated from the initial vacuum takes the form

$$\langle N_k(\tau_2)\rangle_V = \frac{D_k^2}{1-D_k^2} \to \frac{1}{|\alpha_k/\beta_k|^2\exp[4\omega_k(\tau_2)T]-1} , \tag{27}$$

where $T \equiv \tau_2 - \tau_1$ denotes the duration of the instanton in cosmic imaginary time. The last equality in this equation holds only for the case of particle creation on a Euclidean instanton. Were one considering instead the problem of particle creation in a Lorentzian spacetime, $\exp[4\omega_k(\tau_2)T]$ would be replaced by the quantity $|\exp[2i\omega_k(\tau_2)T]|^2 \equiv 1$.

It should be observed that this result looks "nearly thermal." Indeed, neglecting the contribution $|\alpha_k/\beta_k|^2$ one has what is essentially a Bose-Einstein distribution for the average number in each mode, with a temperature $\Theta = 1/4T$. That this should be the case is actually rather easy to understand by returning to the simple toy equation (11). An initial ground state evolved with that equation leads after a time T to a wave function $\chi \propto e^{-HT}$, which implies a probability distribution $\propto e^{-2ET}$ for each energy E. This might naively suggest that the "temperature" in Eq. (27) should be $\Theta = 1/2T$, but that inference would be wrong because of the fact that particles are always created in pairs. If a pair is created with energy E, each of the created particle must itself have only an energy $E/2$.

It should also be noted that, in many cases, one would in fact anticipate a ratio $|\alpha_k/\beta_k|^2 \approx 1$. Physically, it is clear from Eq. (19) that, when realized, this approximate equality reflects a "strong mixing" of positive and negative frequencies. This would be expected to obtain quite generally in the limit of "strongly nonadiabatic instantons." And indeed, it is straightforward to construct examples of instantons for which this "strong mixing" really does arise [cf. Eq. (4.11) in Ref. 8]. More speculatively, when this ratio is not approximately equal to unity it may be interpreted as providing an energy-dependent chemical potential. This possibility has, e.g., been raised in the context of particle creation on an instanton by means of which the Universe "tunneled into being."[7]

It remains to see how "nearly thermal" the fundamentally nonthermal, pure state $\zeta|0\rangle$ really is. If, in Eq. (26), one traces over the mode $-\mathbf{k}$, one ends up with a probability distribution

$$\mathcal{P}(n) = \frac{|D_k|^{2n}}{1-|D_k|^2} \tag{28}$$

which, to the extent that $|D_k|^2$ can be interpreted as a thermal factor, is precisely thermal. This implies that one will in fact compute thermal expectation values for all functions $f(N_k)$. One also verifies further that, as for a thermal state, $\langle a_k^{\dagger p} a_k^q\rangle$ vanishes when $p \neq q$.

One does, however, see clear evidence of the nonthermal character of the final state when one probes mode-mode correlations. For a true thermal state, the modes $+\mathbf{k}$ and $-\mathbf{k}$ are uncorrelated, so that $\langle N_k N_{-k}\rangle=\langle N_k\rangle\langle N_{-k}\rangle$, but, for the final state $\zeta|0\rangle$, this is simply not so. And, even more elementarily, a thermal state implies a vanishing $\langle a_k^\dagger a_{-k}^\dagger\rangle$, whereas this expectation value is nonzero for the state $\zeta|0\rangle$. Indeed, one immediately computes that

$$\langle a_k^\dagger a_{-k}\rangle_V=\langle a_k^\dagger a_k\rangle_V\equiv\langle N_k\rangle \tag{29}$$

and $\langle a_k^\dagger a_{-k}^\dagger\rangle_V^*=\langle a_k a_{-k}\rangle_V=D_k$, so that the product

$$\langle a_k^\dagger a_{-k}^\dagger\rangle_V^*\langle a_k a_{-k}\rangle_V=\frac{\langle N_k\rangle}{1+\langle N_k\rangle}\ . \tag{30}$$

It is also straightforward to write down a comparatively simple formula for the generic expectation value $\langle N_k^p N_{-k}^q\rangle_V$. Specifically, let

$$\sigma(p)\equiv(1-x)\left[x\frac{d}{dx}\right]^p(1-x)^{-1}\ , \tag{31}$$

where $x=|D_k|^2$. In terms of this σ, one then computes that, for a true thermal state,

$$\langle N_k^p N_{-k}^q\rangle_{\text{th}}=\sigma(p)\sigma(q)\ , \tag{32}$$

whereas the final state $\zeta|0\rangle$ leads to

$$\langle N_k^p N_{-k}^q\rangle_V=\sigma(p+q)\ . \tag{33}$$

It is, moreover, easy to see that $\sigma(p+q)\geq\sigma(p)\sigma(q)$, so that Eqs. (32) and (33) imply that number expectation values are systematically larger for the final state generated on the instanton than for the corresponding thermal state. Thus, e.g., $\langle N_k N_{-k}\rangle_{\text{th}}=\langle N_k\rangle^2$, whereas $\langle N_k N_{-k}\rangle_V=2\langle N_k\rangle^2+\langle N_k\rangle$. An "explanation" of this fact will be provided in Sec. IV.

Turn now to the more general case of a "time-homologous" space(time) with a metric

$$ds^2=\pm dt^2+\Omega^2(t)\gamma_{ab}(x^c)dx^a dx^b\ ,$$
$$\Omega\to\Omega_1\ ,\quad \Omega_2 \text{ for } t\to t_1\ ,\quad t\to t_2\ . \tag{34}$$

This corresponds to a space(time), the spatial sections of which may be arbitrarily complicated at each instant, but which are identical to one another at all times except for an overall scale factor $\Omega(t)$. In this space(time), the field equation $\nabla_\mu\nabla^\mu\Phi-m^2=0$ reduces to

$$\pm\frac{\partial^2\Phi}{\partial\tau^2}+\Omega^6(\Omega^{-2}\Delta\Phi-m^2\Phi)\equiv\pm\frac{\partial^2\Phi}{\partial\tau^2}+\mathcal{A}(\tau)\Phi=0\ , \tag{35}$$

where, again, $d\tau\equiv\Omega^{-3}dt$ and the operator Δ denotes the covariant Laplacian associated with the three-metric γ_{ab}. Because the space(time) is not homogeneous and isotropic, one can no longer expand in plane waves. It is, however, natural and convenient to expand in the eigenfunctions of Δ or $\mathcal{A}$ (cf. Refs. 8 and 22). Suppose for specificity that these eigenfunctions form a discrete set (i.e., that they are proper eigenfunctions rather than eigendistributions). Then, at each instant τ, one can introduce a complete set $\{\xi_k^\tau\}$ of real eigenfunctions of $\mathcal{A}(\tau)$ with corresponding eigenvalues $\omega_k^2(\tau)>0$, orthogonal with respect to the inner product

$$(\xi,\eta)\equiv\int d^3x(\gamma)^{1/2}\xi\eta \tag{36}$$

and satisfying the normalization

$$(\xi_k^\tau,\xi_l^\tau)=[2\omega_k(\tau)]^{-1}\delta_{kl}\ . \tag{37}$$

Note that the ξ_k's are functions of time only because of the time-dependent normalization. (Allowing for this time-dependent normalization facilitates the generalization of this analysis[8] to more complicated spacetimes.)

Now introduce the mode expansion

$$\Phi(\tau)=\sum_k[A_k^\dagger g_k(\tau)+A_k f_k(\tau)]\xi_k^\tau\ , \tag{38}$$

where f_k and g_k are solutions to

$$\frac{d^2\Psi_k}{d\tau^2}\mp\omega_k^2(\tau)\Psi_k=0\ . \tag{39}$$

Then impose the boundary conditions

$$g_k(\tau_1)=f_k(\tau_1)=1$$

and, in the Euclidean case,

$$\partial_\tau g_k(\tau_1)=-\partial_\tau f_k(\tau_1)=\omega_k(\tau_1)\ ,$$

or, in the Lorentzian case,

$$\partial_\tau g_k(\tau_1)=-\partial_\tau f_k(\tau_1)=i\omega_k(\tau_1)\ . \tag{40}$$

It follows that $A_k^\dagger$ and A_k represent physical creation and annihilation operators in the initial Fock space defined at time τ_1.

Given this general set up, it is straightforward (cf. Ref. 21) to play the same game as before, evolving the mode functions to the final time τ_2 in terms of Bogoliubov coefficients α_k, β_k, μ_k, and ν_k, and then determining the particle content associated with the final Fock space. Indeed, one can again encapsulate all of the dynamics in terms of an evolution operator ζ. This is particularly trivial in this setting since there exists no mixing of mode pairs $\pm\mathbf{k}$. It thus suffices to observe that an initial state $|p\rangle$ with p particles in some given mode k, as defined in the initial Fock space, will evolve to a final state $\zeta|p\rangle$, where, in terms of $a_k^\dagger$ and a_k, creation and annihilation operators at time τ_2,

$$\zeta\equiv\zeta_1\zeta_2\zeta_3$$
$$=:\exp\left[\sum_k(\tfrac{1}{2}D_k a_k^\dagger a_k^\dagger+F_k a_k^\dagger a_k+\tfrac{1}{2}G_k a_k a_k)\right]:\ . \tag{41}$$

[This corrects a typographical error in Eq. (14) of Ref. 21.] Here the coefficients D_k, F_k, and G_k once again take the forms

$$D_k=\frac{\beta_k}{\alpha_k}e^{-2\Delta_k}\ ,\quad F_k=\frac{e^{-\Delta_k}}{\alpha_k}-1\ ,\quad\text{and } G_k=\frac{\nu_k}{\alpha_k}\ , \tag{42}$$

and Δ_k is again defined as in Eq. (21) or (22).

It is easy to see that Eq. (41) implies a systematic suppression of any initial excitations above the vacuum

$|0\rangle$. Since each mode evolves independently, focus simply on an initial state $|p\rangle$ with p particles in mode k which evolves to a final $\zeta|p\rangle \equiv \zeta_1\zeta_2\zeta_3|p\rangle$ involving the quantity $\zeta_2\zeta_3(a_k^\dagger)^p|0\rangle$. Now commute ζ_2 and ζ_3 past the creation operators, exploiting the relations

$$a_k^\dagger\zeta_2 = z_k\zeta_2 a_k^\dagger \ , \quad a_k^\dagger\zeta_3 = \zeta_3 a_k^\dagger - G\zeta_3 a_k \ ,$$
$$a_k\zeta_2 = z_k^{-1}\zeta_2 a_k \ , \quad \text{and } a_k\zeta_3 = \zeta_3 a_k \ . \tag{43}$$

The net result is an equality

$$\zeta(a_k^\dagger)^p = \alpha_k^p \exp[-p\omega_k(\tau_2)T] \times \zeta_1(a_k^\dagger - \alpha_k\nu_k \exp[2\omega_k(\tau_2)T]a_k)^p|0\rangle \ . \tag{44}$$

This result implies that $\zeta|p\rangle$ can be expanded out as a sum of contributions involving $\zeta_1|p\rangle, \zeta_1|p-2\rangle, \ldots$, the contribution proportional to $\zeta_1|q$ involving a weight $\exp[-q\omega_k(\tau_2)T]$. And thus, if the relative magnitude of these different contributions is determined by this exponential factor, the final state will, at least for $\omega_k(\tau_2)T \gg 1$, be dominated by the lowest $|q\rangle$ term in the sum, namely, $\zeta_1|0\rangle$ or $\zeta_1|1\rangle$. Linear superposition then implies that an initial state $|\text{in}\rangle \equiv \sum_p c_p|p\rangle$ will, unless c_p vanishes for all even p, be dominated at late times by a contribution proportional to the state $\zeta_1|0\rangle = \zeta|0\rangle$ which would have evolved from an initial vacuum, corrections thereunto being suppressed exponentially.

It is also easy to see that, once again, an initial vacuum evolves to a final state with a "nearly thermal" distribution for the average number of particles in each mode. Indeed, one computes that the unnormalized final state

$$|\text{out}\rangle = \exp(\tfrac{1}{2}D_k a_k^\dagger a_k^\dagger)|0\rangle \tag{45}$$

implies an average number

$$\langle N_k(\tau_2)\rangle_V = \frac{D_k^2}{1-D_k^2} \rightarrow \frac{1}{|\alpha_k/\beta_k|^2 \exp[4\omega_k(\tau_2)T]-1} \ . \tag{46}$$

It is, however, clear that, even neglecting all mode-mode correlations, the final state (45) is *not* thermal. Despite the fact that the space need no longer be homogeneous or isotropic, Eq. (41) or (45) still implies that particles can only be generated in pairs, in this case identical particles in the same mode k. This means in particular that, for the final state evolved from an initial vacuum, the probability of there being any odd number of particles vanishes identically.

There is an important lesson in all of this. For the special case of a homogeneous and isotropic space(time), the physics derived here *must* of course agree with the physics derived above, but it is clear that the way in which the physics has been formulated here leads to a rather different picture. If, at the outset of this section, one defines "particle" in terms of plane waves, the individual modes $\pm\mathbf{k}$ behave "nearly thermally" but there will exist correlations between the modes $+\mathbf{k}$ and $-\mathbf{k}$. If, alternatively, one defines "particle" in terms of sines and cosines, one infers that there will exist no correlations between the individual modes, but one finds that the probability distributions $\mathcal{P}(n)$ for the modes are not "thermal."

What is, however, clear is that, as argued in Sec. II, regardless of one's definition of modes, the proportioning of energy must be "nearly thermal." For particle creation on an instanton, the duration of the instanton serves to define an effective "temperature," and, to the extent that $|\alpha_k/\beta_k|^2 \approx 1$, this implies that an initial vacuum will evolve to a "nearly thermal" state.

One final point should be stressed. Both definitions of "particle" introduced above coincide in the sense that, with respect to either definition, there is no ambiguity as to whether or not particles are present, and, moreover, there will be complete agreement as to how much energy is present in the form of particles. The ambiguity observed here does *not* reflect a question of observers seeing different amounts of energy in the matter sector: it is fundamentally unrelated to the question of which is the "right" observer. That was determined by the restriction to a statically bounded instanton, and the use of the time translation symmetry associated therewith. Rather, the ambiguity reflects upon how the observer chooses to do his or her bookkeeping, i.e., whether it is deemed convenient to use real or complex modes.

IV. ENTROPY GENERATION ON INSTANTONS

There are two absolutely crucial points to be understood about the propagation of test fields on a gravitational instanton. (1) At least to the extent that any back reaction on the geometry is neglected, the matter wave function χ corresponds to a pure state, so that one must introduce some sort of "coarse graining" in order to obtain a nonvanishing measure of entropy. This is a generic feature of both Lorentzian and Euclidean space(time)s. (2) Because there is no notion of a unitary evolution for a (pure or mixed state) density matrix ρ propagating on an instanton, manifest probability conservation is lost, so that, in a Euclidean setting, one can no longer conclude that the quantity $\mathrm{Tr}f(\rho)$ is conserved for any f.

The possible sorts of "coarse grainings" that one might envision are essentially identical in the Euclidean and Lorentzian settings. As will be evidenced below, it is only when one focuses on the problem of evolution that fundamental differences are manifest.

At least two different sorts of "coarse grainings" have been envisioned in the context of quantum field theory in a cosmological setting. One of these entails the splitting of the total field(s) into two pieces, a "system" and a "bath," and then defining a reduced density matrix ρ_S for the system by tracing over the bath variables. In this context one can then define a "system entropy"

$$S_S = -\mathrm{Tr}_S\rho_S \ln\rho_S$$

or, in principle, an analogous "bath entropy"

$$S_B = -\mathrm{Tr}_B\rho_B \ln\rho_B \ , \tag{47}$$

in terms of the reduced "bath" matrix ρ_B. Note in particular that, in the context of quantum physics, S_S and

S_B will in general be nonvanishing even for a pure state ρ, this in sharp contrast to the situation arising in classical physics.

Another possibility is to define a measure of entropy relative to some complete (or less than complete) set of observables $\{\mathcal{O}\}$. The idea here is to compute the probabilities $P(\mathcal{O}_i)$ for each possible outcome, and then to define an informational or von Neumann entropy[23]

$$S_O = -\sum_i P(\mathcal{O}_i)\ln P(\mathcal{O}_i) \ . \tag{48}$$

The problem of defining a "system entropy" S_S has been considered quite generally in the context of Lorentzian field theory,[24] and, as will be seen below, this *definition* (if not its implications) carries over immediately to a Euclidean setting. One particularly subtle implementation of this picture entails choosing the "system" as corresponding simply to that piece of the wave function or density matrix which does not reflect mode-mode correlations.[25] In a certain sense, this is analogous to defining a classical Boltzmann entropy in terms of a one-, rather than N-particle, distribution function.

The informational S_O construction has been considered in a Lorentzian[18] setting for the special (and particularly physical?) case in which the observables of interest are the numbers of particles in each mode of the field. The obvious difficulty in implementing such a prescription is that there exists no compelling definition of a "physical particle" while the system is evolving, either in a Euclidean or Lorentzian manner, so that there can be no natural notion of a continuous evolution for a "number" entropy S_N. One *does*, however, have a natural physical definition of particle at the initial and final boundaries, relative to some *given* mode decomposition, so that it is meaningful to compute and compare $S_N(t_1)$ and $S_N(t_2)$.

For the special case in which the initial and final boundaries are *flat*, one can also define Wigner functions via three-dimensional Fourier transforms and then interpret them as providing quantum distribution functions $f(\mathbf{x},\mathbf{p})$. The net result is a Wigner function entropy (up to overall scale factors) of the form

$$S_W = -\int d^3x \int d^3p[f\ln f-(1+f)\ln(1+f)] \ . \tag{49}$$

By assuming that f is spatially homogeneous one then concludes (cf. Ref. 26) that

$$f(\mathbf{x},\mathbf{p}) = \mathrm{Tr}\rho a^\dagger(\mathbf{p})a(\mathbf{p}) = \langle N(\mathbf{p})\rangle \ . \tag{50}$$

The Wigner function entropy S_W is just a probe of the average number of particles in each mode.

One could of course also seek to exploit the quasilocal construction of Wigner functions in curved spaces developed by Calzetta, Habib, and Hu,[27] but it is clear that the Lorentzian evolution equation for their f_W will be altered signifcantly in a Euclidean setting.

The aforementioned constructions all work equally well at a given instant in either a Euclidean or a Lorentzian setting, provided that one imposes suitable normalizations on P or ρ. Given, however, that probability is not conserved in the Euclidean case, the behavior of these "entropies" as the system evolves differs critically in these two settings.

Because the simple models in Sec. III entail no mode-mode couplings, they cannot be used to study coarse-grained "correlational" entropies as in Ref. 25. The basic physical point can, however, be understood very simply by considering an isolated system assumed to consist of only two modes, (say) k and l, characterized by a density matrix $\rho(k,l)$. One "obvious" coarse graining then arises by neglecting all mode-mode correlations and focusing on the reduced matrices $\rho(k)$ and $\rho(l)$. In such a setting it is natural to contrast the "true" entropy

$$S = -\mathrm{Tr}\rho(k,l)\ln\rho(k,l) \tag{51}$$

with the "coarse-grained" entropy

$$S_S = -\mathrm{Tr}\rho(k)\rho(l)\ln\rho(k)\rho(l) \tag{52}$$

that obtains by neglecting the correlations embodied in the "bath" contribution $\rho_B \equiv \rho(k,l)-\rho(k)\rho(l)$. The crucial point then is that, in both Lorentzian and Euclidean settings, one can conclude that, for fixed reduced matrices $\rho(k)$ and $\rho(l)$, the coarse-grained $S_S \geq S$, with equality if and only if the total density matrix factorizes, so that $\rho_B \equiv 0$.[28]

In a Lorentzian setting this fact has a beautiful implication: If there exist no mode-mode correlations at some given time t_0, then

$$S_S(t) \geq S_S(t_0) \tag{53}$$

for all other times. That this be true follows as a direct consequence of the unitary evolution of the field. The absence of correlations at t_0 implies that $S_S(t_0)=S(t_0)$. However, the "true" S is conserved, so that $S(t)=S(t_0)$. And thus, since $S_S(t)\geq S(t)$, it follows that $S_S(t)\geq S(t)=S(t_0)=S(t_0)$. The physical implication of this result is obvious: In a Lorentzian setting, the entropy S_S must grow if an initially uncorrelated configuration evolves mode-mode correlations.

It is important to stress that this "natural" conclusion does *not* remain valid in a Euclidean setting. The absence of a unitary evolution means that there is no guarantee that the "true" entropy is conserved, i.e., in general $S(t_1)\neq S(t_2)$. In the Euclidean case, the growth of correlations does *not* guarantee a commensurate growth of the entropy.

Consider now the information entropy

$$S_N = -\sum_k \sum_{N_k\in k} P(\{N_k\})\ln P(\{N_k\}) \equiv \sum_k S_k \tag{54}$$

associated with the number of particles in each mode. It is clear that the probabilities $P(\{N_k\})$ are nothing other than the diagonal components of the density matrix in a numbers representation. And thus, it is easy to see[28] that $S_N = S$ if ρ is diagonal, but that, otherwise, $S_N > S$. A simple variational argument shows that, for fixed diagonal components $P(\{N_k\})$, S_N is minimized by that particular density matrix ρ which is itself diagonal.

In a Lorentzian setting this has a striking implication. If, at some instant, ρ is diagonal in a numbers representation, thus corresponding to a "random phase" density matrix, the true entropy $S(t_0)=S_N(t_0)$, whereas, at all other times, $S_N(t)\geq S(t)$. Unitarity implies, however, that $S(t_0)=S(t)$, so that $S_N(t)\geq S_N(t_0)$. This result may be interpreted physically by saying that particles are necessarily created with "phase correlations," this implying that the field is losing its random phase character and generating entropy.

This result can also be formulated in terms of the uncertainty principle.[19] Initially, the field is in a random phase state characterized by a maximal uncertainty as to the "phase information" complementary to particle number. The subsequent evolution implies a generation of nontrivial phase information, so that the "spread" in the N_k's would be expected to grow. This is manifest[18] by an increase in S_N as well as, e.g., the dispersion ΔN_k.

In a Euclidean setting, one can still use the uncertainty principle to "compare" two different states at some given instant of time. However, because of the lack of a unitary evolution, one cannot use it to compare states at different times. There is simply no guarantee that, even if the system is in a random phase state at t_0, $S_N(t)\geq S_N(t_0)$ for all other times.

The moral of the story is in fact quite simple: Comparing states on some given $t=$const surface is equally reasonable in either a Euclidean or a Lorentzian setting. However, comparing states on different surfaces is much less natural in a Euclidean setting because of the absence of a unitary evolution.

The obvious point is that the aforementioned theorems about the evolution of a quantum field in a Lorentzian setting rely only upon properties of the field, such as a "random phase" initial state or an absence of initial mode-mode correlations. They do *not* depend upon how "particle" is defined or on any special properties of the background spacetime. It does not, e.g., matter whether the Universe is expanding or contracting, i.e., in some sense, whether time is running forward or backward. This is of course fundamentally different from the case of fields propagating on an instanton. Tunneling solutions most definitely *do* care about the direction of time: it is the choice of an "outgoing" instanton, i.e., one in which the Universe is expanding, that leads to a suppression of initial excitations and an evolution in the $|\alpha_k|\approx|\beta_k|$ limit toward a "nearly thermal" end state.

For a Lorentzian field theory defined on a fixed background, any effective "arrow of time" that is observed reflects entirely the choice of initial conditions for the quantum field. However, in a Euclidean field theory this "arrow" must reflect properties of the background space as well, as manifest, e.g., by a sign choice in the ansatz $\exp(\pm m_P^2 S)$ leading to an imaginary time Tomonaga-Schwinger equation.

The final question is to ascertain "how thermal" the end state of the field really is. Does one, e.g., obtain a thermal entropy in the $|\alpha_k|\approx|\beta_k|$ limit? As a concrete example, consider a homogeneous and isotropic instanton of the form given by Eq. (12). In this case, one finds that, in terms of a complex plane wave decomposition, the individual probability distributions $P(N_k)$ and $P(N_{-k})$ are thermal but that the joint probability $P(N_k,N_{-k})$ is *not*. Indeed, one sees that

$$S(k,-k)\equiv-\sum P(N_k,N_-k)\ln P(N_k,N_{-k}) = \tfrac{1}{2}S_{\rm Th}\ , \tag{55}$$

whereas

$$S(k)+S(-k)\equiv-\sum P(N_k)\ln P(N_k)\ln P(N_k) -\sum P(N_{-k})\ln P(N_{-k}) = S_{\rm th}\ . \tag{56}$$

The end state has correlations between the modes k and $-k$, so that any probe of such possible correlations will yield a lower entropy than that attributable to a true thermal state. That such correlations exist is of course indicative of nontrivial phase coherence among the created particles, so that, in view of the uncertainty principle, one might also anticipate a higher than thermal particle constant. This is indeed manifest [cf. Eqs. (32) and (33)] by such quantities as the expectation value $\langle N_k N_{-k}\rangle$.

If instead one were to consider an expansion in real modes, even the individual probabilities $P(N_k)$'s would no longer be thermal since particles are always created in pairs. And thus, one would conclude again that the entropy is smaller than the thermal value $S_{\rm th}$.

The Wigner function entropy *does* of course yield a precisely thermal result. As noted already, the "distribution function" $f(\mathbf{x},\mathbf{p})$ only probes $\langle N_k\rangle$, and is thus guaranteed to reproduce the correct thermal result whenever $\langle N_k\rangle=\langle N_k\rangle_{\rm th}$. In this sense one might argue that, since the "nearly thermal" results for particle creation on an instanton reflect energetic considerations, which are manifest directly by the average number of particles in each mode, the Wigner function entropy may be the most natural measure of entropy to consider for particle creation on instantons.

V. GENERIC PROPERTIES OF PARTICLE CREATION ON INSTANTONS

The preceding sections of this paper are somewhat restricted, in that they have considered the problems of particle creation and entropy generation on instantons only for some rather specialized models. It is therefore important to ascertain the extent to which the conclusions are in fact generic, rather than model dependent.

In this connection, the first important point to observe is that the assumption of a scalar Bose field was not crucial. The basic picture works equally well for a spinor field, the only difference being that, as one would anticipate, the "nearly thermal" results now reflect a Fermi-Dirac, rather than Bose-Einstein, distribution. Consider, e.g., a spinor field propagating on an instanton of the form considered in Sec. III. Here an initial vacuum will once again evolve to a final state of the form[29,30]

$$|{\rm out}\rangle=\zeta\exp(D_k a_k^\dagger a_{-k}^\dagger)|0\rangle\ , \tag{57}$$

where D_k is defined as in Eq. (24). However, because the field now satisfies anticommutation relations, one finds upon expanding out the exponential that only the first two terms will contribute. This means that, in contrast to Eq. (26), the joint probability distribution

$$\mathcal{P}(n,m)=\frac{|D_k|^{2n}}{1+|D_k|^2}\delta_{nm} \tag{58}$$

when both n and $m\leq 1$, and vanishes otherwise. One thus infers that

$$\langle N_k(\tau_2)\rangle_V=\frac{D_k^2}{1+D_k^2} \to \frac{1}{|\alpha_k/\beta_k|^2\exp[4\omega_k(\tau_2)T]+1}\ , \tag{59}$$

which is the appropriate "near thermal" form for particles obeying Fermi-Dirac, rather than Bose-Einstein, statistics.

Rather than the choice of particle statistics, the really special feature of the simple models considered in Secs. III and IV was that one could speak of a single fixed duration (in imaginary time) for the instanton. It was this fact that implied the possible existence of a *uniquely* defined temperature inversely proportional to this duration. If different "parts" of the instanton last for different lengths of (imaginary) time, there is no obvious sense in which one can hope to speak of a single temperature for the tunneling process.

It is, however, clear that, despite this difficulty, there are certain generic properties of particle creation on an instanton that *are* independent of the assumption of one single duration. Most obvious, perhaps, is the fact that the choice of a minus sign in the Tomonaga-Schwinger equation always implies that the system is trying to "forget" its initial conditions. This conclusion relies only on the assumption that $\tau(x)$ is systematically increasing for all x during the tunneling.

Less obvious, but still most likely true, is the expectation that, even for a (mildly) inhomogeneous instanton, one should be able to speak of a "nearly thermal" state for modes with wavelengths short compared with the scale of the inhomogeneity. Physically, one can think of evolution on the instanton as involving a continuous creation and destruction of virtual pairs until the final boundary, at which point the pairs become real. And, to the extent that these virtual pairs see a spatial environment that is nearly homogeneous and isotropic, one might anticipate that they would once again end up manifesting a "nearly thermal" distribution. The basic "near thermal" behavior derived here should be robust with respect to at least small deviations from homogeneous tunneling.

In this regard it is worth emphasizing explicitly that the formal problem of calculating particle creation on an instanton works quite generally, regardless of any assumptions regarding the metric or topological properties of the instanton. The nonunitary Bogoliubov transformation technique[8] works for *any* instanton with product topology $\Sigma\times R$. And, for more complicated instantons one can always use a functional integral approach.[9,10]

When allowing for the possibility of more complicated, topology-changing instantons, where this functional integral approach becomes essential, one can of course envision all sorts of new possibilities. Thus, e.g., there is the possibility of "fluctuations" in topology, as reflected by "handles" on a basic instanton of product topology $\Sigma\times R$; and there is even the possibility of a single Universe "bifurcating" into two or more separate entities, which *a priori* need not contain matter fields at the same temperature. The nature of particle creation on such instantons is clearly far more complicated than for the simple models considered in this paper, but, whatever the complications that arise, it *is* clear that the basic physics necessarily manifests an "arrow of time" that results from the very fact that one is considering an instanton.

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