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Charged black hole in a grand canonical ensemble

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A spherical charged black hole in thermal equilibrium is considered from the perspective of a grand canonical ensemble in which the electrostatic potential, temperature, and surface area are specified at a finite boundary. A correspondence is established between the boundary-value data of a well-posed problem in a finite region of Euclidean spacetime and the freely chosen thermodynamic data specifying the ensemble. The Hamiltonian and Gauss's-law constraints are solved and eliminated from the Einstein-Maxwell action, producing a "reduced action" that depends upon two remaining degrees of freedom (two free parameters), as well as on the thermodynamic data. The black-hole temperature, entropy, and corresponding electrostatic potential then follow from relations holding at the stationary points of the reduced action with respect to variation of the free parameters. Investigation of an appropriate eigenvalue problem shows that the criteria for local dynamical and thermodynamical stability are the same. The ensemble can be either stable or unstable, depending upon a certain relation involving mean charge, gravitational radius, and boundary radius. The role of the reduced action in determining the grand partition function, the thermodynamics of charged black holes, and the density of states is discussed.

I. INTRODUCTION

Progress in deriving black-hole thermodynamics from the statistical mechanics of gravitational fields has required explicit recognition that self-gravitating systems in thermal equilibrium are spatially inhomogeneous.[1] In accordance with the principle of equivalence, intensive variables such as temperature and chemical potential vary in the same way as does wave frequency in the gravitational redshift. In setting up an ensemble, one must therefore specify where such variables take their values. In this paper we consider a grand canonical ensemble under the condition of spherical symmetry and specify the boundary radius r_B of the system together with the values of the inverse temperature β and chemical potential ϕ determined by an observer at rest at r_B. The thermodynamic data can thus be viewed as boundary-value data, and indeed they do play that role in the correspondence between statistical thermodynamics and certain well-posed boundary-value problems in Euclidean spacetime.[2] This correspondence is fundamental in the present analysis.

With thermodynamic data such as $\{\beta,\phi,r_B\}$ given, the appropriate Massieu function[3] is $\ln Z_{\rm GC}=-\beta\Omega$, where $Z_{\rm GC}$ is the grand canonical partition function and $\Omega(\beta,\phi,r_B)$ is the grand potential. Our method for obtaining an approximation to $\ln Z_{\rm GC}$ is based upon finding the stationary points of an action from which the constraints have been eliminated. From $\ln Z_{\rm GC}$ one can derive the various thermodynamic quantities and response functions by differentiation. Of special interest for self-gravitating systems is the issue of the stability of equilibrium in a given type of ensemble, that is, thermodynamical stability with respect to certain fixed boundary conditions. This question can be settled by an examination of the local and global behavior of the appropriate Massieu function.

The main purpose of this paper is to generalize previous work[4,5] on gravitational statistical mechanics to a case in which gravity is coupled to a gauge field, namely, the electromagnetic field. The system is described in a grand canonical ensemble in which the boundary radius and the temperature and chemical potential at that radius are fixed. The chemical potential is conjugate to electric charge and is therefore an electrostatic potential energy difference per unit charge. The charge itself is not fixed. Our main results are, first, a complete derivation from an action principle of the temperature, entropy, and electrostatic potential of a charged black hole and, second, a demonstration of the equivalence of local dynamical and thermodynamical stability using the same action principle. We find that the ensemble can be either locally stable or unstable according to an explicit relation among the system parameters.

We have chosen the boundary conditions of a grand canonical ensemble because, first, they are the ones corresponding to the usual Einstein-Maxwell action in which $g_{\mu\nu}$ and A_μ are fixed on the boundary and, second, our boundary conditions lead to a well-posed problem and a stable ensemble. We are aware that particular problems involving black holes may call for boundary conditions other than the ones we have adopted here. But even though different boundary conditions will generally yield ensembles with inequivalent behaviors, it is nevertheless possible to obtain the existence or nonexistence (stability or instability) of the various ensembles from one well-defined ensemble. For example, for Schwarzschild-like black holes the usual canonical ensemble can be stable, and this can be used to show that the petit canonical ensemble in which pressure and temperature are fixed is always unstable.[6] (This, in fact, is the unique sense in which a black hole in thermal equilibrium can be said unambiguously to be unstable.)

In this work we do not consider the effects of the equilibrating thermal radiation that must reside between the surface of the black hole at $r=r_+$ and the surface of the cavity wall that encloses the system at $r=r_B$; that is, we do not consider the back reaction of the ambient radiation near the hole.[7] However, the added complexity entailed by passing from the absence to the presence of electric fields will be seen to be nontrivial and very instructive.

We work in the Euclidean picture[8] corresponding to regular black-hole topologies with static spherical geometries and electrostatic fields, as described in Sec. II. In Sec. III we consider the Einstein-Maxwell action appropriate for fixing the metric and one-form potential on the boundary, and show that these variables also fix the thermodynamic data of the grand canonical ensemble described above. We simplify this action using the geometry and fields of Sec. II and verify that the resulting simplified classical variational principle is consistent.

In Sec. IV we introduce the "reduced action" I_*, which is obtained from the simplified classical action I in two steps. First, the gravitational Hamiltonian constraint and the electromagnetic Gauss's-law constraint are solved. The solutions depend on two constants of integration that are shown to be the gravitational radius r_+ and a charge parameter e defined by the flux $4\pi e$ of the electrostatic field through the sphere $r=r_B$. These two free parameters represent the two physical degrees of freedom for fields of the given geometrical and topological type that are compatible with the constraints. The second step in obtaining the reduced action is to insert the solutions of the constraints into I and carry out the integrations to obtain $I_*=I_*(\beta,\phi,r_B;r_+,e)$, a function of five independent variables. Variation of I_* with respect to the physical degrees of freedom r_+ and e reveals that when I_* has a nontrivial real extremum, the previously arbitrary ensemble data β and ϕ are the equilibrium inverse temperature and associated electrostatic potential, evaluated at $r=r_B$, of a black hole of gravitational radius r_+ and charge e. (In Sec. V we show explicitly that the ensemble mean charge is $\langle Q \rangle = e$.) Thus the principle of stationary reduced action for the grand canonical ensemble leads to a simple derivation not only of the thermal equilibrium temperature, but also of the electrostatic potential of a black hole.

Examination of the conditions for the existence of real nontrivial stationary points of the reduced action leads to a cubic equation that determines r_+ as a function of β, ϕ, and r_B. From this equation it follows that real local extrema must occur in pairs and that they exist if and only if a certain quantum electrogeometrical inequality is satisfied. Mathematically, the inequality expresses the nonpositivity of the discriminant of the cubic equation. We determine the physical distinction between the two solutions in Sec. V by studying an appropriate eigenvalue problem.

In Sec. V we obtain an expression for the Massieu function of the grand canonical ensemble by introducing the zero-loop approximation[9] $\ln Z_{GC} \approx -\tilde{I}(\beta,\phi,r_B)$, where $\tilde{I}$ is I_* evaluated at an extremum in the parameters r_+ and e. From $\tilde{I}$ all of the thermodynamic relations can be obtained; in particular, we show that $\langle Q \rangle = e$ and we find an expression for the mean thermal energy $\langle E \rangle$. The entropy is found to have the value πr_+^2, as expected on the basis of black-hole thermodynamics. This procedure makes sense if the grand canonical ensemble is stable; we demonstrate that local stability indeed holds when the system configuration is described by a particular one of the two solutions mentioned above. This is shown by examining the eigenvalue problem associated with the matrix of second derivatives of I_* with respect to e and r_+ evaluated at a stationary point. Our demonstration establishes simultaneously the equivalence of local dynamical and thermodynamical stability.[10] We also give explicitly the criterion guaranteeing that the locally stable extremum is a global minimum of I_*.

In Sec. VI we indicate how to obtain the canonical ensemble in which charge, rather than electrostatic potential, is fixed at the boundary. We also give a brief discussion of the density of states, whose principal features follow from the existence of stable configurations in the grand canonical ensemble.

II. TOPOLOGY AND GEOMETRY

We consider gravitational fields in the black-hole topological sector with metrics of the form

$$ds^2=b^2d\tau^2+\alpha^2dy^2+r^2(d\theta^2+\sin^2\theta\, d\varphi^2)\ , \tag{2.1}$$

where b, α, and r are functions only of the radial coordinate $y\in[0,1]$. The "Euclidean time" τ is chosen to have period 2π; the angles θ and φ are the usual coordinates of the unit sphere. By "black-hole topology" we mean that the four-geometries (M,g) are regular with product topology $R^2\times S^2$, boundary $S^1\times S^2$ at $y=1$, and Euler number $\chi=2$.

The boundary at $y=1$ has a standard round two-sphere S^2 with area $4\pi r_B^2$, where $r_B=r(1)$. This sphere may be thought of as a cavity wall at r_B enclosing a spherical system with a black hole at the center and through which heat can flow in either direction to maintain a temperature at the wall of $T\equiv T(r_B)\equiv\beta^{-1}$. The inverse temperature is related to the proper length of the round S^1 of the boundary by

$$\beta=T^{-1}=\int_0^{2\pi}b(1)d\tau=2\pi b(1)\ . \tag{2.2}$$

(We use units with $G=\hbar=k_B=c=1$ unless otherwise noted.) Of course, there must be radiant energy between the surface of the hole at $r_+=r(0)$ and the cavity wall r_B in order even to speak of a temperature β^{-1} describing the system. The explicit effect of this radiation ("back reaction") will be neglected in the present work,[7] but this in no way affects the statement of the boundary conditions, the method of analysis, or the qualitative conclusions. We see that these boundary conditions specify the three-geometry of the boundary $S^1\times S^2$ and furnish the thermodynamic data β and r_B that are needed for a canonical or grand canonical description.

The center $y=0$, $r(0)=r_+$, which represents the horizon of the black hole, is not a component of the three-dimensional boundary of M; rather, it is a regular two-sphere with nonzero area $4\pi r_+^2$. We can think of $y=0$ as a degenerate leaf of a foliation of M by three-dimensional hypersurfaces $y=$const. These leaves have topology $S^1\times S^2$. As $y\rightarrow 0$, the S^1 shrinks smoothly to zero to give the S^2 at the center; thus we have

$$b(0)=0 \ . \tag{2.3}$$

This description in the context of a smooth geometry requires that the y-τ plane near $y=0$ be isometric to a flat disk. The corresponding regularity condition is

$$(\alpha^{-1}b')|_{y=0}=1 \ , \tag{2.4}$$

where a prime denotes $\partial/\partial y$. Combining (2.3) and (2.4) shows that for a point $y=\bar{y}$ very near $y=0$, the proper radial distance is

$$\int_0^{\bar{y}}\alpha\, dy \cong b(\bar{y}) \ . \tag{2.5}$$

The regularity conditions (2.3) and (2.4) will not be varied in the action principles.

From the Gauss-Bonnet-Chern formula applied to the metrics (2.1), the Euler number of M is deduced to be

$$\begin{aligned}\chi&=2\{(\alpha^{-1}b')[1-\alpha^{-2}(r')^2]\}_{y=0}\\ &=2[1-\alpha^{-2}(r')^2]_{y=0} \ ,\end{aligned} \tag{2.6}$$

where (2.4) was used to obtain the second form of (2.6). In order to have $\chi=2$, the condition

$$[\alpha^{-2}(r')^2]_{y=0}=0 \tag{2.7}$$

must hold. This condition will be used in Sec. IV. Condition (2.7) is not used in the variational principles.

The electromagnetic field will be described by the one-form $A_\mu dx^\mu$. With static spherical symmetry we have $A_\mu dx^\mu = A_\tau(y)d\tau + A_y(y)dy$. The second term plays no essential role and can be removed by a gauge transformation to yield the form we shall employ: namely,

$$A_\mu dx^\mu = A_\tau(y)d\tau \ . \tag{2.8}$$

Let us examine the regularity condition on A_μ at $y=0$. Note that the proper orthonormal frame component of the potential one-form is $A_{\hat{\tau}}=b^{-1}A_\tau$. From elementary physics it follows that $A_{\hat{\tau}}$ will be the electrostatic potential multiplied by a constant k. If we require that this potential be bounded, then because $b(0)=0$, we must impose the regularity condition

$$A_\tau(0)=0 \ . \tag{2.9}$$

Our boundary condition on the electromagnetic field is to fix $A_\tau(1)$ to some constant value. The discussion above and (2.2) show that

$$A_\tau(1)=b(1)A_{\hat{\tau}}(1)=k\beta\phi(2\pi)^{-1}$$

where k is a constant and ϕ is the difference of potential between $y=0$ and 1, i.e., $\phi\equiv\phi(0)-\phi(1)$. In Sec. IV, from an examination of the Gauss's-law constraint, we will show that $k=-i$. Therefore, the boundary condition is

$$A_\tau(1)=\frac{\beta\phi}{2\pi i} \ , \tag{2.10}$$

with ϕ fixed. Both $A_\tau(0)$ and $A_\tau(1)$ are held fixed in our variational principles.

Fixing both the three-geometry and the electromagnetic potential one-form at the boundary $y=1$ gives the thermodynamic data $\{\beta, r_B, \phi\}$ required for a discussion of the grand canonical ensemble. The regularity conditions (2.3), (2.4), and (2.9), together with (2.7), correspond to the topology of a black hole that can be electrically charged. [By contrast the case of "trivial" topology $R^3\times S^1$ (with the same boundary and boundary conditions) would correspond to the conditions $r(0)\equiv 0$, $r'(0)\equiv\alpha(0)$, and $A'_\tau(0)=0$.]

III. ACTION AND FIELD EQUATIONS

The Euclidean gravitational action suitable when the three-geometry of the boundary is fixed is given as in previous works by[1,8]

$$I_g=-\frac{1}{16\pi}\int_M d^4x\,\sqrt{g}\,R+\frac{1}{8\pi}\int_{\partial M}d^3x\,\sqrt{\gamma}(K-K^0) \ , \tag{3.1}$$

where the subtraction term (the boundary term with K^0) leads to the "normalization" $I_g=0$ for flat spacetime with the given boundary data. It should be emphasized that the subtraction term does not affect the derivation of temperature, electrostatic potential, or entropy. As we shall see, however, the subtraction term does affect our expression for the mean thermal energy $\langle E\rangle$, ensuring that $\langle E\rangle\rightarrow M$ as $r_B\rightarrow\infty$, where M is the Arnowitt-Deser-Misner (ADM) mass of the black hole.

Using the metric (2.1), one can show that

$$R=-\frac{2}{\alpha b r^2}\left[\frac{r^2b'}{\alpha}\right]'-2G^\tau_\tau \ , \tag{3.2}$$

where

$$G^\tau_\tau=\frac{1}{r^2r'}\left[r\left[\frac{(r')^2}{\alpha^2}-1\right]\right]' \ . \tag{3.3}$$

The other nonvanishing components of the Einstein tensor are

$$G^y_y=\frac{2b'r'}{\alpha^2br}+\left[\frac{r'}{\alpha r}\right]^2-\frac{1}{r^2} \ , \tag{3.4}$$

$$G^\theta_\theta=G^\varphi_\varphi=\frac{1}{\alpha b r^2}\left[\frac{r^2b'}{\alpha}\right]'+\frac{b}{\alpha}\left[\frac{r'}{\alpha b r}\right]'+\left[\frac{r'}{\alpha r}\right]^2 \ . \tag{3.5}$$

The trace of the extrinsic curvature of a $y=$const hypersurface is

$$K=-\frac{1}{\alpha b r^2}(br^2)' \ .$$

Upon substitution, simplification, and integration over θ

and φ, we obtain

$$I_g=-\frac{1}{2}\int d\tau\, dy\left[\frac{2}{\alpha}rr'b'+\frac{b}{\alpha}(r')^2+\alpha b-2(br)'\right]$$
$$-\frac{1}{2}\int_{y=0}d\tau\left[\frac{1}{\alpha}(br^2)'-2br\right]. \tag{3.6}$$

Variation of (3.6) with respect to b, α, and r yields

$$\frac{\delta I_g}{\delta b}=\pi\alpha r^2 G^\tau_\tau\,, \tag{3.7}$$

$$\frac{\delta I_g}{\delta \alpha}=\pi b r^2 G^y_y\,, \tag{3.8}$$

$$\frac{\delta I_g}{\delta r}=2\pi\alpha b r G^\theta_\theta\,. \tag{3.9}$$

The Euclidean action for the electromagnetic field with A_μ fixed on the boundary is

$$I_A=\frac{1}{16\pi}\int d^4x\,\sqrt{g}\,F_{\mu\nu}F^{\mu\nu}\,, \tag{3.10}$$

with $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. Because $A_\mu dx^\mu=A_\tau(y)d\tau$, the only nontrivial Maxwell equation is the Gauss's-law constraint

$$\left[\frac{r^2}{\alpha b}A'_\tau\right]'=0\,. \tag{3.11}$$

The nonvanishing components of the Maxwell stress-energy tensor

$$T^{\mu\nu}=\frac{1}{4\pi}\left[F^{\mu\sigma}F^\nu{}_\sigma-\frac{1}{4}g^{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}\right] \tag{3.12}$$

are

$$T^\tau_\tau=T^y_y=-T^\theta_\theta=-T^\varphi_\varphi=\frac{1}{8\pi}\left[\frac{A'_\tau}{\alpha b}\right]^2. \tag{3.13}$$

The Maxwell action can be simplified as was the gravitational action, and we find

$$I_A=\frac{1}{2}\int d\tau\, dy\left[\frac{r^2}{\alpha b}A'_\tau\right]A'_\tau\,. \tag{3.14}$$

Variation with respect to A_τ, respecting the boundary and regularity conditions, yields correctly the Gauss's-law constraint (3.11). Variation of the metric functions in (3.14) yields

$$\frac{\delta I_A}{\delta b}=-\pi\frac{r^2}{b^2\alpha}(A'_\tau)^2\,, \tag{3.15}$$

$$\frac{\delta I_A}{\delta \alpha}=-\pi\frac{r^2}{b\alpha^2}(A'_\tau)^2\,, \tag{3.16}$$

$$\frac{\delta I_A}{\delta r}=2\pi\frac{r}{b\alpha}(A'_\tau)^2\,. \tag{3.17}$$

Taking the total action to be

$$I=I_g+I_A\,, \tag{3.18}$$

with I_g in the form (3.6) and I_A in the form (3.14), the functional derivatives (3.7)–(3.9) and (3.15)–(3.17) show that this simplified Einstein-Maxwell action produces the correct field equations and is stationary when the field equations hold and the boundary and regularity conditions are fixed. This demonstrates that we have a consistent framework from which to pass to the reduced action.

IV. REDUCED ACTION AND STATIONARY POINTS

To obtain the reduced action I_* from the simplified action described in Sec. III, we begin by solving the constraints. The Gauss's-law constraint is given by (3.11). It is evident that the first integral is

$$\frac{r^2}{b\alpha}A'_\tau=\text{const}\,. \tag{4.1}$$

The constant can be obtained from the physical consistency requirement that the analytic continuation of the fields A_μ and metric $g_{\mu\nu}$ from Euclidean to Lorentzian spacetime should yield the elementary integral form of Gauss's law with a real charge, say e, and electric flux $4\pi e$ through a sphere surrounding the center. For static field configurations, the passage from Lorentzian to Euclidean variables can be realized by the analytic continuation $t=-i\tau$ between Lorentzian time t and Euclidean time τ. In this way, it is readily seen that the only nonzero physical component of the real Lorentzian electrostatic field is the familiar radial Coulomb field

$$E^{\hat{y}}=\alpha^{-1}(-g_{tt})^{-1/2}A'_t=\frac{e}{r^2}\,. \tag{4.2}$$

However, from $d\tau=i\,dt$, it follows that

$$A_\tau(y)=-iA_t(y)\,, \tag{4.3}$$

so that $A_\tau d\tau=A_t dt$ is invariant. Combining (4.2) and (4.3) and noting that $(-g_{tt})^{1/2}$ continues to b shows that (4.1) becomes

$$\frac{r^2}{b\alpha}A'_\tau=-ie\,. \tag{4.4}$$

A similar argument fixes the same multiplicative constant $(-i)$ in $A_\tau(1)$ that we wrote down in (2.10).

The Hamiltonian constraint is

$$G^\tau_\tau-8\pi T^\tau_\tau=0\,. \tag{4.5}$$

Using (3.3), (3.13), and (4.4), this constraint becomes

$$\frac{1}{r^2r'}\left\{r\left[\left[\frac{r'}{\alpha}\right]^2-1\right]\right\}'+\frac{e^2}{r^4}=0\,, \tag{4.6}$$

which is integrated easily to yield

$$\left[\frac{r'}{\alpha}\right]^2=1-\frac{C}{r}+\frac{e^2}{r^2}\,. \tag{4.7}$$

The integration constant C is found from the demand that the Euler number is $\chi=2$. With $r_+\equiv r(0)$, enforcing (2.7) implies

$$C=r_+ + \frac{e^2}{r_+} \ , \tag{4.8}$$

so that

$$\left[\frac{r'}{\alpha}\right]^2 = 1-\frac{r_+}{r}+\frac{e^2}{r^2}-\frac{e^2}{r_+ r} = \left[1-\frac{r_+}{r}\right]\left[1-\frac{e^2}{r_+ r}\right] . \tag{4.9}$$

To relate the above result to the more familiar form of the radial contravariant metric component of a charged black hole, it may be helpful to recall the relation between the gravitational radius and the ADM mass:

$$r_+ = M+(M^2-e^2)^{1/2} . \tag{4.10}$$

Then, substituting (4.10) into (4.9) reveals the familiar form

$$g^{rr} = \left[\frac{r'}{\alpha}\right]^2 = 1-\frac{2M}{r}+\frac{e^2}{r^2} . \tag{4.11}$$

It is interesting to note that M plays essentially no role in our work; we have displayed (4.11) only to make the natural variables of the problem seem more familiar. Also note that we have not yet established any relationship whatever between the unconstrained dynamical degrees of freedom r_+ and e and the freely prescribed thermodynamic boundary data $\{\beta,\phi,r_B\}$.

The reduced action $I_* = I_*(\beta,\phi,r_B;e,r_+)$ can now be obtained by substituting the solutions (4.4) and (4.9) into the sum $I_g + I_A$ and performing the integrations over y and τ. After a straightforward calculation we find, using also $\beta\phi = 2\pi i A_\tau(1)$ from (2.10),

$$I_* \hbar^{-1} = G^{-1}\beta r_B \times \left[1-\left[1-\frac{r_+}{r_B}\right]^{1/2}\left[1-\frac{Ge^2}{r_+ r_B}\right]^{1/2}\right] -(G\hbar)^{-1}\pi r_+^2 - \beta\phi e \ . \tag{4.12}$$

Note that in (4.12) we have temporarily restored Newton's constant G and Planck's constant $\hbar$ (keeping $k_B = c = 1$). It is evident from the appearance of the term $(G\hbar)^{-1}\pi r_+^2$ that the "classical" action of the topologically nontrivial Euclidean fields produces a decidedly quantum-mechanical term—the term that will be shown to be the entropy of the system. [Observe that no other expression on the right-hand side of (4.12) contains Planck's constant.]

To obtain the stationary points of I_* with respect to e and r_+ is a simple matter of differentation with β, ϕ, and r_B held fixed. We find (reverting to absolute units)

$$\frac{\partial I_*}{\partial e} = \frac{\beta e}{r_+}\left[1-\frac{r_+}{r_B}\right]^{1/2}\left[1-\frac{e^2}{r_+ r_B}\right]^{-1/2} - \beta\phi \ , \tag{4.13}$$

$$\frac{\partial I_*}{\partial r_+} = \frac{1}{2}\beta\left[1-\frac{r_+}{r_B}\right]^{-1/2}\left[1-\frac{e^2}{r_+ r_B}\right]^{1/2} - \frac{1}{2}\beta\frac{e^2}{r_+^2}\left[1-\frac{r_+}{r_B}\right]^{1/2}\left[1-\frac{e^2}{r_+ r_B}\right]^{-1/2} - 2\pi r_+ \ . \tag{4.14}$$

Setting (4.13) to zero yields

$$\phi = \frac{e}{r_+}\left[1-\frac{r_+}{r_B}\right]^{1/2}\left[1-\frac{e^2}{r_+ r_b}\right]^{-1/2} , \tag{4.15}$$

which is the difference in electrostatic potential between r_+ and r_B "blueshifted" from infinity to r_B. Letting $r_B \to \infty$, one easily recovers the usual electrostatic potential of a charged black hole, $\Phi = e r_+^{-1}$. Recalling that charge is not specified in the grand canonical ensemble, we shall show in Sec. V that at equilibrium the mean value of the charge is indeed given by $\langle Q \rangle = e$. Setting (4.14) to zero yields

$$\beta = 4\pi r_+ \left[1-\frac{e^2}{r_+^2}\right]^{-1}\left[1-\frac{r_+}{r_B}\right]^{1/2}\left[1-\frac{e^2}{r_+ r_B}\right]^{1/2} . \tag{4.16}$$

This is the inverse of the temperature of a charged black hole as found by Hawking,[11] blueshifted from infinity to r_B.

Using (4.16) and (4.15), r_+ and e can be written as functions of β, ϕ, and r_B, as required in expressing a thermal equilibrium configuration in the grand canonical ensemble. Combining these equations yields

$$(1-\phi^2)x^3 - x^2 + (1-\phi^2)^2\sigma^2 = 0 \ , \tag{4.17}$$

and

$$q = \frac{\phi x^2}{\sigma(1-\phi^2)} \ , \tag{4.18}$$

where

$$x \equiv \frac{r_+}{r_B}, \quad q \equiv \frac{e}{r_B} \ , \tag{4.19}$$

and

$$\sigma \equiv \frac{\beta}{4\pi r_B} \ . \tag{4.20}$$

We assume that the free parameters satisfy $e^2 < r_+^2$ and that the specified potential satisfies $\phi^2 < 1$. The cubic equation (4.17) has two real and positive solutions satisfying the inverse temperature relation (4.16) if and only if the discriminant of the cubic is nonpositive.[12] This condition yields the quantum electrogravitational inequality

$$3\sqrt{3}\beta\hbar(1-G\phi^2)^2 \le 8\pi r_B \ , \tag{4.21}$$

where G and $\hbar$ have been restored. These two solutions, denoted $x_>$ and $x_<$, can be expressed as

$$x_>(1-\phi^2) = \frac{1}{3}\left[1+2\cos\frac{\alpha}{3}\right] , \tag{4.22a}$$

$$x_<(1-\phi^2)=\frac{1}{3}\left[1-2\cos\left[\frac{\alpha}{3}+\frac{\pi}{3}\right]\right], \tag{4.22b}$$

where

$$\cos\alpha=1-\tfrac{27}{2}\sigma^2(1-\phi^2)^4 , \tag{4.23}$$

with $0\le\alpha\le\pi$. These solutions generalize Eqs. (3) and (4) of Ref. 1. When equality holds in (4.21), the discriminant vanishes and the two solutions coincide. Studying this case reveals that $\phi^2<\frac{1}{3}$ is necessary in order that $x_>\,<1$, which is required in order that it can be interpreted as a black hole inside a thermal cavity, that is, $r_+<r_B$. J. Louko has pointed out to us that the third root of (4.17) can also be interpreted as a solution of our boundary-value problem. However, $r_+<0$ for this solution, and so the corresponding metric must be complexified in order to avoid a curvature singularity at $r=0$. The action of this solution is real, but because it is greater than that of either (4.22a) or (4.22b), we shall not consider it further in this paper. (A solution with similar features also arises in the canonical ensemble, discussed below in Sec. VI.) When the inequality (4.21) is violated, the reduced action has no real positive stationary points. However, (4.17) does possess a conjugate pair of complex solutions and a real negative solution with the given real values of β, ϕ, and r_B. We shall not consider solutions other than (4.22a) and (4.22b) any further in this article, although they could ultimately prove to be of interest.

Having obtained the two real solutions $x_>$ and $x_<$ as indicated above, we can now substitute them into (4.18) to obtain $q\equiv er_B^{-1}$. Thus one finds $r_+=r_+(\beta,\phi,r_B)$ and $e=e(\beta,\phi,r_B)$ when $\partial I_*/\partial e=0$ and $\partial I_*/\partial r_+=0$ at equilibrium.

V. DYNAMICAL AND THERMODYNAMICAL STABILITY

We consider here an approximation to the statistical mechanics of gravitational fields in the black-hole topological sector—the level of approximation that corresponds to black-hole thermodynamics. These results are obtained by what can be called a "zero-loop approximation" to the partition function calculated as a path integral in which back-reaction effects are ignored.[7] This corresponds in the present case to adopting as the Massieu function for the grand canonical ensemble the expression $\ln Z_{GC}\approx-\tilde{I}(\beta,\phi,r_B)$, where $\tilde{I}$ is I_* evaluated at a locally stable stationary point. We show below that one of the two stationary points is indeed locally stable.

The mean value of the charge is found from the standard expression

$$\langle Q\rangle=\beta^{-1}\left[\frac{\partial(\ln Z_{GC})}{\partial\phi}\right]_{\beta,r_B}$$
$$\approx-\beta^{-1}\left[\frac{\partial\tilde{I}}{\partial\phi}\right]_{\beta,r_B} . \tag{5.1}$$

Because $\partial I_*/\partial r_+=\partial I_*/\partial e=0$ at equilibrium, the derivative in (5.1) is easily obtained from differentiating $I_*(\beta,\phi,r_B;e,r_+)$ and evaluating the result using $e(\beta,\phi,r_B)$ and $r_+(\beta,\phi,r_B)$ as determined through Eqs. (4.15) and (4.16). Other derivatives of $\tilde{I}$ are evaluated similarly. For the charge (5.1) we obtain

$$\langle Q\rangle=e , \tag{5.2}$$

as anticipated. The entropy is obtained from

$$S\approx\beta\left[\frac{\partial\tilde{I}}{\partial\beta}\right]_{\phi,r_B}-\tilde{I}=\pi r_+^2 , \tag{5.3}$$

where it is understood that $r_+=r_+(\beta,\phi,r_B)$. This of course is the result expected from black-hole thermodynamics,[11,13] as expressed in the grand canonical ensemble.

The mean thermal energy in this ensemble is defined by

$$\langle E\rangle\approx\left[\frac{\partial\tilde{I}}{\partial\beta}\right]_{\phi,r_B}-\phi\beta^{-1}\left[\frac{\partial\tilde{I}}{\partial\phi}\right]_{\beta,r_B}$$
$$=\left[\frac{\partial\tilde{I}}{\partial\beta}\right]_{\beta\phi,r_B} . \tag{5.4}$$

This produces a new expression for the thermal energy of a charged black hole, namely,

$$\langle E\rangle=r_B-r_B\left[1-\frac{r_+}{r_B}\right]^{1/2}\left[1-\frac{\langle Q\rangle^2}{r_Br_+}\right]^{1/2} , \tag{5.5}$$

where again r_+ and $\langle Q\rangle=e$ are functions of β, ϕ, and r_B. The significance of this quasilocal energy[14] is perhaps best seen by recalling (4.10) and solving (5.5) for the ADM mass M. This results in the physically natural relationship

$$M=\langle E\rangle-\frac{\langle E\rangle^2}{2r_B}+\frac{\langle Q\rangle^2}{2r_B} , \tag{5.6}$$

asserting that the "mass at infinity" is the thermal energy plus the (negative) gravitational binding energy plus the (positive) electrostatic binding energy.

From (5.5) and the previous results, one can now establish the thermodynamic identity

$$d\langle E\rangle=T\,dS-\lambda\,dA+\phi\,d\langle Q\rangle , \tag{5.7}$$

where $T=\beta^{-1}$, $A=4\pi r_B^2$, and the non-negative "surface pressure" λ is given by

$$\lambda=\frac{1}{8\pi r_B}\left\{\left[1-\frac{r_+}{r_B}\right]^{-1/2}\left[1-\frac{\langle Q\rangle^2}{r_+r_B}\right]^{-1/2}\right.$$
$$\left.\times\left[1-\frac{r_+}{2r_B}\left[1+\frac{\langle Q\rangle^2}{r_+^2}\right]\right]-1\right\} . \tag{5.8}$$

Observe that, if A and $\langle Q\rangle$ are fixed, a transfer of a small amount of "heat" determined locally at $r=r_B$ is given by $d\langle E\rangle=T\,dS\equiv T(r_B)dS$. (This experiment could be done with a calorimeter.) This reversible pure heat transfer corresponds directly to a small change of the quasilocal energy $\langle E\rangle$, and so, indirectly, by (5.6) will also lead to a small change in the ADM mass M. One finds that $d\langle E\rangle$ is dM divided by the square root of (4.9) or, equivalently, divided by the square root of (4.11), as expected. Observe that the square root of (4.9) or (4.11) is simply $(1-\langle E\rangle r_B^{-1})$.

The above results make sense if $e(\beta,\phi,r_B)$ [equivalently, $\langle Q\rangle(\beta,\phi,r_B)$] and $r_+(\beta,\phi,r_B)$ [equivalently $S(\beta,\phi,r_B)$] correspond to a locally stable solution. This can be decided by examining the signs of the eigenvalues of the real symmetric matrix $I_{*,ij}$ at the stationary points, with the indices i,j ranging over $\langle Q\rangle$ and S (or e and r_+).

First, note that at equilibrium we have [using $x\equiv r_+r_B^{-1}\equiv(S/\pi)^{1/2}r_B^{-1}$ and $q\equiv er_B^{-1}\equiv\langle Q\rangle r_B^{-1}$]

$$\frac{\partial^2 I_*}{\partial\langle Q\rangle^2}\equiv I_{*,QQ}=4\pi(1-x)\left[1-\frac{q^2}{x^2}\right]^{-1}\left[1-\frac{q^2}{x}\right]^{-1}, \tag{5.9}$$

$$\frac{\partial^2 I_*}{\partial\langle Q\rangle\partial S}\equiv I_{*,QS}=-\frac{2q}{r_Bx^2}\left[1-\frac{1}{2}\left[x+\frac{q^2}{x}\right]\right]\left[1-\frac{q^2}{x^2}\right]^{-1}\left[1-\frac{q^2}{x}\right]^{-1}, \tag{5.10}$$

$$\begin{aligned}\frac{\partial^2 I_*}{\partial S^2}\equiv I_{*,SS}={}&-\frac{1}{2\pi r_B^2x^2}+\frac{1}{4\pi r_B^2x}\left[1-\frac{q^2}{x^2}\right]^{-1}(1-x)^{-1}\left[1-\frac{q^2}{x}\right]+\frac{q^2}{\pi r_B^2x^4}(1-x)\left[1-\frac{q^2}{x^2}\right]^{-1}\\&+\frac{q^2}{2\pi r_B^2x^3}\left[1-\frac{q^2}{x^2}\right]^{-1}+\frac{q^4}{4\pi r_B^2x^5}\left[1-\frac{q^2}{x^2}\right]^{-1}\left[1-\frac{q^2}{x}\right]^{-1}(1-x)\,.\end{aligned} \tag{5.11}$$

The eigenvalues could be calculated by a standard straightforward method. This produces unwidely expressions. There is a much better procedure, however, because only the signs of the eigenvalues are needed. For this purpose, it suffices to compute the pivots of the matrix $I_{*,ij}$, that is, the ratios of its principal minors. These, it turns out, are precisely the thermodynamic response functions that are relevant to the thermal stability of the grand canonical ensemble.

Thermal stability in an ensemble with a black hole must apply to the entire system, because such systems obviously cannot be subdivided into spatially separate parts as is usually done in treating questions of thermodynamic stability. The response functions relevant to the thermal stability of a given type of ensemble, therefore, are those which can be obtained by variation of the extensive variables that are *not* fixed by the boundary conditions defining the given ensemble. Accordingly, for the grand canonical ensemble, we consider second variations of the action with respect to the charge and entropy. We do not vary the size of the system, as defined by the area $4\pi r_B^2$. By relating the relevant response functions to the pivots of the matrix $I_{*,ij}$, we can treat simultaneously the issues of local dynamical and thermodynamical stability.[15]

Observe that the physical range of our variables is

$$x<1,\quad q^2<x^2\,, \tag{5.12}$$

by which we avoid, respectively, the case in which r_+ and r_B coincide and the case of extreme charged holes $|e|=r_+$. The two pivots of

$$I_{*,ij}=\begin{bmatrix}I_{*,QQ} & I_{*,QS}\\ I_{*,QS} & I_{*,SS}\end{bmatrix} \tag{5.13}$$

are, first,

$$I_{*,QQ}=\beta\left[\frac{\partial\phi}{\partial\langle Q\rangle}\right]_{S,r_B}, \tag{5.14}$$

which is positive by inspection of (5.9), and, second,

$$\frac{1}{I_{*,QQ}}\det(I_{*,ij})\equiv\frac{1}{C_{\phi,r_B}}, \tag{5.15}$$

where C_{ϕ,r_B} is the heat capacity at constant electrostatic potential difference and cavity radius. The heat capacity (5.15) can be computed directly from the standard definition

$$C_{\phi,r_B}\equiv-\beta\left[\frac{\partial S}{\partial\beta}\right]_{\phi,r_B}, \tag{5.16}$$

from which we find

$$\frac{1}{C_{\phi,r_B}}=\frac{3x^2-2x-q^2}{4\pi r_B^2x^3(1-x)}\,. \tag{5.17}$$

From (5.14) and (5.17) it follows that when

$$3x^2-2x-q^2>0\,, \tag{5.18}$$

the eigenvalues are both positive, for the signs of the pivots are in one-to-one correspondence with the signs of the eigenvalues.[16] The condition (5.18) for local dynamical and thermodynamical stability generalizes earlier results for a canonical ensemble with a black hole and $\phi=0$.[1,10] The stability requirement (5.18) holds for the solution (4.22a) denoted by $x_>$.

The solution (4.22b) denoted $x_<$ violates the inequality (5.18) and is therefore unstable. Also, when

$$3x^2-2x-q^2=0\,, \tag{5.19}$$

one of the eigenvalues of $I_{*,ij}$ is zero, and the corresponding heat capacity is $\pm\infty$. This corresponds to

$$3\sqrt{3}\beta(1-\phi^2)^2=8\pi r_B\,, \tag{5.20}$$

and

$$x_> = x_< = x_{\rm crit} \ . \tag{5.21}$$

The infinite discontinuity as one passes from $x_< \uparrow x_{\rm crit}$ and $C_{\phi,r_B} \to -\infty$ to $x_> \downarrow x_{\rm crit}$ and $C_{\phi,r_B} \to +\infty$ does not correspond directly to a phase transition. It signals the conditions under which locally stable contact between the system containing a charged black hole is, or is not, possible when it is in thermal and "diffusive" (charge-nonconserving) contact with its environment. We reiterate that because r_B is fixed in the grand canonical ensemble (and in the canonical and microcanonical ensembles as well), any mechanical response functions ("compressibilities") are irrelevant to present issues of stability.[17]

The stationary points corresponding to $x_>$ are the *local* minima of I_*, but not necessarily the global minima, over the space of variables $\{e, r_+\}$ for fixed data $\{\beta,\phi,r_B\}$. One can see this from the fact that I_* approaches zero as e and r_+ approach zero, whereas I_* can be positive at the local minimum corresponding to the given data $\{\beta,\phi,r_B\}$. In such cases we would not expect the zero-loop approximation to $Z_{\rm GC}$ to be accurate. Thus, following Ref. 4, suppose that the grand partition sum is written as

$$Z_{\rm GC}(\beta,\phi,r_B) = \int d\mu(e,r_+) \exp[-I_*(\beta,\phi,r_B;e,r_+)] \ , \tag{5.22}$$

where we assume the measure $d\mu(e,r_+)$ is not exponential in either variable. If the local minimum of I_* is positive, $Z_{\rm GC}$ is dominated by points near the origin $(e=0, r_+=0)$. But these points do not describe thermal equilibrium as it is normally understood, precisely because they are not stationary points of the action. We expect that the correct physics for such a situation would be described in another topological sector.[4] The locally stable stationary point $x_>$ would describe in this case a metastable black hole.

However, for some boundary data the local minimum of I_* is indeed a global minimum, as illustrated in Fig. 1. These cases, in which I_* is negative at its local minimum and the black hole dominates $Z_{\rm GC}$, can be characterized as follows. Let the data $\{\beta,\phi,r_B\}$ be such that the locally stable solution satisifes $0 < x_> < 1$. Then the corresponding local minimum $\tilde{I}$ is negative, and is also a global minimum, if and only if

$$\frac{\beta}{4\pi r_B}(1-\phi^2)^2 < \frac{8}{27} \ . \tag{5.23}$$

When $\phi=0$, this reduces to the result $\beta/4\pi r_B < \frac{8}{27}$ found in Refs. 1 and 4. Combining $x_> < 1$, (4.22a), and (5.23), we see that for I_* to be a global minimum, we must have $\phi^2 < \frac{1}{9}$ and $\sigma < \frac{3}{8}$. Restoring G and $\hbar$, we can state these necessary conditions as

$$G\phi^2 < \tfrac{1}{9} \ , \tag{5.24}$$

$$\beta\hbar < \tfrac{3}{2}\pi r_B \ . \tag{5.25}$$

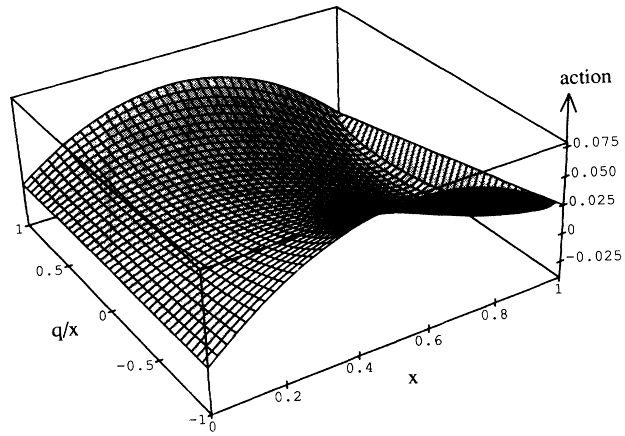

FIG. 1. Action $I_*(4\pi r_B^2)^{-1}$ as a function of $x = r_+/r_B$ and q/x, where $q = e/r_B$, for given thermodynamic data $\beta/4\pi r_B = 0.25$ and $\phi = 0.1$. Note the presence of the unstable saddle point at $x \approx 0.29$ and $q/x \approx 0.12$. The local minimum at $x \approx 0.94$ and $q/x \approx 0.38$ is a global minimum.

VI. CANONICAL ENSEMBLE AND DENSITY OF STATES

In the canonical ensemble the charge $Q \equiv \langle Q \rangle = e$ is fixed, rather than the potential difference ϕ. The simplest way to treat this problem and relate it to the analysis carried out above is to calculate the canonical partition function Z_C in the zero-loop approximation, where $\ln Z_C = -\beta F$, with F the Helmholtz free energy. That is, employ the approximation $\ln Z_C \approx -\tilde{I}_C$, where $\tilde{I}_C(\beta, r_B, Q)$ is a suitable action evaluated at a stable stationary point of a corresponding reduced action $I_{*C}(\beta, r_B, Q; r_+)$. This reduced action is obtained just as before, by elimination of constraints and integration of an appropriate action functional, but in this case the action functional should be tailored to the changed boundary condition on the electromagnetic field. Thus the canonical action from which one begins is

$$I_C = I_g + I_F \ , \tag{6.1}$$

where I_g is the same gravitational action as before, but I_F is the usual Maxwell action I_A augmented by a boundary term:

$$I_F = \frac{1}{16\pi}\int_M d^4x \sqrt{g} F_{\mu\nu}F^{\mu\nu} + \frac{1}{4\pi}\int_{\partial M} d^3\Sigma_\nu F^{\mu\nu} A_\mu \ . \tag{6.2}$$

The action I_F is stationary when the Maxwell equations hold and when the electromagnetic field $F^{\mu\nu}$, rather than its potential A_μ, is fixed on the boundary. This action allows us to fix directly, as given thermodynamic boundary data, the flux of the electric field, that is, the charge Q.

Simplifying and reducing the action (6.1) gives

$$I_{*C} = \beta r_B \left[1 - \left[1 - \frac{r_+}{r_B} \right]^{1/2} \left[1 - \frac{Q^2}{r_+ r_B} \right]^{1/2} \right] - \pi r_+^2 \ . \tag{6.3}$$

This reduced canonical action $I_{*C}(\beta,r_B,Q;r_+)$ has r_+ as its only degree of freedom. Extremizing I_{*C} with respect to r_+ yields the black-hole inverse temperature β given just as before by (4.16). To produce $\tilde{I}_C(\beta,r_B,Q)$, one must invert (4.16) to obtain $r_+=r_+(\beta,r_B,Q)$, which involves solving an algebraic equation of seventh order. (Recall that the analogous problem produced a cubic equation in the grand canonical ensemble.) We will only remark about this equation that it has two physically acceptable solutions, of which at least one is stable (that is, for which I_{*C} has a local minimum with respect to r_+ and the relevant response function C_{Q,r_B} is positive). One can show that the canonical ensemble has greater stability than the grand canonical one in the sense that a larger set of the (r_+,Q) plane corresponds to locally stable solutions. This result is expected because the canonical ensemble has a conserved charge Q, while the grand canonical ensemble does not.

From $\tilde{I}_C$ one obtains an expression for the quasilocal energy $\langle E\rangle=\partial\tilde{I}_C/\partial\beta$ equivalent to the previous one [Eq. (5.5)], except that now r_+ depends on β, r_B, and Q, and $Q\equiv\langle Q\rangle$ is a given conserved constant. Finally, in the canonical ensemble one obtains the electrostatic potential difference ϕ from

$$\begin{aligned}\langle\phi\rangle&=\frac{1}{\beta}\frac{\partial\tilde{I}_C}{\partial Q}\\&=\frac{Q}{r_+}\left[1-\frac{r_+}{r_B}\right]^{1/2}\left[1-\frac{Q^2}{r_+r_B}\right]^{-1/2},\end{aligned}\tag{6.4}$$

with r_+ understood as above. This agrees with (4.15) because $e=\langle Q\rangle$ in the grand canonical ensemble.

The density of states $\nu(E,Q,r_B)$ could be obtained by carrying out inverse "Laplace" transforms on the grand partition function Z_{GC} through integrations of β and ϕ on suitable complex contours, analogously to what was done for the case $\phi=0$ in Ref. 9. Alternatively, one can obtain in a straightforward manner the zero-loop approximation to the density of states by following a recent treatment of thermodynamic ensembles and gravitation that allows one to express E (as well as Q and r_B) directly as *boundary* data.[2] This latter method, in fact, is a generalization of the way we obtained above the canonical $\tilde{I}_C$ from the grand canonical $\tilde{I}$. One obtains

$$\nu(E,Q,r_B)\approx\exp(\pi r_+^2)\ ,\tag{6.5}$$

where r_+ is expressed in terms of E, Q, and r_B using (4.10) and (5.6).

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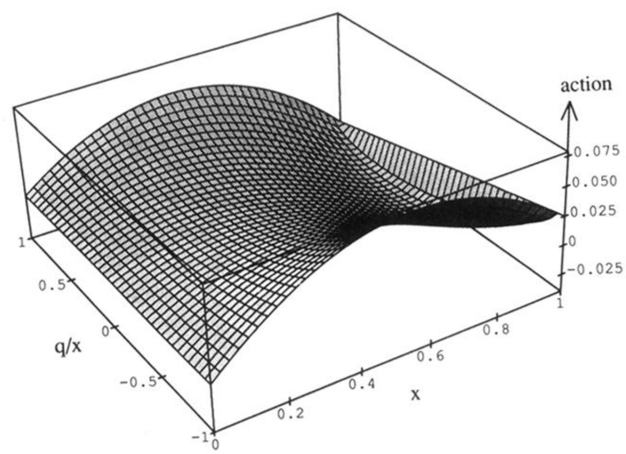
[14]The following purely "mechanical" interpretation of the quasilocal energy was obtained by one of us [James W. York, Jr. (unpublished)]. The ADM surface integral for the total energy can be written as a proper two-surface integral on a constant time slice of $N(k-k^0)$, where N is the lapse function, k is the trace of the extrinsic curvature of the two-surface embedded in a three-space of constant time, and k^0 is this trace as computed when the two-surface is embedded in flat Euclidean three-space. One obtains the quasilocal energy by setting $N=1$. Thus the quasilocal energy is a "proper" total energy obtained from the Hamiltonian. The result can be derived just as the boundary term involving K in the action (3.1) was derived by J. W. York, Jr., Phys. Rev. Lett. **28**, 1082 (1972); Found. Phys. **16**, 249 (1986). The term k^0 is inserted

to ensure that the ADM and quasilocal energies of flat three-space are zero. A detailed treatment of the relation of the quasilocal energy to the action principle has been given by J. David Brown and James W. York, Jr. (unpublished).

[15]The relationships among eigenvalues, pivots, and response functions that enter into the determination of various types of partition functions and the integral transforms that connect them have been elaborated elsewhere [J. David Brown and James W. York, Jr. (unpublished)]. The partition functions in question are those described in Ref. 2 and generalizations of them to include matter fields in addition to gravity.

[16]See, for example, G. Strang, *Linear Algebra and Its Applications* (Harcourt Brace Jovanovich, San Diego, 1988), p. 339. This is the "law of inertia" of quadratic forms.

[17]In Ref. 1, the discussion of compressibility is incorrect, but it is irrelevant to the conclusions of that paper.



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