PHYSICAL REVIEW D VOLUME 42, NUMBER 10 15 NOVEMBER 1990

Arbitrariness of inflationary fluctuation spectra

Hardy M. Hodges*
Santa Cruz Institute for Particle Physics, University of California, Santa Cruz, California 95064

George R. Blumenthal
*Santa Cruz Institute for Particle Physics, University of California, Santa Cruz, California 95064
and University of California Observatories/Lick Observatory, Board of Studies in Astronomy and Astrophysics, University of California, Santa Cruz, California 95064*
(Received 28 December 1989)

We find that the simplest model of inflation, a single scalar field undergoing slow roll, is consistent with nearly arbitrary fluctuation power spectra, each corresponding to a unique family of potentials which we determine. Deviations from a Zel'dovich spectrum become increasingly dramatic in the limit that the slow-roll approximation breaks down. Examples are given. Possible features in the power spectrum could reconcile galaxy formation theories with observations of large-scale structure. We further derive new constraints on the reheating temperature.

I. INTRODUCTION

Inflation is possibly the most popular scenario for producing large-scale fluctuations in the Universe. Perhaps the simplest model (out of the many, reviewed in Ref. 1) is chaotic inflation,[2] described by the dynamics (often assumed to be friction dominated or "slow roll") of a single scalar field ϕ. Even within this subclass of inflationary models, there is still great freedom since a potential $V(\phi)$ must be specified. It is frequently claimed that a Harrison-Zel'dovich spectrum (HZS) is the generic outcome of inflation. The standard examples involve simple potentials of the form $V(\phi)=m^2\phi^2/2$ or $V(\phi)=A\phi^4/4$, which do lead to a near HZS. However, there is no particle-physics reason why the potential should be so simple, or even renormalizable, and other potentials should be explored. Nonstandard spectra have been found in two models that go beyond the standard "slow-roll" approximation, and use more complicated potentials. The most general renormalizable potential $V(\phi)=A\phi^4/4+B\phi^3/3+C\phi^2/2+V_0$ has been shown to yield strong non-Zel'dovich features (valleys in the HZS).[3] Also, power-law power spectra, of academic interest only, can be generated in models that contain an exponential potential.[4] For completeness, we mention that interesting spectral features (a mountain, or possibly a plateau, in the HZS) can arise in models with several scalar fields.[5] However, some of these models appear to be unlikely within the standard framework of chaotic inflation.[6]

Upon confronting the observations, it appears questionable that a HZS with Gaussian fluctuations could explain the large-scale structure in our Universe. Such observations include the galaxy and cluster angular correlations, bulk motions, and other very-large-scale structures such as voids or the "cosmic picket fence."[7] In one of the most attractive theories of structure formation, the cold-dark-matter (CDM) scenario, the HZS can be normalized to fit a great number of observations on small scales $\lesssim 10$ Mpc, but the predicted large-scale features are not nearly as prominent as the observations would suggest. Presumably, the scenario could be rescued with a power spectrum that contains additional power on large scales.

The possibility that inflation might lead to a wide variety of spectra is obviously of great importance, as non-Zel'dovich spectra might help reconcile observations of large-scale structure with theories of galaxy formation. Here, we will show that a very large class of non-HZS $P(\lambda)$ exists within one of the simplest models of inflation, a result which goes well beyond the singular findings of non-HZS in previous studies, by demonstrating the existence of potentials $V(\phi)$ that lead to such spectra. We show through examples that the potentials corresponding to interesting non-HZS do not appear to be terribly contrived. In addition, we use our formalism to provide a transparent, and very stringent, constraint on the reheating temperature.

II. TRANSFORMATION OF A POWER SPECTRUM INTO A POTENTIAL

A uniform scalar field, which is presumed to be responsible for inflation, has the equation of motion

$$\ddot{\phi}+3H\dot{\phi}+V'=0 , \tag{2.1}$$

where overdots and primes denote derivatives with respect to time and ϕ, respectively, and the Hubble parameter $H\equiv\dot{R}/R$ is given by

$$H^2\simeq\frac{8\pi}{3m_{\rm Pl}^2}[V(\phi)+\dot{\phi}^2/2+k_{\rm curv}/R^2+\rho_{\rm rad}] , \tag{2.2}$$

where $m_{\rm Pl}$ is the Planck mass, R is the cosmic scale factor, $k_{\rm curv}$ is the curvature, and $\rho_{\rm rad}$ is the radiation energy density. (Throughout we use units with $\hbar=c=k_B=1$.) Inflation occurs if $\ddot{R}>0$, in which case scales move outside of the horizon and the curvature term becomes small. For a wide class of inflationary models, the potential energy density is the dominant contribution to the Hubble parameter, and the motion of ϕ is friction dominated and therefore given by the slow-roll equation of motion

$$3H\dot{\phi}+V'=0\ . \tag{2.3}$$

The slow-roll approximation is appropriate if[8]

$$|V''|\lesssim 24\pi V/m_{\rm Pl}^2\ ,\quad |V'|\lesssim\sqrt{48\pi}V/m_{\rm Pl}\ , \tag{2.4}$$

and becomes increasingly worse as the inequalities approach equality. Inflation occurs ($\ddot{R}>0$), within this approximation, if

$$|V'|\lesssim\sqrt{16\pi}V/m_{\rm Pl}\ . \tag{2.5}$$

This is nearly (apart from a numerical factor of $\sqrt{3}$) the same as the second slow-roll condition in Eq. (2.4).

Fluctuations on various scales are determined by microphysics, i.e., quantum fluctuations. In the slow-roll approximation, horizon-scale density fluctuations after inflation are simply related to the properties of the potential when the scale left the horizon during inflation:[9]

$$P^{1/2}(k)\simeq 8\sqrt{6\pi}V^{3/2}/V'm_{\rm Pl}^3\ , \tag{2.6}$$

where ϕ at horizon crossing can be expressed in terms of the wave number k via the relation $k\simeq H(\phi)R(\phi)$. Here, $P\equiv k^3\langle\zeta_k^2\rangle/2\pi^2\propto(\delta\rho/\rho)^2$ is the power spectrum associated with the gauge-invariant variable $\zeta\simeq\delta\rho/(\rho+p)$.

The physical scale of a fluctuation leaving the horizon at epoch ϕ is just $\sim H^{-1}$. To connect the value of ϕ at horizon crossing with present-day length scales λ, we need to know how much expansion takes place between horizon crossing and today. It is customary to describe this amount of expansion in terms of the number of e-folds $N\equiv\int H\,dt\propto\ln(R)$. We shall split N into two pieces: $N=N_*+N(\phi)$, where N_* is the number of e-folds from the end of inflation (at $\phi=\phi_{\rm end}$) to now, and $N(\phi)$ is the number of e-folds from ϕ to $\phi_{\rm end}$. Using the slow-roll approximation, $N(\phi)$ can be expressed in terms of the potential:

$$N(\phi)=\frac{8\pi}{m_{\rm Pl}^2}\int_{\phi_{\rm end}}^{\phi}\frac{V}{V'}d\phi\ . \tag{2.7}$$

The scale $\lambda(\phi)$ is then determined from the relation

$$\lambda\simeq 2\pi H^{-1}(\phi)\exp[N_*+N(\phi)]\ . \tag{2.8}$$

We now invert Eqs. (2.6)–(2.8) and derive an expression for the potential for any fluctuation spectrum $P^{1/2}(\lambda)$ consistent with slow roll. Equation (2.6) can be rewritten as

$$\frac{V'm_{\rm Pl}^3}{V^{3/2}}d\phi=8\sqrt{6\pi}\frac{d\phi(\lambda)}{d\lambda}\frac{1}{P^{1/2}(\lambda)}d\lambda\ . \tag{2.9}$$

From Eq. (2.8) we find that

$$\frac{d\lambda(\phi)}{d\phi}\simeq\frac{8\pi}{m_{\rm Pl}^2}\frac{V}{V'}\lambda=\frac{\sqrt{\pi}m_{\rm Pl}\lambda P^{1/2}(\lambda)}{\sqrt{6}V^{1/2}}\ . \tag{2.10}$$

In this expression, the derivative of the $H^{-1}(\phi)$ prefactor in Eq. (2.8) has been ignored, in accordance with Eq. (2.5). If we substitute Eq. (2.10) into (2.9) and multiply both sides of the equation by $V^{-1/2}$, then the right-hand side of Eq. (2.9) becomes independent of V and integrates to

$$V(\lambda)=\left[V_0^{-1}-\frac{48}{m_{\rm Pl}^4}\int_{\lambda_0}^{\lambda}\frac{d\lambda'}{\lambda'P(\lambda')}\right]^{-1}\ , \tag{2.11}$$

where V_0 is the potential energy density to be specified at λ_0. The expression for $V(\phi)$ is then parametrized by λ, and $\phi(\lambda)$ is obtained from integrating Eq. (2.10) and using the above expression for $V(\lambda)$:

$$\phi(\lambda)=\phi_0+\frac{\sqrt{6}}{\sqrt{\pi}m_{\rm Pl}}\int_{\lambda_0}^{\lambda}\frac{V^{1/2}(\lambda')d\lambda'}{\lambda'P^{1/2}(\lambda')}\ . \tag{2.12}$$

After one determines $V(\phi)$, the self-consistency of the slow-roll approximation is easily checked via conditions (2.4) and (2.5). A manifestation of the breakdown of these conditions is that, if the integral in (2.11) becomes too large, $V(\lambda)$ may go negative. We note that other possible inconsistencies do not arise: (1) if P is finite, there are no false vacua in $V(\phi)$; (2) $V(\phi)$ is a single-valued function according to Eq. (2.12).

To show that a wide variety of *interesting* non-HZS are possible we must further demonstrate that the slow-roll assumption does not significantly restrict the form of $P(\lambda)$ over 3–4 orders of magnitude in λ. A variation in $P^{1/2}(\lambda)$ by a factor of a few is sufficient to be of astrophysical importance. Using (2.5) and (2.6) we find that

$$V(\lambda)/m_{\rm Pl}^4\lesssim P(\lambda)/24 \tag{2.13}$$

and the first condition in (2.4) can be expressed as

$$\left|\frac{24V}{Pm_{\rm Pl}^4}-\frac{\lambda}{3P^{1/2}}\frac{dP^{1/2}}{d\lambda}\right|\lesssim 1\ . \tag{2.14}$$

The first term in this expression is guaranteed to satisfy the constraint if (2.13) is satisfied, so one of the slow-roll conditions depends only upon the power spectrum:

$$\frac{\lambda}{P^{1/2}}\left|\frac{dP^{1/2}}{d\lambda}\right|\lesssim 3\ . \tag{2.15}$$

If the slow-roll conditions are strongly satisfied, $P(\lambda)$ should therefore correspond to a nearly HZS. However, we presently assume (and later check) that the inequality in (2.15) need not be well-satisfied, and hence that significant variations in $P(\lambda)$ can be achieved within the slow-roll approximation.

If one calculates $P(\lambda)$ from a potential $V(\phi)$, the structure of the potential during the last $\simeq 60$ e-folds of inflation is needed so that one can locate the regime corresponding to large-scale structure scales. Working in reverse, we are only interested in specifying the power spectrum over a limited range of astrophysical interest, e.g., 1 Mpc $<\lambda<3000$ Mpc. Hence, we obtain a small piece of the overall potential $V(\phi)$ upon using the transformations (2.11) and (2.12). More precisely, there is a family of potential pieces corresponding to different constants of integration V_0. This occurs because there is not enough in-formation to connect V_0 to a scale λ. The constant of integration ϕ_0 reflects the translation invariant properties of any single-field potential, and is unimportant.

III. EXAMPLES

At present, there are a limited number of analytic expressions for both $V(\phi)$ and $P(\lambda)$. They were found by taking simple forms of the potential, which allowed one to calculate the fluctuation spectrum, e.g., $V \propto \phi^2$, ϕ^4 which leads to nearly HZS (up to logarithmic terms). There are a number of analytic cases in which the transformations (2.11) and (2.12) can be used to calculate $V(\phi)$. Rather than amassing a large table of such solutions, we consider only the simplest analytic case which involves simple powers of λ, i.e., $P^{1/2} \propto \lambda^{-n}$. We first note that condition (2.15) restricts the power n in which the slow-roll approximation is valid: $|n| \lesssim 3$. The case of $n=0$, a pure HZS with $P^{1/2}=\epsilon$, yields the potential piece

$$V(\phi)/V_0=\left[1-\frac{4\sqrt{6\pi}V_0^{1/2}}{m_{\rm Pl}^3\epsilon}(\phi-\phi_0)\right]^{-2} . \tag{3.1}$$

The case of negatively sloped spectra $[P^{1/2}=\epsilon(\lambda/\lambda_0)^{-n}$ with $n>0]$ can also be written analytically:

$$V(\phi)=\beta^{-1}\{\cos[2\sqrt{\pi n}\,(\phi-\phi_0)/m_{\rm Pl} +\arcsin(\epsilon^{-1}m_{\rm Pl}^{-2}\sqrt{24/n\beta})]\}^{-2} , \tag{3.2}$$

where $\beta \equiv V_0^{-1}+24/n\epsilon^2 m_{\rm Pl}^4$. This potential has the oddity that there is a wavelength cutoff λ_c (corresponding to where the potential approaches infinity) determined from the relation $\lambda_c/\lambda_0=(\beta m_{\rm Pl}^4\epsilon^2 n/24)^{1/2n}$. This merely reflects a breakdown of the slow-roll condition (2.13), which must occur at some point since $P(\lambda)$ decreases with wavelength while $V(\lambda)$ increases with λ (similar reasoning can be applied to the pure HZS, where a cutoff also occurs). This solution remains valid as $n \rightarrow 0$ (where it reduces to 3.1) and as n becomes negative. When n is negative, Eq. (3.2) corresponds to a generalization of the exponential potential discussed in Ref. 4, where essentially a specific choice of V_0 was made.

It is also rather easy to evaluate Eqs. (2.11) and (2.12) numerically, either by direct numerical integration or by differentiating them and numerically solving the resulting two coupled first-order equations. Although λ_0 is arbitrary, we have set $\lambda_0=1$ Mpc and solved the equations over the range 1 Mpc $\leq \lambda \leq 3000$ Mpc corresponding to large-scale structure scales. For each solution, one must specify a value for the potential $V_0=V(\lambda_0)$. We chose $V_0=10^{-12}m_{\rm Pl}^4$. In Fig. 1 we show a number of examples of power spectra that we have converted into potentials.

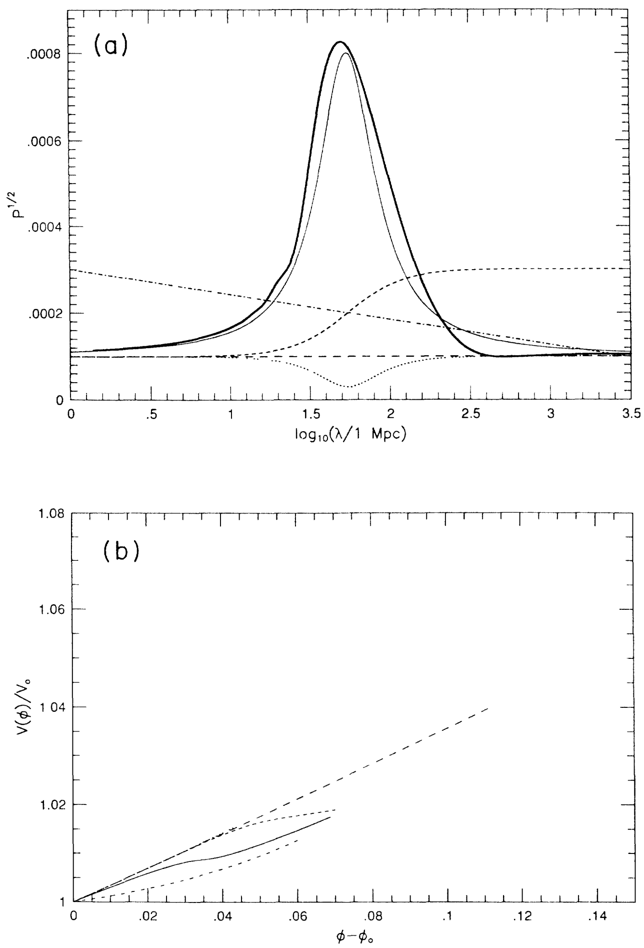

FIG. 1. (a) Plotted are fluctuation spectra as a function of scale, that we chose as examples to transform into potentials. The derived potential for the "mountain spectrum" was used to generate a spectrum (heavy-weight curve) with slow roll relaxed. (b) The potentials $V(\phi)$, where ϕ is in Planck mass units, corresponding to the fluctuation spectra in (a). The initial value of the potential, for each of these examples, was taken to be $V_0=V(1\ \text{Mpc})=10^{-12}m_{\rm Pl}^4$. Only the range of $P^{1/2}(\lambda)$ shown in (a) is converted into $V(\phi)$, to illustrate the mapping of λ space into ϕ space.

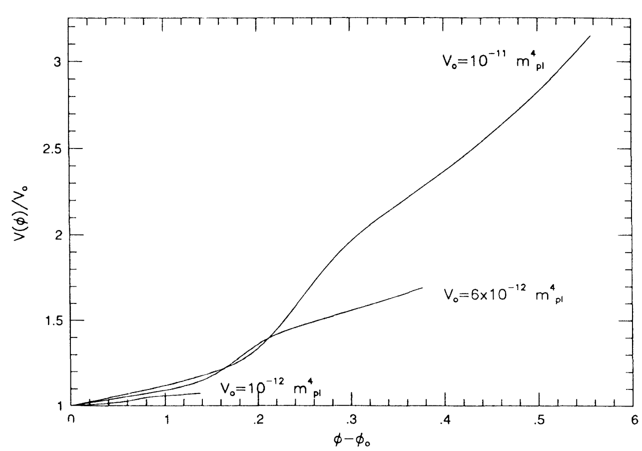

FIG. 2. The valley spectrum of Fig. 1(a) is transformed into potential pieces $V(\phi)$, where ϕ is in Planck mass units, for several choices of V_0: $10^{-12}m_{\rm Pl}^4$, $6\times10^{-12}m_{\rm Pl}^4$, and $10^{-11}m_{\rm Pl}^4$. The qualitative features of the potential appear to be the same in each case.

In each case, we verified that the slow-roll conditions were satisfied, and hence that our analysis was self-consistent. We further checked our results by calculating the spectra from the potentials in Fig. 1(b), with slow roll relaxed (see Ref. 3 for the techniques used here), and comparing them with the original spectra. All cases agreed rather well—in Fig. 1(a) we have plotted one of the worst "matchups." An important point is that the transformations (2.11) and (2.12) applied to interesting spectra lead to reasonable looking potentials. We also examined how the potentials changed with different choices of V_0. Qualitatively, there was not much difference. Larger choices of V_0 typically correspond to expanding, in both the ϕ and $V(\phi)$ direction, potentials derived from smaller choices of V_0. In Fig. 2 we illustrate how the potential corresponding to the "valley spectrum" in Fig. 1(a) changes with V_0.

IV. CONSTRAINTS ON THE REHEATING TEMPERATURE

When there is a constraint on the power spectrum, the *assumption* that the slow-roll approximation is appropriate allows one to derive bounds on the energy density during inflation. Equation (2.13) provides such a constraint, and it has frequently been used in conjunction with limits on P based on the microwave background dipole or quadrupole anisotropy constraints,[10] and on estimates of the power needed to get rms mass fluctuations of unity on scales of $8h^{-1}$ Mpc (h is the present Hubble constant divided by 100 km/sec Mpc.[11] The potential $V(\phi)$ should monotonically decrease from values of ϕ corresponding to our present horizon, down to where inflation ends, in order to avoid problems that would arise with a false vacuum. Then, the maximum possible temperature T_r of the Universe after inflation can be calculated by taking the energy density associated with ϕ, corresponding to some scale λ where a constraint on P is applied, and equating it to the thermal radiation density $\rho_{\rm rad}=\pi^2 g(T)T^4/30$, where g is the effective number of particle degrees of freedom.

The inversion Eq. (2.11) can provide even stronger constraints than Eq. (2.13). The potential, for values of ϕ corresponding to $\lambda \le \lambda_0$, is bounded by

$$V(\phi) \le \frac{m_{\rm Pl}^4}{48}\left[\int_{\lambda_0}^{\lambda_1}\frac{d\lambda'}{\lambda' P(\lambda')}\right]^{-1}, \tag{4.1}$$

where λ_1 is an arbitrary wavelength with $\lambda_1 > \lambda_0$. If one then assumes that the power is constrained so that $P < P_{\rm limit}$ = const over the wavelength range $\lambda_0 < \lambda < \lambda_1$, then the constraint (4.1) yields

$$V(\lambda \le \lambda_0) \le 0.009\frac{m_{\rm Pl}^4 P_{\rm limit}}{\log_{10}(\lambda_1/\lambda_0)} . \tag{4.2}$$

For a wavelength range of only one decade, (4.2) provides about a factor of 5 stronger a constraint than does Eq. (2.13). For example, if one uses the microwave constraints of Ref. 12, i.e., $P^{1/2} < 10^{-3.6}$ on scales a tenth of a horizon up to the horizon, Eq. (4.2) yields $V(\lambda \lesssim 600h^{-1}$ Mpc$) \lesssim 5.7\times 10^{-10} m_{\rm Pl}^4$.

It was pointed out in Ref. 11 that a stronger bound may be obtained from estimates of P when the normalization scale reentered the horizon. The normalization of the HZS in the standard CDM scenario is $P_n^{1/2} = 10^{-4}/b$, where b is a biasing factor thought to be in the range $\sim 1.4-2.5$. For the constraint, we take $P_n^{1/2} \le 10^{-4}$. Since this constraint operates over a very small range of wavelengths, Eq. (4.1) provides a weaker constraint on the potential than does Eq. (2.13), even including the fact that Eq. (2.15) limits the rate at which $P(\lambda)$ can rise. However, if the standard CDM scenario is valid over at least three decades in λ, with an assumed HZS over this range, Eq. (4.2) yields

$$V \le 3\times 10^{-11} b^{-2} m_{\rm Pl}^4$$

and (4.3)

$$T_r \lesssim 3\times 10^{16}/g^{1/4}\ \text{GeV} .$$

In the standard model $g = 106.75$, and at the temperatures under consideration one would expect further contributions from GUT physics, and possibly supersymmetry. However, if we let g assume its lowest possible value, one finds that $T_r \lesssim 9\times 10^{15}$ GeV. Assuming a HZS over all scales, or $P(\lambda \le \lambda_n) \le P_n$ (which occurs in a wide class of models), leads to the constraint $T_r \le 6 \times 10^{15}(106.75/b^2 g)^{1/4}$ GeV.

In general, astrophysical observations provide constraints not on the power spectrum directly, but on an integral of the power spectrum over some appropriate window function. This is certainly the case for constraints based on the microwave-background isotropy, the fluctuation amplitude at $8h^{-1}$ Mpc, and bulk streaming flows in the Universe. Then, the appropriate way to limit the reheating temperature is to minimize the integral in Eq. (4.1) while satisfying (1) all integral constraints based on the observations, and (2) the requirement that P rise no faster than $\lambda^{\pm 6}$ from Eq. (2.15).

V. DISCUSSION

We have reposed the problem of inflationary density fluctuations in terms of specifying a power spectrum, and then calculating a potential. When the fluctuation spectrum on very large scales is determined, it can provide a window to very early particle physics, and yield a family of potential pieces that represent a portion of the overall potential $V(\phi)$—assuming inflation is described by a single scalar field. A number of examples, both analytic and numerical, were given to illustrate that, in fact, nearly any spectrum is possible given the freedom to choose the potential $V(\phi)$. It is necessary to invoke an ill-defined concept of naturalness, that potentials must be simple and featureless, in order to obtain Harrison-Zel'dovich inflationary spectra. Non-Zel'dovich spectra have been advocated as a possible solution to problems encountered in models of galaxy formation. For example, the cold-dark-matter theory is very successful on scales $\lesssim 10$ Mpc, but predicts less large-scale structure than is observed. This could be fixed by putting more power on large scales, e.g., cluster scales, and then having the spectrum dip to evade the microwave constraints. Observations of large-scale structure and constraints on the anisotropy of the cosmic microwave background impose strong constraints on the energy density during, and hence after, inflation. We have presented a new, stringent, way to calculate such constraints.

ACKNOWLEDGMENTS

We thank Joel Primack for useful discussions. This work was supported by NSF Grant No. PHY88-00801, and by a faculty research grant at the University of California, Santa Cruz.

---

*Present address: Harvard-Smithsonian Center for AstroPhysics, Cambridge, MA 02138.

[1]A. Linde, Rep. Prog. Phys. **47**, 925 (1984); R. Brandenberger, Rev. Mod. Phys. **57**, 1 (1985); M. Turner, in *Fundamental Interactions and Cosmology,* proceedings of the Cargese Summer School, Cargese, France, 1989, edited by J. Audouze and J. Tran Thanh Van (Editions Frontières, Gif-sur-Yvette, France, 1985); K. Olive, Phys. Rep. **190**, 307 (1990).

[2]A. Linde, Phys. Lett. **129B**, 177 (1983).

[3]H. Hodges, G. Blumenthal, L. Kofman, and J. Primack, Nucl. Phys. **B335**, 197 (1990).

[4]L. F. Abbott and M. B. Wise, Nucl. Phys. **B244**, 541 (1984); F. Lucchin and S. Matarrese, Phys. Rev. D **32**, 1316 (1985).

[5]A. A. Starobinsky, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 124 (1985) [JETP Lett. **42**, 152 (1985)]; J. I. Silk and M. S. Turner, Phys. Rev. D **35**, 419 (1987); L. A. Kofman and A. D. Linde, Nucl. Phys. **B282**, 555 (1987); L. A. Kofman and D. Y. Pogosyan, Phys. Lett. B **214**, 508 (1988); D. S. Salopek, J. R. Bond, and J. M. Bardeen, Phys. Rev. D **40**, 1753 (1989).

[6]H. Hodges, Phys. Rev. Lett. **64**, 1080 (1990).

[7]T. Broadhurst, R. Ellis, D. Koo, and A. Szalay, Nature (London) **343**, 726 (1990).

[8]P. J. Steinhardt and M. S. Turner, Phys. Rev. D **29**, 2162 (1984).

[9]A. H. Guth and S. Pi, Phys. Rev. Lett. **49**, 1110 (1982); S. W. Hawking, Phys. Lett. **115B**, 295 (1982); A. A. Starobinsky, *ibid.* **117B**, 175 (1982); J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1983).

[10]D. H. Lyth, Phys. Lett. **147B**, 403 (1984); S. W. Hawking, *ibid.* **150B**, 339 (1985); C. Kim and P. Murphy, *ibid.* **167B**, 43 (1986); E. Vishniac, K. Olive, and D. Seckel, Nucl. Phys. **B289**, 717 (1987).

[11]D. H. Lyth, Phys. Lett. B **196**, 126 (1987).

[12]I. A. Strukov, D. P. Skulachev, and A. A. Klypin, in *Large Scale Structures of the Universe,* edited by J. Audouze *et al.* (Kluwar, Dordrecht, 1988).