Predictive hierarchical model of quark and lepton masses

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Details of a recently proposed model of quark and lepton masses are presented, including a thorough discussion of technical issues relating to the naturalness, self-consistency, and realism of the model. Also discussed are the essential underlying ideas, which are compared to other ideas in the literature, and possible variants of the model.

I. INTRODUCTION

In a recent Letter[1] we proposed a simple model of quark and lepton masses that predicts a heavy top quark in the neighborhood of 100 GeV. In this paper we shall present a more extended and detailed discussion of that model and possible variants of it.

In Sec. II we give a very brief review of the model, recapitulating what is to be found in Ref. 1. In Sec. III we discuss the ideas of the model and how they are related to existing ideas on quark and lepton masses such as grand unified mass relations, "radiative mass hierarchy," " factorization" of mass matrices, proportionality of mass matrices, and the so-called "Fritzsch form." In Sec. IV we discuss how the model can be made fully realistic and various technical points that arise such as the stability of the pattern of vacuum expectation values assumed, right-handed neutrino masses, etc. In Sec. V we discuss SO(10) or Pati-Salam versions of the model. In Sec. VI we discuss neutrino masses. And finally in Sec. VII we make some general remarks about the consequences of imposing low-energy supersymmetry on this and similar grand-unified-theory (GUT) models which have a radiative hierarchy of light fermion masses.

II. A BRIEF REVIEW OF THE MODEL

The model is based on the group $E_6 \times Z_2$. (As we shall see in Sec. IV, making the model fully realistic may entail additional symmetries. We shall give an example with an additional $Z_2 \times Z_2$.) The fermions include three **27** representations that are even under Z_2 denoted $\mathbf{27}_i$, $i=1,2,3$, and an extra vectorlike pair of a family plus a mirror family[2] that are odd under Z_2 denoted $\mathbf{27}+\overline{\mathbf{27}}$. The Higgs-field representations include an adjoint denoted $\mathbf{78}_H$ and a fundamental denoted $\mathbf{27}_H$. Both of these are odd under Z_2. We say the content "includes" these fields because other fields will be necessary to perform the following tasks: (a) give right-handed neutrinos a large mass, (b) give large masses to the exotic fermions of E_6, (c) generate radiative masses for those quarks and leptons which in our model are massless at the tree level (the e, u, and d), and (d) break E_6 down to the standard model in a realistic way (i.e., a way that makes the proton lifetime and $\sin^2\theta_W$ come out satisfactorily. For purposes of discussion let us divide the fields of the theory into those relevant for our discussion of the tree-level quark and lepton masses which we enumerated above ($\mathbf{27}_i$, **27**, $\overline{\mathbf{27}}$, $\mathbf{78}_H$, and $\mathbf{27}_H$), and those which are required to make the model fully realistic by performing the tasks listed above which we will call the "additional fields." We can then divide the Lagrangian into terms which involve the additional fields, $\mathcal{L}_{AF}$, and those which do not, $\mathcal{L}_0$. In this section we will discuss $\mathcal{L}_0$ and the predictions for quark and lepton masses and mixing that arise from it. In Sec. IV we will present a specific example of a set of additional fields and an $\mathcal{L}_{AF}$ which is realistic and technically natural, and will also show that in that example the couplings in $\mathcal{L}_{AF}$ do not disturb the tree-level results we derive from $\mathcal{L}_0$. As will be seen $\mathcal{L}_{AF}$ need contain only a few additional fields in small representations of E_6 (**1**'s of fermions and **27**'s and **78**'s of scalars).

The most general form consistent with $E_6 \times Z_2$ for the Yukawa and fermion-mass terms of $\mathcal{L}_0(\mathbf{27}_i, \mathbf{27}, \overline{\mathbf{27}}, \mathbf{27}_H, \mathbf{78}_H)$ is

$$\mathcal{L}_{0,\text{mass+Yukawa}} = M\,\mathbf{27}\,\overline{\mathbf{27}} + \sum_{i=1}^{3} b_i \mathbf{27}_i \overline{\mathbf{27}}\,\mathbf{78}_H + \sum_{i=1}^{3} a_i \mathbf{27}_i \mathbf{27}\,\mathbf{27}_H + \text{H.c.} \quad (1)$$

M is (naturally) assumed to be of the unification scale. For purposes of discussion it will prove convenient to refer to the decomposition of E_6 representations under the following sequence of subgroups:

$$E_6 \supset SO(10)\times U(1)_R$$
$$\supset SU(5)\times U(1)_X \times U(1)_R$$
$$\supset SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X \times U(1)_R \; .$$

R is normalized so that $\mathbf{27} \to \mathbf{16}^1 + \mathbf{10}^{-2} + \mathbf{1}^4$, and X so that $\mathbf{16} \to \mathbf{10}^1 + \overline{\mathbf{5}}^{\,3} + \mathbf{1}^5$. We will refer to an SO(10) p contained in an E_6 q as a $p(q)$ representation. We will assume (see Sec. IV) that all light fermions are in $\mathbf{16}(\mathbf{27})$ representations, and that the exotic fermions $\mathbf{10}(\mathbf{27})$ and $\mathbf{1}(\mathbf{27})$ become superheavy. As these latter are in real representations under the standard-model group this is what one would generally expect. We assume that the $\mathbf{27}_H$ has nonzero vacuum expectation values (VEV's) in the $\mathbf{10}(\mathbf{27}_H)$ and possibly the $\mathbf{1}(\mathbf{27}_H)$, but *not* in $\mathbf{16}(\mathbf{27}_H)$ [or else the fermions in $\mathbf{16}(\mathbf{27})$ and $\mathbf{10}(\mathbf{27})$ would mix which would complicate matters.] There are *two* independent VEV's in $\mathbf{10}(\mathbf{27}_H)$ (because the $\mathbf{10}_H$ is complex) which we call v and v'. v gives mass to the up quarks and v' to the down quarks and leptons. We also assume that the $\mathbf{78}_H$ only has nonvanishing VEV's in the $\mathbf{45}(\mathbf{78}_H)$ and $\mathbf{1}(\mathbf{78}_H)$, but *not* in the $\mathbf{16}(\mathbf{78}_H)$ or $\overline{\mathbf{16}}(\mathbf{78}_H)$. (These assumptions about the pattern of VEV's will be shown in Sec. IV to be consistent in a concrete realization).

As the $\mathbf{78}_H$ is in the adjoint its VEV can be written as a linear combination of generators of E_6. $\langle \mathbf{78}_H \rangle = \sum_{a=1}^{78} C_a \lambda_a$. However, because this VEV is superlarge, it must not break $SU(3)_C$ or $SU(2)_L$. This means that it can be written without loss of generality as the linear combination of the generators of $U(1)_Y$, $U(1)_X$, and $U(1)_R$ [recall that $\langle \mathbf{16}(\mathbf{78}_H)\rangle = \langle \overline{\mathbf{16}}(\mathbf{78}_H)\rangle = 0$]:

$$\langle \mathbf{78}_H \rangle = \tfrac{1}{5}\Omega(X + 6zY/2 + wR) \ . \tag{2}$$

The parameters z and w are dimensionless and have group-theoretical significance as specifying the direction in which $\langle \mathbf{78}_H \rangle$ points in E_6 space. Ω has dimensions of mass and is assumed to be of the GUT scale.

Without loss of generality we can choose our axes in the three-dimensional family space of the $\mathbf{27}_i$ to point so that $b_i = (0,0,1)b$ and $a_i = 0, \sin\theta, \cos\theta)a$. Denote an $SU(3)_C \times SU(2)_L \times U(1)_Y$ fermion multiplet contained in $\mathbf{27}_i$ by F_i, one contained in $\mathbf{27}$ by F, and the conjugate representation in $\overline{\mathbf{27}}$ by F^c. [F can be, for example, $\binom{u}{d}$, u^c, d^c, $\binom{\nu}{l^-}$, l^+, or $\bar{\nu}$.] Then the first two terms in Eq. (1) can be rewritten

$$M\left[F + \frac{\Omega}{5} b(X + 6zY/2 + wR)_F F_3\right] F^c \ .$$

Let us call $b\Omega/M \equiv T$ and $(X + 6zY/2 + wR)_F \equiv \alpha(F)$. Then the superheavy family contains the linear combinations

$$F_h = [F + \tfrac{1}{5}T\alpha(F)F_3]/N_{\alpha(F)} \ . \tag{3}$$

The three light families are therefore F_1, F_2, and

$$F_{3'} \equiv [F_3 - \tfrac{1}{5}T\alpha(F)F]/N_{\alpha(F)} \ . \tag{4}$$

where $N_\alpha \equiv (1 + \frac{1}{25}\alpha^2 T^2)^{1/2}$. With these definitions it is now a simple matter to read off from the third term in Eq. (1) the light-quark and lepton mass matrices. Consider, for example, the leptons

$$\sum_{i=1}^{3} a_i \mathbf{27}_i \mathbf{27} \langle \mathbf{27}_H \rangle = a\langle \mathbf{27}_H \rangle (\sin\theta \mathbf{27}_2 + \cos\theta \mathbf{27}_3)\mathbf{27} \supset av'[\sin\theta l_2^+ + \cos\theta l_3^+) l^- + (\sin\theta l_2^- + \cos\theta l_3^-) l^+]$$

$$\supset av'\left[\left(\sin\theta l_2^+ + \frac{\cos\theta}{N_{\alpha(l^+)}} l_{3'}^+\right)\left(\frac{-\frac{1}{5}T\alpha(l^-)}{N_{\alpha(l^-)}} l_{3'}^-\right) + \left(\sin\theta l_2^- + \frac{\cos\theta}{N_{\alpha(l^-)}} l_{3'}^-\right)\left(\frac{-\frac{1}{5}T\alpha(l^+)}{N_{\alpha(l^+)}} l_{3'}^+\right)\right] \ .$$

This gives the matrix

$$\sum_{i,j} l_i^- M_{ij}^{\text{lepton}} l_j^+ = (l_1^- \, l_2^- \, l_{3'}^-)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sin\theta \dfrac{\alpha(l^+)}{N_{\alpha(l^+)}} \\ 0 & \sin\theta \dfrac{\alpha(l^-)}{N_{\alpha(l^-)}} & \cos\theta \dfrac{\alpha(l^+)+\alpha(l^-)}{N_{\alpha(l^+)}N_{\alpha(l^-)}} \end{pmatrix} \begin{pmatrix} l_1^+ \\ l_2^+ \\ l_{3'}^+ \end{pmatrix} \frac{aT}{5} v' \ . \tag{5}$$

The matrices for the up and down quarks are of the same form, with $\alpha(l^\pm)$ being replaced by the appropriate α of u, $\bar{u}$, d, or $\bar{d}$. The matrix for the up quarks is proportional to v rather than v'. [It should be emphasized that we are not making any assumption here about whether there are one or two light Higgs doublets. In the former case then

$$\phi_{\text{light}} = \frac{1}{\sqrt{v^2 + v'^2}}(v\phi + v'\phi') \ ,$$

$$\phi_{\text{heavy}} = \frac{1}{\sqrt{v^2 + v'^2}}(-v'\phi + v\phi') \ .$$

And $\langle \phi_{\text{heavy}} \rangle = 0$.] Altogether the tree-level mass ratios and mixing angles depend on just five real parameters: θ, z, w, T, and $|v/v'|$ [these are real because any phases in the couplings of Eq. (1) can be rotated away by field redefinitions]; and the dependence on T is only through the quantities N, and is therefore weak if $T < 1$.

The $\alpha(F)$ have the values

$$\alpha(l^+) = 1 + 6z + w \ , \quad \alpha(l^-) = -3 - 3z + w \ ,$$

$$\alpha(u) = 1 + z + w \ , \quad \alpha(\bar{u}) = 1 - 4z + w \ ,$$

$$\alpha(d) = 1 + z + w, \quad \alpha(\bar{d}) = -3 + 2z + w \ .$$

It is interesting to note that for $N_\alpha \cong 1$, $m_b^0 \cong \cos\theta[\alpha(d) + \alpha(\bar{d})] = \cos\theta(-2 + 3z + 2w)$, and $m_\tau^0 \cong \cos\theta[\alpha(l^{-1}) + \alpha(l^+)] = \cos\theta(-2 + 3z + 2w)$. (Here and throughout the superscript zero refers to running masses at the GUT scale.) This is not an accident but rather the result of the fact that down quarks and leptons get mass from the same Higgs field ϕ', so that $\alpha(l^+) + \alpha(l^-) = -\alpha(\phi') = \alpha(d) + \alpha(\bar{d})$. On the other hand because different α's appear in the off-diagonal entries of the mass matrices it is *not* true that $m_\mu^0 \simeq m_s^0$. Thus for $T<1$, the old puzzle that $m_b^- \cong m_\tau^0$, while $m_s^0 \ncong m_\mu^0$ and $m_d^0 \ncong m_e^0$ emerges automatically and quite naturally.

For $\theta<1$, $T<1$ one can write down the following approximate expressions for the predictions of the model:

$$m_b^0/m_\tau^0 \simeq 1 \,, \tag{6a}$$

$$m_s^0/m_\mu^0 \cong \frac{(1+z+w)(-3+2z+w)}{(1+6z+w)(-3-3z+w)} \,, \tag{6b}$$

$$V_{ts}^0 \cong \tan\theta \left[\frac{-3+2z+w}{2-3z+2w} - \frac{1-4z+w}{-2+3z+2w} \right] . \tag{6c}$$

$$m_\mu^0/m_\tau^0 \cong \tan^2\theta \frac{(1+6z+w)(-3-3z+w)}{(-2+3z+2w)^2} \,, \tag{6d}$$

$$(m_c^0/m_t^0)(m_s^0/m_b^0) \cong \frac{(1-4z+w)(-2+3z+2w)}{(-3+2z+w)(2-3z+2w)} \,, \tag{6e}$$

$$m_t^0/m_b^0 = \left| \frac{v}{v'} \right| \frac{2-3z+2w}{-2+3z+2w} \,. \tag{6f}$$

If one fits the known quark and lepton masses using the exact tree-level forms of Eq. (5) one find the results plotted in Fig. 1. (We have made the same simplifying assumptions about the running of the couplings as in Ref. 1. Namely, we have used the one-loop β functions ignoring the effects of the Yukawa couplings on the running of the masses, and have assumed E_6 breaks the standard model at a single GUT scale.)

If one constructs an SO(10) version of this model one obtains the same expressions as in Eqs. (5) and (6) with the parameter w set to zero. That leads to one more prediction. The fit to the known masses and angles is no longer exact (since one is now fitting five quantities with only the four parameters, θ, z, T and $|v/v'|$). And for $m_{\rm top}$ one predicts a value ≈ 45 GeV, which is too small.

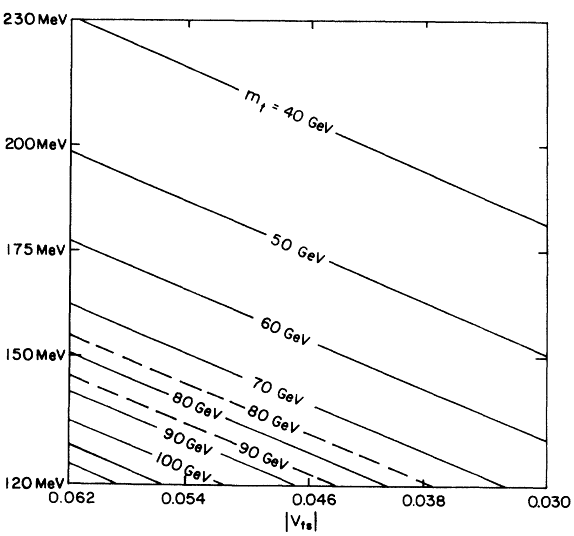

FIG. 1. A contour plot of m_t(phys) $|V_{ts}|$ and m_s (1 GeV). [Here m_s is the tree-level value of the strange-quark mass which one expects to differ from the true mass by $O(\sin\theta_c)$.]

The fits which give $m_{\rm top} \simeq 100$ GeV have $\theta \cong 0.322$, $z \cong 0.89$, $w \cong 2.92$, and $T \cong 0.24$. Note that with this value of T, the N_α are indeed very close to unity and can be neglected. The values of z and w have a group-theoretical significance. From Eq. (2)

$$\begin{aligned}\langle 78_H \rangle &= \frac{\Omega}{5}[X+6z(Y/2)+wR] \\ &= 4\Omega[(\tfrac{2}{5})^{1/2}\tilde{X} + (\tfrac{3}{5})^{1/2} z\tilde{Y}/2 + (\sqrt{6}/5)w\tilde{R}] \\ &= 4\Omega\left[\frac{\sqrt{6}}{5}(1+z)(\widetilde{B-L}) + \left[\frac{-2+3z}{5}\right]\tilde{I}_{3R} + \frac{\sqrt{6}}{5}w\tilde{R} \right] ,\end{aligned}$$

where the generators with tildes are consistently normalized so that $\sum_{27}\tilde{\lambda}^2 = 300$. Substituting the fitted values of z and w one finds that $\langle 78_H \rangle \propto [(0.93(\widetilde{B-L}) + (0.14)\tilde{I}_{3R} + (1.4)\tilde{R}]$. The breaking of $SU(2)_R$ is much weaker than that of $SU(4)_C$. This suggests (though it does not imply—see the discussion in Sec. IV) that E_6 breaks via the chain $E_6 \to SO(10) \to SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$.

III. THE IDEAS OF THE MODEL

In this section several major ideas for explaining aspects of the quark and lepton masses and mixings are renewed, grand unification,[3] radiative mass hierarchy,[4] factorization of mass matrices,[5] and the "Fritzsch form."[6] The great majority of models of light fermion masses that have been proposed in the literature are based on one or more of these approaches. It will be shown how the model described in the last section incorporates features of all of these ideas while avoiding some of their difficulties.

A. Grand unification and "proportionality"

That the masses of the leptons, down-quarks, and up-quarks all exhibit similar geometrical hierarchies, and that the mass eigenstates of the up and down quarks are nearly aligned ($V_{\rm KM} \cong I$) suggest that a close connection exists between the three mass matrices. Such a connection can be imposed "by hand" through some family symmetry under which the same family quantum number is assigned to t, b, and τ, say. However, grand unification is a principle well motivated on other grounds which already provides for a quark-lepton connection.

Minimal SU(5) predicts $M_{ij}^{\rm down(0)} = M_{ij}^{\rm lepton(0)}$ which works well for the third generation ($m_b^0/m_\tau^0 \cong 1$) but rather poorly for m_s^0/m_μ^0 and m_d^0/m_e^0, though even these ratios are of order unity (about $\frac{1}{2}$ and 4, respectively). SU(5) does not relate $M^{\rm down}$ to $M^{\rm up}$, however, and so does not explain the smallness of the KM angles. Minimal SO(10) with a real 10 of Higgs fields would give $M_{ij}^{\rm up(0)} = M_{ij}^{\rm down(0)} = M_{ij}^{\rm lepton(0)}$. With a complex 10 of Higgs fields (and a Peccei-Quinn symmetry to allow

16 16 10$_H$ but not **16 16 10**$^*_H$) SO(10) gives $M_{ij}^{\mathrm{up}(0)} \propto M_{ij}^{\mathrm{down}(0)} = M_{ij}^{\mathrm{lepton}(0)}$ since now a different VEV is responsible for the up-quark masses (v) and the down-quark and lepton masses (v'). [If SO(10) is embedded in E_6, E_6 itself can play the role of the Peccei-Quinn symmetry since **27**$_H$ contains a complex **10**$_H$ and E_6 allows **27 27 27**$_H$ but not **27 27 27**$^*_H$.] Such "proportionality"[7] of up- and down-mass matrices [which can arise also in left-right or Pati-Salam subgroups of SO(10)] would imply that all the KM angles vanish and that $m_t^0 m_s^0 / m_b^0 m_c^0 = 1$. The KM angles are indeed very small, but for realistic values of m_t (between 80 and 200 GeV) $m_t^0 m_s^0 / m_b^0 m_c^0 \sim 3$ to 7; values not so far from unity as to produce hope that some kind of approximate SO(10) may be viable. (We should keep in mind that the fermion masses span an enormous range—at least 2×10^5—and in that light consider numbers like 3 as "close to unity.") We regard the relations $m_b^0 \cong m_\tau^0$, $m_s^0 \sim m_\mu^0$, $m_d^0 \sim m_e^0$, $V_{\mathrm{KM}} \cong I$, and $m_t^0 m_s^0 / m_b^0 m_c^0 \sim 1$ (where $\sim$ means "of the same order of magnitude") as evidence for approximate or *broken* SU(5) and SO(10) symmetry.

The key question is how this breaking of SU(5) or SO(10) which occurs at superlarge scales shows up in the fermion masses which arise from SU(2)$\times$U(1) breaking at small scales. Our answer is to introduce superheavy fermions which derive mass directly from the SU(5) and SO(10)-breaking Higgs VEV's and which mix with the known light fermions. Such fermions must be in a real representation of the GUT group which contains leptons, up quarks, and down quarks. The simplest choice is clearly a family plus an antifamily.

B. "Radiative hierarchy" and "factorization"

The large ratios among the masses of the different generations strongly suggests a radiative origin to this hierarchy. This ideas has a long history,[4] dating from the observation that $m_e/m_\mu = O(\alpha)$. Now that the third generation is known the case is even more compelling. Such a hierarchy can be implemented using family symmetries. A more elegant approach, we believe, is what we call "factorization".[5] If the fermion mass matrices at the tree level are given not by Yukawa couplings which are matrices in "family space" but rather by expressions which factorize into vectors in family space, as, for example, $M_{ij} \propto f_i f_j'$, then the tree-level matrices can be brought to the desirable form

$$\begin{bmatrix} 0 & & \\ & 0 & \\ & & A \end{bmatrix} .$$

But a Yukawa coupling which is a vector in family space must couple one of the ordinary three families of fermions to some exotic fermion. Again this is just the role that is played by the $\mathbf{27}+\overline{\mathbf{27}}$ in our model. Because the couplings a_i and b_i span only a two-dimensional subspace of the three-dimensional family space one family perforce is massless at the tree level without any *ad hoc* family symmetry having to be imposed to single it out. The generational hierarchy appears as a consequence of the simplicity of the Yukawa part of the theory: there are only two types of Yukawa terms in Eq. (1) which is not enough to give mass at the tree level to all three generations. We would emphasize the economy of having the extra families $\mathbf{27}+\overline{\mathbf{27}}$ play two roles. Their mixing with the usual families, both generates the hierarchy and introduces SU(5) and SO(10) breaking into the tree-level relations. It is a remarkable bonus that the good relation $m_b^0 \cong m_\tau^0$ is left relatively undisturbed.

A symmetry is needed in our model to forbid the term $\sum_{ij} f_{ij} \mathbf{27}_i \mathbf{27}_j \mathbf{27}_H$ which has a family matrix of couplings. That symmetry is Z_2 which is broken at the GUT scale by $\langle \mathbf{78}_H \rangle \neq 0$. Thus terms can arise radiatively that are of the form $\sum f_{ij} \mathbf{27}_i \mathbf{27}_j \mathbf{27}_H \mathbf{78}_H$ which will generate masses for e, u, and d of order $(g^2/16\pi^2)(\langle \mathbf{78}_H \rangle / M) v$, where g is some coupling of order one and $\langle \mathbf{78}_H \rangle / M$ is a ratio of GUT scales that we assume also to be of order one. Such higher dimension operators can also arise at the tree level from integrating out some of the heavy "additional fields". This does not happen in the example given in Sec. IV. But for some choices of additional fields it could happen. (In such a case one could still have a tree-level hierarchy given by small ratios of VEV's or masses. Such tree-level hierarchies have been studied in the literature.[8])

C. Fritzsch form

The so-called "Fritzsch form"[6] for the quark and lepton mass matrices is an *Ansatz* leading to relations between the KM angles and the ratios of fermion masses. There appears to be no strong argument for the Fritzsch form or other "forms" proposed in the literature[9] to be taken as exact descriptions of the fermion mass matrices. The zeros that appear in these forms are chosen to be zero simply to reduce the number of free parameters rather than for some fundamental reason. Indeed, one would not expect them to remain zero to all orders, nor would the qualitative successes of these *Ansätze* be jeopardized if these entries were simply small rather than exactly zero. An objection to taking the Fritzsch from too literally is that it is difficult if not impossible to reconcile an exact Fritzsch form with the idea of a radiative hierarchy. Radiative contributions to the matrices would not generally fit the *Ansatz* exactly. Nevertheless, the "Fritzsch form" and other forms similar to it have the great virtue of relating the smallness of the KM angles to the smallness of the mass ratios among generations. The relation obtained is typically of the form $\theta_{ij} \sim \sqrt{m_i/m_j}$. Furthermore, in the "Fritzsch form" the ratio of masses goes as the *square* of the ratio of nonzero elements in the mass matrices. This means that there may be less of a "small number problem" among the fundamental parameters appearing in the mass matrices than the hierarchy of mass eigenvalues would suggest.

It can be seen that the tree-level matrices given in Eq. (5) have a quasi-Fritzschian form (though the matrices are not symmetric). This is reflected in the fact that $V_{ts} \sim \sqrt{m_2/m_3}$ as appears from Eqs. (6c) and (6d). Since the form of the tree-level matrices is approximately (i.e., for small T)

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & s_\theta A \\ 0 & s_\theta B & c_\theta(A+B) \end{bmatrix}$$

one finds that $m_2/m_3 \cong \tan^2\theta AB/(A+B)^2$ which for $AB>0$ is less than $\frac{1}{4}\tan^2\theta$. Thus one achieves ratios such as $m_s/m_b \approx \frac{1}{30}$ and $m_c/m_t \approx \frac{1}{100}$ with $\theta \approx \frac{1}{3}$ which is not such a small value. (Note that $|m_\mu/m_\tau| \cong \frac{1}{17} > \frac{1}{4}\tan^2\theta$ because for the lepton matrix $AB<0$.) The model realizes, then, the essential qualitative features of the Fritzsch form for the second and third generations. This is also true to some extent for the first-generation mixings. For example, if one assumes the (radiatively generated) entries M_{12} and M_{13} are of roughly the same order then one would find that $V_{td} \ll V_{cd}$ as observed experimentally. This raises an interesting question. The "Fritzsch form" would suggest that $M_{12}^{\text{down}} \simeq \sin\theta_C m_s$. But M_{12}^{down} is a loop effect and m_s a tree effect in our model. However, as noted, m_s is suppressed by of order $\frac{1}{4}\tan^2\theta \ll 1$ even though it is a tree effect. It is more sensible to compare M_{12}^{down} to m_b. If there is any truth in the Fritzsch form then $M_{12}^{\text{down}}/m_b \simeq \sin\theta_C m_s/m_b \approx \frac{1}{150} \approx 1/16\pi^2$ which is not unreasonable for a loop effect.

D. Successes of the model

In summary, the model incorporates some features of all the ideas listed in the headings of this section. The main qualitative successes of the model are the following. (1) One generation (e, u, and d) that gets mass only radiatively and hence is very light. (2) The good relation $m_b^0/m_\tau^0=1$ is only slightly modified while the bad relations $m_s^0/m_\mu^0=1$ and $m_d^0/m_e^0=1$ are corrected by factors of order one. (3) A form naturally emerges that is reminiscent of the Fritzsch form and leads to $\theta_{ij} \sim \sqrt{m_i/m_j}$. (4) The top-quark mass is predicted to be even larger than predicted by simple proportionality and consistent with present limits.

Inevitably, some price has been paid for these successes in the form of new particles and symmetries. But the essential new features, the mirror pair of families and the Z_2 parity, are of a particular simplicity.

IV. THE ADDITIONAL FIELDS REQUIRED FOR A REALISTIC MODEL

As was emphasized in Sec. II, the fields which appear in Eq. (1) are insufficient to produce a realistic model. Additional fields are required to perform the following tasks: (a) generate superlarge masses for the right-handed neutrinos, (b) generate radiative masses for the first-generation fermions, (c) generate large masses for the exotic [$\mathbf{10}(\mathbf{27})$ and $\mathbf{1}(\mathbf{27})$] fermions of E_6 and (d) break E_6 all the way down to the standard model, in a way consistent with the $\sin^2\theta_W$ and the bounds on the proton lifetime. There are also certain conditions that this additional sector must satisfy for consistency and naturalness: (e) It must be shown that there are no tree-level contributions to the first-generation masses or indeed to the masses of the other light quarks and leptons beyond those given by Eq. (1); and (f) the pattern of VEV's assumed must be shown to be stable. We will address each of these issues in turn in the context of a specific choice for the additional fields of the theory and their interactions, $\mathcal{L}_{\text{AF}}$. This is meant as an example and, as it were, a consistency proof of the ideas of Sec. II. From the point of view of the model of the tree-level light-quark and lepton masses, it does not matter what $\mathcal{L}_{\text{AF}}$ is chosen as long as it succeeds in being realistic and does not change the results at tree level given in Eq. (5).

In addition to the fields which appear in Eq. (1) let there be the following fields: a set of several E_6 singlet fermions, denoted $\mathbf{1}_K''$, $K=1,\ldots,N$; two fundamental representations of Higgs fields, denoted $\mathbf{27}_H'$ and $\mathbf{27}_H''$; and an adjoint Higgs field denoted $\mathbf{78}_H'$. Let the full symmetry of the theory be $E_6 \times Z_2 \times Z_2' \times Z_2''$ where the fields transform as shown in Table I. (Note that Z_2 is just the same symmetry discussed in Sec. II where we have now made all additional fields even under it. Under Z_2' only the primed fields $\mathbf{27}_H'$ and $\mathbf{78}_H'$ are odd. And under Z_2'' only the double-primed fields $\mathbf{27}_H''$ and $\mathbf{1}_K''$ are odd.) The most general fermion mass and Yukawa terms in $\mathcal{L}_{\text{AF}}$ invariant under $E_6 \times Z_2 \times Z_2' \times Z_2''$ are

$$\mathcal{L}_{\text{AF}}^{\text{mass+Yukawa}} = \sum_{i=1}^{3}\sum_{K=1}^{N} c_{iK}(\mathbf{27}_i \mathbf{1}_K'')\mathbf{27}''^{\dagger}_H + \sum_{K,L=1}^{N} M_{KL}(\mathbf{1}_K''\mathbf{1}_L'') , \tag{7}$$

We will assume that in the $\mathbf{27}_H''$ *only* the SO(10)-spinor component $\mathbf{16}(\mathbf{27})$ acquires a VEV and that this includes a superlarge VEV in the SU(5)-singlet direction. Both the $\mathbf{27}_H$ and $\mathbf{27}_H'$ are assumed to have $\langle \mathbf{16}(\mathbf{27})\rangle=0$. That these assumptions are consistent will be shown shortly. With that VEV the $\mathbf{27}_H''$ directly gives the right-handed neutrinos superlarge masses through the terms in Eq. (7). This meets requirement (a) of those listed above. The

TABLE I. Particle transformations under symmetry $Z_2 \times Z_2' \times Z_2''$.

	Fermion fields				Higgs fields				
	$\mathbf{27}_i$	$\mathbf{27}$	$\overline{\mathbf{27}}$	$\mathbf{1}_K''$	$\mathbf{27}_H$	$\mathbf{27}_H'$	$\mathbf{27}_H''$	$\mathbf{78}_H$	$\mathbf{78}_H'$
Z_2	+	−	−	+	−	+	+	−	+
Z_2'	+	+	+	+	+	−	+	+	−
Z_2''	+	+	+	−	+	+	−	+	+

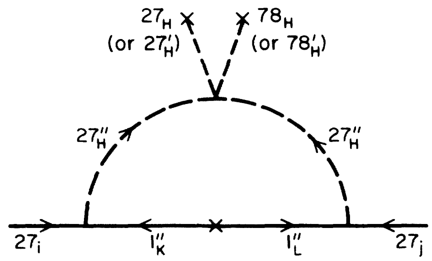

FIG. 2. A diagram generating radiative masses for the electron, u and d quarks. The same diagram also gives superlarge radiative masses to the exotic **10**(**27**) fermions of E_6.

first-generation fermions acquire mass radiatively [requirement (b)] through the diagrams shown in Fig. 2 which induce terms of the form $\mathbf{27}_i\mathbf{27}_j\mathbf{27}_H\mathbf{78}_H$ and $\mathbf{27}_i\mathbf{27}_j\mathbf{27}'_H\mathbf{78}'_H$, which contain the terms $\mathbf{16}(\mathbf{27}_i)\mathbf{16}(\mathbf{27}_j)\langle \mathbf{10}(\mathbf{27}_H)\rangle\langle \mathbf{1}$ or $\mathbf{45}(\mathbf{78}_H)\rangle$. They *also* contain the terms $\mathbf{10}(\mathbf{27}_i)\mathbf{10}(\mathbf{27}_j)\langle \mathbf{1}(\mathbf{27}_H)\rangle\langle \mathbf{1}$ or $\mathbf{45}(\mathbf{78}_H)\rangle$ which give superlarge masses to the exotic fermions of E_6, thus satisfying requirement (c), as well. Note, that is necessary to have VEV's in both the **10**(**27**) and **1**(**27**) directions. This would be difficult to achieve, in general, in one and the same multiplet. That is why both the $\mathbf{27}_H$ and $\mathbf{27}'_H$ are introduced. Generally both the $\mathbf{27}_H$ and $\mathbf{27}'_H$ will have VEV's in the **10**(**27**) and **1**(**27**) directions. (Because of terms like $\mathbf{78}_H\mathbf{78}'_H\mathbf{27}^{\dagger}_H\mathbf{27}'_H$, which satisfy all symmetries, it would be unnatural to prevent this.) However, since at the tree level only the $\mathbf{27}_H$ and *not* the $\mathbf{27}'_H$ couples to $(\mathbf{27}_i\mathbf{27})$ (because of Z_2) the results of Sec. II are not affected.

Requirement (d) is that E_6 break all the way to $SU(3)_C\times SU(2)_L\times U(1)_Y$ in a way consistent with τ_p and $\sin^2\theta_W$. One nice possibility is that the breaking occurs in the sequence

$$
\begin{aligned}
E_6 \xrightarrow{M_1} SO(10) &\xrightarrow{M_2} SU(3)_C\times SU(2)_L\times SU(2)_R\times U(1)_{B-L}\\
&\xrightarrow{M_3} SU(3)_C\times SU(2)_L\times U(1)_{3R}\times U(1)_{B-L}\\
&\xrightarrow{M_4} SU(3)_C\times SU(2)_L\times U(1)_Y\ .
\end{aligned}
$$

The first breaking at M_1 is done by the $\langle \mathbf{1}(\mathbf{27})\rangle$ in the $\mathbf{27}_H$ and $\mathbf{27}'_H$. The breakings at M_2 and M_3 are done by the VEV's of the $\mathbf{78}_H$ and $\mathbf{78}'_H$. Our fit of light fermion masses suggested that $\langle \mathbf{78}_H\rangle$ points more in the $B-L$ direction than in the I_{3R} direction (see Sec. II). Since we have *two* adjoint Higgs fields this does not necessarily tell us about the sequence of breaking (*a priori* we know nothing about the direction of $\langle \mathbf{78}'_H\rangle$), but it is certainly consistent with SO(10) breaking first to 3 2 2 1. This sequence gives the proton a longer lifetime than minimal SU(5) (all else being equal) because proton decay is controlled by the scale M_2. Finally, the breaking at M_4 is done by the $\mathbf{27}''_H$ acquiring a VEV in the 1[**16**(**27**)] direction. All of this is somewhat decoupled from the question of tree-level light fermion masses since only the $\langle \mathbf{78}_H\rangle$ is involved in that. There is some considerable freedom therefore in what is assumed about the pattern of GUT symmetry breaking and in particular about the values of the M_i. There are three masses (M_2, M_3, and M_4) which can be adjusted to fit τ_p, $\sin^2\theta_W$, and Λ_{QCD}, so there is no problem in principle in obtaining a realistic model. However, since the spectrum of the superheavy particles can be very complicated there is an irremovable source of uncertainty in performing the renormalization-group running of the quark and lepton masses from the GUT scales down to the weak scale. This issue will be discussed more in a forthcoming paper.

It is easy to see that the $\mathcal{L}_{AF}$ assumed here satisfies requirement (e) that there are no additional tree-level contributions to the light-quark and lepton masses beyond those which arise from Eq. (1), and in particular that m_e, m_u, and m_d vanish at the tree level. If there is a *tree-level* contribution to a fermion mass which arises from integrating out heavy fields it must come from diagrams of the type shown in Fig. 3. Electric charge must not be violated so the fermions which are being integrated out that appear as internal lines in Fig. 2 must have the same charge as the external light fermions that are being given mass. Turning to the present case, if the fermions being given mass are up quarks, down quarks, or charged leptons, then obviously the internal fermion lines cannot be the electrically neutral fermions $\mathbf{1}''_K$. But then the Yukawa terms in $\mathcal{L}_{AF}$ [Eq. (7)] cannot affect the tree-level masses of the charged quarks and leptons since they all contain the $\mathbf{1}''_K$.

More generally one sees that it is necessary simply that the "additional fields" not contain any fermions with the right quantum numbers to mix with the ordinary charged light fermions, or that in some other way such mixing is avoided naturally.

Next we come to the issue [requirement (f)] of stability of the pattern of VEV's. The crucial assumptions that we have made are that in the $\mathbf{27}''_H$ *only* the **16**(**27**) acquires a nonvanishing VEV, while in the $\mathbf{27}_H$, $\mathbf{27}'_H$, and $\mathbf{78}_H$ the SO(10) spinor components have vanishing VEV's. We shall demonstrate that symmetry prevents any linear term from arising to any order in perturbation theory that would destabilize such a minimum. The most general term in the effective potential is of the form $(\mathbf{27}_H)^a(\mathbf{27}'_H)^b(\mathbf{27}''_H)^c(\mathbf{78}_H)^d(\mathbf{78}'_H)^e$. Here a is the number

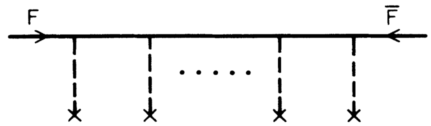

FIG. 3. A diagram that could generate higher-dimensional operators giving tree-level contributions to the mass of some fermion F.

of powers of 27_H *minus* the number of powers of 27_H^+, and similarly for the integers $b,\ldots,e$. Thus the dangerous terms linear in $\mathbf{16}(\mathbf{27}_H)$ would have the form $\mathbf{16}(\mathbf{27}_H)\,[1 \text{ or } \mathbf{10}(\mathbf{27}_H)]^{a-1}\,[1 \text{ or } \mathbf{10}(\mathbf{27}'_H)]^b\,[\mathbf{16}(\mathbf{27}''_H)]^c\,[1 \text{ or } \mathbf{45}(\mathbf{78}_H)]^d\,[1 \text{ or } \mathbf{45}(\mathbf{78}'_H)]^e$. Z_2'' symmetry requires that c be even. The center of SO(10) is a Z_4 which contains a Z_2 under which spinors are odd and tensors are even. This Z_2 requires for this term that $1+c=$even. These two conditions are incompatible, so the dangerous term cannot arise. The same type of argument applies to show that terms linear in $\mathbf{16}(\mathbf{27}'_H)$, $\mathbf{16}(\mathbf{78}_H)$, $\mathbf{16}(\mathbf{78}'_H)$, $\overline{\mathbf{16}}(\mathbf{78}_H)$, $\overline{\mathbf{16}}(\mathbf{78}_H)$, $\overline{\mathbf{16}}(\mathbf{78}'_H)$, $\mathbf{1}(\mathbf{27}''_H)$, or $\mathbf{10}(\mathbf{27}''_H)$ do not arise either.

There is one more technical point to be considered. By imposing the discrete symmetries $Z_2\times Z_2'\times Z_2''$ it is possible that accidental global symmetries might arise giving unwanted Goldstone or pseudo-Goldstone bosons. It is simple to check that this does not in fact occur.

V. SMALLER GAUGE GROUPS

If one constructed the simplest SO(10) model along the same lines as our E_6 model the analogue of Eq. (1) would be

$$\mathcal{L}_{0,\text{mass+Yukawa}}=M\,\mathbf{16}\,\overline{\mathbf{16}}+\sum_{k=1}^{3} b_i\,\mathbf{16}_i\,\overline{\mathbf{16}}\,\mathbf{45}_H + \sum_{i=1}^{3} a_i\,\mathbf{16}_i\,\mathbf{16}\,\mathbf{10}_H+\text{H.c.} \tag{8}$$

The $\langle \mathbf{45}_H\rangle$ can be written as $\frac{1}{5}\Omega[X+6z(Y/2)]$. In the E_6 model the adjoint Higgs field's VEV has more possible directions in which to point without breaking SU(3)$\times$SU(2)$\times$U(1): in addition to the X and $Y/2$ directions in $\mathbf{45}(\mathbf{78}_H)$, there are the $\mathbf{1}(\mathbf{78}_H)$ direction and the standard model singlet directions in $\mathbf{16}(\mathbf{78}_H)$ and $\overline{\mathbf{16}}(\mathbf{78}_H)$. In Sec. II we constrained $\langle \mathbf{16}(\mathbf{78}_H)\rangle$ and $\langle \overline{\mathbf{16}}(\mathbf{78}_H)\rangle$ to vanish. (This was shown in Sec. IV to be achievable in a technically natural and simple way.) But that still leaves the $\mathbf{1}(\mathbf{78}_H)$ direction which corresponds to the generator we called R. If $\langle \mathbf{1}(\mathbf{78}_H)\rangle$ also vanished that would mean that $w=0$ in Eq. (2) and the $\langle \mathbf{78}_H\rangle$ lay entirely in the adjoint of SO(10). Effectively, then, $w=0$ in the E_6 model reduces to the SO(10) model shown in Eq. (8). As was noted in Ref. 1 and Sec. II, this leads to too small a value of $m_t(\approx 45$ GeV). It was precisely to allow the superheavy singlet VEV $\langle(\mathbf{78}_H)\rangle$ and thus get a realistic m_t that in Ref. 1 the group was enlarged from SO(10) to E_6. An obvious alternative to consider is simply to introduce a singlet Higgs field $\mathbf{1}_H$ into the SO(10) model. Then one has a new coupling in addition to those in Eq. (8):

$$\Delta\mathcal{L}_{\text{Yukawa}}=\sum_{k=1}^{3} c_i\,\mathbf{16}_i\,\overline{\mathbf{16}}\,\mathbf{1}_H+\text{H.c.} \tag{9}$$

If $c_i=b_i$ this would reproduce the satisfactory results of the E_6 model. But if the group is only SO(10) in general $c_i\neq b_i$. As we shall now see most of the desirable qualitative features of the E_6 model that are enumerated in Sec. III nevertheless remain. Any prediction for m_t is, however, lost.

One might expect that since a_i, b_i, and c_i in general span the entire three-dimensional family space no fermion would remain massless at the tree level. However, the point is that the SU(2)$\times$U(1)-breaking masses still come only from the $a_i\mathbf{16}_i\mathbf{16}\,\mathbf{10}_H$ term and this when decomposed under the standard-model group still gives *two* terms for each type of fermion: $a_i(F_i\overline{F}+F\overline{F}_i)\langle \mathbf{10}_H\rangle$. The form of the tree-level matrices is

$$\begin{bmatrix} 0 & 0 & C\\ 0 & 0 & B\\ C' & B' & A\end{bmatrix}$$

which is still of rank *two*. By rotating in the 1-2 planes one can bring this to the same quasi-Fritzschian form

$$\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & B\\ 0 & B' & A\end{bmatrix}$$

as in the E_6 model. Moreover the same 1-2 rotation does this for the left-handed up and down quarks so that it is still true that $V_{td}=V_{ub}=0$ at the tree level. Finally, the relation $m_b^0\cong m_\tau^0$ which followed if $(\alpha/5)T\equiv b\langle \mathbf{78}_H\rangle/M<1$ in the E_6 model still follows if $b\langle \mathbf{78}_H\rangle/M$ and $c\langle \mathbf{1}_H\rangle/M$ are both smaller than one.

In the SO(10) model with the Higgs singlet there are three new complex parameters $c_i\langle \mathbf{1}_H\rangle$, so clearly no definite prediction for m_t arises. However, a heavy m_t is certainly possible since the special choice $c_i=b_i$ just reproduces the (still) satisfactory results of E_6.

One could also consider smaller groups than SO(10). For example, a model with the Pati-Salam group SU(4)$_C\times$SU(2)$_L\times$SU(2)$_R$ constructed along the same lines would have to have the Higgs fields $(\mathbf{15},\mathbf{1},\mathbf{1})_H$, $(\mathbf{1},\mathbf{1},\mathbf{3})_H$, and $(\mathbf{1},\mathbf{1},\mathbf{1})_H$ which now would have Yukawa coupling matrices b_i, c_i, and d_i. Although even more parameters would enter, the good qualitative features would remain [including $m_b^0\cong m_\tau^0$ as long as M were larger than $b\langle(\mathbf{15},\mathbf{1},\mathbf{1})_H\rangle$, etc.].

In Appendix A we give the form of the light quark and lepton mass matrices which arise in the SO(10) version of the model with general $c_i\neq b_i$.

There are several advantages that the SO(10) version has over the E_6 version. In the latter one had to enforce the conditions that $\langle \mathbf{16}(\mathbf{27}_H)\rangle=\langle \mathbf{16}(\mathbf{78}_H)\rangle=\langle \overline{\mathbf{16}}(\mathbf{78}_H)\rangle=0$ [as well as the related conditions that $\langle \mathbf{1}(\mathbf{27}''_H)\rangle=\langle \mathbf{10}(\mathbf{27}''_H)\rangle=0$]. One also required the $\mathbf{27}'_H$

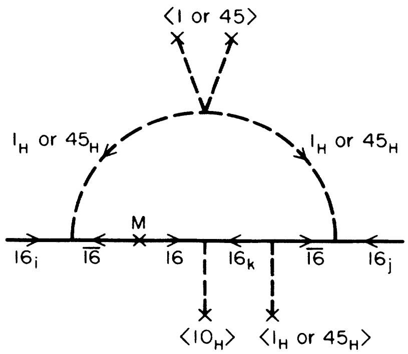

FIG. 4. A diagram involving only the couplings in Eqs. (8) and (9) that could generate radiative masss for the first generation in an SO(10) model.

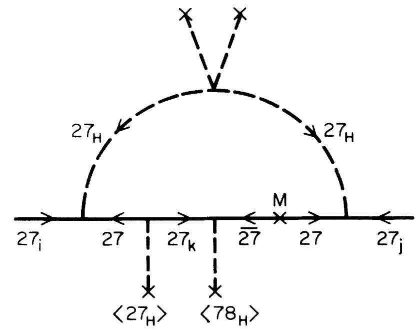

FIG. 5. A diagram contributing to radiative masses for the first generation. Even though this diagram involves the breaking of Z_2 at superlarge scales by $\langle \mathbf{78}_H \rangle \neq 0$ it can include (when expanded in terms of mass eigenstates) diagrams in which all the internal lines are light particles.

in addition to the $\mathbf{27}_H$ to achieve a VEV in the $\mathbf{1}(\mathbf{27})$ direction to give mass to the exotic $\mathbf{10}(\mathbf{27}_i)$ fermions. These issues disappear in the SO(10) model. Furthermore though the coupling of the $\mathbf{1}_H$ introduces the new parameters c_i and precludes an m_t prediction it also allows radiative contributions to arise for the first generation fermions from diagrams involving only the fields appearing in Eqs. (8) and (9). Figure 4 shows such a diagram.

VI. NEUTRINO MASSES

The masses of the light neutrinos arise from a seesaw mechanism. Not surprisingly one finds that typically $m(\nu_\tau):m(\nu_\mu):m(\nu_e) \approx m_t^2:m_c^2:m_u^2$. The mass of the τ neutrino crudely should be of order 10^{-2} eV$\times(10^{15}$ GeV$/M)$ where M is a typical right-handed neutrino mass. The leptonic KM mixing between the μ and τ generation, is, of course, given by the difference between $\theta_{\mu\tau}^-$ and $\theta_{\mu\tau}^0$, the angles in the charged and neutral sectors. $\theta_{\mu\tau}^-$ can be computed from our model [Eq. (5)]. For small mixing

$$\tan 2\theta_{\mu\tau}^- \cong 2 \left[\frac{m_\mu}{m_\tau} \right]^{1/2} \left[\frac{\alpha(l^+)}{\alpha(l^-)} \right]^{1/2} \left[1 + 2 \left| \frac{\alpha(l^+)}{\alpha(l^-)} \frac{m_\mu}{m_\tau} \right| \right] .$$

Note that there is an enhancement from the naive Fritzschian estimate of $2\sqrt{m_\mu/m_\tau}$. Interestingly, this enhancement is quite significant. For the values of θ, z, and w required to fit the known masses and mixings ($\theta = 0.322$, $z = 0.89$, $w = 2.9$) it comes out to be about 2.4. So $\theta_{\mu\tau}^- \cong 26°$. However, one cannot calculate $\theta_{\mu\tau}^0$ without a knowledge of the full mass matrix of the neutrinos including the many superheavy ones (e.g., in $\mathbf{1}(\mathbf{27})$], $\mathbf{1}[\mathbf{16}(\mathbf{27})]$, and E_6 singlets). However, it is possible to show that if the ratios of certain elements of M^{-1}, where M is the mass matrix of the superheavy neutrinos, are not in ratios much different from unity then

$$\theta_{\mu\tau}^0 \approx \theta \cong 18.5° .$$

One expects therefore that $\theta_{\mu\tau} \approx 7.5°$. However, as this is the difference between two comparable numbers one of which is very poorly known, $\theta_{\mu\tau}$ could be much larger; clearly values of 26° or more are not unreasonable.

VII. LOW-ENERGY SUPERSYMMETRY

The ideas presented here and in Ref. 1 depend on grand unified symmetries so one must confront the gauge hierarchy problem. The question arises whether the essential ideas are compatible with low-energy supersymmetry (SUSY). There seems to be no reason why low-energy SUSY would substantially alter the picture as far as the tree-level mass matrices are concerned. However, as noted by Ibañez,[10] radiative fermion masses in SUSY theories are suppressed by $m_{\rm SSB}/M_{\rm loop}$ where $m_{\rm SSB}$ is the SUSY-breaking scale and $M_{\rm loop}$ is the typical mass running around the loop. For us, then, $M_{\rm loop}$ must be of order $m_{\rm SSB}$, i.e., the weak scale. At first sight this would seem problematic. Since the Z_2 symmetry that prevents a tree-level m_e, m_u, and m_d is broken by the GUT-scale VEV $\langle \mathbf{78}_H \rangle$ one might expect that the loops giving rise to the effective operators $1/M_{\rm GUT}\ \mathbf{27}_i \mathbf{27}_j \mathbf{27}_H \mathbf{78}_H$ would have to involve superheavy particles in them. The diagram shown in Fig. 5 shows why this is not so. When this diagram is expanded in terms of mass eigenstates it leads to a sum of diagrams some of which could involve only light particles. The explicit factors $(M\langle \mathbf{78}_H \rangle)/(M_{\overline{27}} M_{27})$ that appear in Fig. 5 would then be simply a function of dimensionless mixing angles describing the amount of the light $\mathbf{27}$ to be found in $\mathbf{27}_K$.

This is rather a significant point about radiative fermion mass hierarchies in general. The symmetry responsible for the hierarchy can be broken at a large scale, while the radiative masses can come from loops involving only light particles. The symmetry of course renders the diagrams involving only light particles finite, by canceling the divergent part against diagrams involving superheavy particles.

ACKNOWLEDGEMENTS

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APPENDIX

In the SO(10) model given in Eqs. (8) and (9) we may without loss of generality take $b_i=(0,0,1)b$; $a_i=(0, \sin\theta, \cos\theta)a$, and $c_i=(\sin\psi\sin\chi, \sin\psi\cos\chi, \cos\psi)c$. Define as before $\langle \mathbf{45}_H\rangle=\frac{1}{5}\Omega(X+6zY/2)\equiv\frac{1}{5}\Omega\alpha$ and $T\equiv b\Omega/M$. Further define $\langle \mathbf{1}_H\rangle\equiv\frac{1}{5}\Delta$ and $S\equiv c\Delta/M$. Then the superheavy fields, in the same notation as Sec. II, are

$$F_h=\left[\frac{\alpha(F)}{5}T\sin\psi\sin\chi, \frac{\alpha(F)}{5}T\sin\psi\cos\chi,\ \frac{1}{5}S+\frac{\alpha(F)}{5}T\cos\psi, 1\right]/N_F ,$$

where now

$$N_F\equiv[1+\tfrac{1}{25}(S^2+2\alpha(F)\cos\psi TS+\alpha(F)^2T^2)]^{1/2}$$

and three orthogonal linear combinations which we denote $F_{1'}$, $F_{2'}$, $F_{3'}$ remain light.

Straightforward algebra allows us to write the fields $F(\mathbf{16})$ and $F_i(\mathbf{16}_i)$ in terms of $F_{i'}$ and F_h as

$$F=(F_h-\sqrt{N_F^2-1}\,F_{3'})/N_f ,$$

$$F_3=\frac{\frac{1}{5}[S+\alpha(F)T\cos\psi]}{\sqrt{N_F^2-1}}\left[\frac{\sqrt{N_F^2-1}\,F_h+F_{3'}}{N_F}\right]-\frac{\frac{1}{5}\alpha(F)T\sin\psi}{\sqrt{N_F^2-1}}F_{2'} ,$$

$$F_2=\frac{\cos\chi}{\sqrt{N_F^2-1}}\left[\frac{1}{5}\alpha(F)T\sin\psi\left[\frac{\sqrt{N_F^2-1}}{N_F}F_h+\frac{1}{N_F}F_{3'}\right]+\tfrac{1}{5}[S+\alpha(F)T\cos\psi]F_{2'}\right]-\sin\chi F_{1'} ,$$

$$F_1=\frac{\sin\chi}{\sqrt{N_F^2-1}}\left[\frac{1}{5}\alpha(F)T\sin\psi\left[\frac{\sqrt{N_F^2-1}}{N_F}F_h+\frac{1}{N_F}F_{3'}\right]+\tfrac{1}{5}[S+\alpha(F)T\cos\psi]F_{2'}\right]+\cos\chi F_{1'} .$$

By algebra that parallels that given in Sec. II one finds that the up-quark mass matrix has the form

$$(Q_{1'}Q_{2'}Q_{3'})\begin{bmatrix}0 & 0 & c_{1'3'}\\ 0 & 0 & c_{2'3'}\\ c_{3'1'} & c_{3'2'} & c_{3'3'}\end{bmatrix}\begin{bmatrix}u^c_{1'}\\ u^c_{2'}\\ u^c_{3'}\end{bmatrix}\frac{1}{5}av ,$$

where

$$c_{3'3'}=\frac{\sqrt{N^2_{(u^c)}-1}}{N_{(u^c)}}\left[\frac{\cos\theta[S+\alpha(Q)T\cos\psi]+\sin\theta\cos\chi\sin\psi\alpha(Q)T}{N_{(Q)}\sqrt{N^2_{(Q)}-1}}\right]+\frac{\sqrt{N^2_{(Q)}-1}}{N_{(Q)}}\left[\frac{\cos\theta[S+\alpha(u^c)T\cos\psi]+\sin\theta\cos\chi\sin\psi\alpha(u^c)T}{N_{(u^c)}\sqrt{N^2_{(u^c)}-1}}\right] ,$$

$$c_{2'3'}=\frac{\sqrt{N^2_{(u^c)}-1}}{N_{(u^c)}}\left[\frac{\sin\theta\cos\chi[S+\alpha(Q)T\cos\psi]-\cos\theta\sin\psi\alpha(Q)T}{\sqrt{N^2_{(Q)}-1}}\right] ,$$

$$c_{1'3'}=\frac{\sqrt{N^2_{(u^c)}-1}}{N_{(u^c)}}(-\sin\theta\sin\chi) ,$$

with $c_{3'2'}$ and $c_{3'1'}$ being given by $c_{2'3'}$ and $c_{1'3'}$ with $N_{(u^c)}\leftrightarrow N_{(Q)}$. At first sight these expressions look horrible but some simple remarks can be made. They should reduce to the E_6 expressions for $\psi=0$. In that limit $N_F^2-1=[\frac{1}{5}(S+\alpha(F)T]^2$. Identifying $S\equiv TwR=Tw$ one has $S+\alpha(F)T=T(X+6zY/2+wR)_F\equiv T\bar{\alpha}(F)$ where $\bar{\alpha}(F)$ is the E_6 version of $\alpha(F)$. Then $\sqrt{N_F^2-1}=\frac{1}{5}\bar{\alpha}(F)T$ and $N_F=[1+\frac{1}{25}\bar{\alpha}(F)^2T^2]^{1/2}$, which is the same as the $N_{\alpha(F)}$ used in Sec. II. We find that, for $\psi=0$,

$$c_{3'3'}=\frac{\cos\theta}{N_{(u^c)}N_{(Q)}}[\bar{\alpha}(u^c)+\bar{\alpha}(Q)]T,\quad c_{2'3'}=\frac{\sin\theta}{N_{(u^c)}}[\bar{\alpha}(u^c)]T\cos\chi,\quad c_{1'3'}=\frac{-\sin\theta}{N_{(u^c)}}[\bar{\alpha}(u^c)]T\sin\chi .$$

Except for the rotation of the 1′2′ axes by an angle χ this is the same as the E_6 result of Eq. (5).

The expressions for the down-quark and lepton mass matrices are of the same form with the obvious replacements for the subscripts Q and u^c. Note that $c_{1'3'}/c_{2'3'}$ is the same for the up- and down-quark mass matrices (for any ψ) so that $V_{ub}=V_{td}=0$ at the tree level. The relation $m_b^0 \cong m_\tau^0$ no longer holds in general, but if we expand in ψ we find that

$$m_b^0/m_\tau^0 = 1 + O(\sin\theta \sin\psi, \sin^2\psi) \ .$$

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