Narrow resonances in the diquonium system

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We study the diquonium $qq\bar{q}\,\bar{q}$ system within the chromopotential confining model. Unexpectedly, we find narrow resonances for S-wave intercluster angular momenta. These resonances, with widths of the order of 10 MeV, are located just below the radially excited meson levels, indicating that these new states are bound states ("molecules") of $(q\bar{q})_{N=1}$ with $(q\bar{q})_{N=0}$, for example, for the first resonance. Although we start from a confining potential, we show that the scattering problem can be treated with the usual methods of scattering theory, essentially because the residual potential between the asymptotic meson-meson states (van der Waals force) decreases fast enough. We are dealing with a multichannel scattering problem for which we use a truncated four-quark Schrödinger equation. Our choice of trial functions in the variational calculation ensures us that the resonances that we find are not artifacts of the method: we only introduce meson-meson states in which the system can apparently decay without any problem by quark rearrangement. We particularize to the case of the harmonic-oscillator potential, where the separate conservation of intercluster orbital angular momenta simplifies the problem. Moreover, the computation of the T matrix reduces then to the inversion of a matrix of finite dimension. We discuss the limits of the model: in particular, decays of these states by quark pair creation will enlarge their widths.

I. INTRODUCTION

We study the system $qq\bar{q}\,\bar{q}$ in the confining chromopotential model with the aim of looking at possible resonances in this system. We did present a resume of our results at the 1989 Hadron Spectroscopy Conference.[1]

Weinstein and Isgur[2] have recently pointed out that the old discussions on the states $\{3\}\{\bar{3}\}$ and $\{6\}\{\bar{6}\}$ did not take into account the strong coupling between these colored states and hence did not include the coupling to the unconfined mesons $\{1\}\{1\}$. This remark is very reasonable and seems to suggest that there cannot be $qq\bar{q}\,\bar{q}$ bound states or narrow resonances, as these states would be widened by the strong coupling to the meson-meson channels. This is indeed the result found by Weinstein and Isgur in their method of an effective potential between ground-state mesons (note, on the other hand, that δ and S^* are below the $K\bar{K}$ threshold).

On the contrary, taking fully into account the interaction between the different color channels, we find narrow resonances in this system. More precisely, studying the sector where all intercluster orbital angular momenta are $L=0$, we find very narrow resonances just below the threshold of the radial excitations, suggesting that these states are bound states or "molecules" of the ground-state mesons and the radially excited ones or of the radially excited ones among themselves. For example, the first resonance that we find seems to be a bound state of $(q\bar{q})_{N=1}$ with $(q\bar{q})_{N=0}$. These states are found taking fully into account the coupling to the open meson-meson channels, and contradict therefore the objection of Weinstein and Isgur. Their appearance remains nevertheless rather surprising. Maybe the effective potential method between ground-state mesons of Weinstein and Isgur misses the attraction involving excited mesons. Note that our narrow resonances are obtained taking into account the coupling to the meson-meson channels, but ignoring the possible coupling, by $q\bar{q}$ pair creation, to the baryon-antibaryon channels, as we will discuss below.

The present narrow resonances seem of similar nature as the bound states that we found long time ago in the chromoharmonic potential.[3] The difference is that the latter were uncoupled to the open channels by peculiar angular momentum selection rules, while the former are coupled to some of these open channels.

These old results can be summarized as follows: (1) in the $\{6\}\{\bar{6}\}$ structure, one could find resonances only for the very large angular momenta ($L \geq 9$); (2) in the $\{3\}\{\bar{3}\}$ structure we found *bound states* for $L \geq 2$. We recall by a simple argument how these bound states appear (not the whole variational calculation of Ref. 3) in Appendix A. Assuming a more realistic potential, one could expect these bound states to be narrow resonances. Note that this calculation did take fully into account the coupling to the meson-meson channels, as in the present work.

This calculation left two questions unanswered. (1) One could question the degree of generality of these bound states, since one could suspect that their existence was specific, by some potential barrier mechanism, to high angular momenta. (2) Since these bound states seemed linked to the angular momentum selection rules of the chromoharmonic potential model, it could be that, adopting a more realistic potential, these bound states could completely disappear in the continuum: since the angular momentum selection rules would not hold anymore, these states would couple to the open channels.

The present paper answers these two questions: (1) we make the calculation for S-wave intercluster angular mo-

menta, so that there is no angular momentum barrier; (2) in the S wave, no selection rule forbids the coupling to the continuum, even for the chromoharmonic potential model, and we can expect these states to appear in more general schemes than the particular model that we consider.

A word of caution is necessary concerning the large van der Waals force induced between neutral color clusters by a chromoconfining potential. We did show[4] that a color-confining potential r^α induces a long-distance potential behaving like $r^{\alpha-4}$. This behavior is too strong for any positive α and is excluded by the data. For example, the phase shift for large orbital angular momentum is sensitive to the long-distance piece and too strong. Does this mean that this potential model is completely useless? There are quantities that are however rather insensitive to this long-range force: for instance, the phase shifts at low orbital angular momentum and not too low energy (see, for example, the work of Maltman and Isgur[5] on the NN potential). In the present calculation, still more caution is needed because a large negative tail of the potential could well give spurious bound states which probably would not survive in more realistic schemes. But we have reasons to think that such bound states are not found within our approximation. The interaction in the truncated model is effectively cut off at a distance of some mesonic radii, suppressing the effect of the van der Waals force.

We have been informed of a whole current of work on multiquark systems, based on the so-called flip-flop string models.[6] In these models, resonances quite similar to the present ones are found. We will discuss at the end of the paper the relation of this type of model to our work.

The paper is planned as follows. In Sec. II we set up our chromopotential model. We demonstrate the positivity of the potential in Appendix B. In Sec. III we write down the Hamiltonian for the diquonium system as a matrix in color space. The general theory of multichannel scattering applies to our problem. However, there are some unusual aspects as compared to the usual diffusion by a potential, and in Secs. IV and V we discuss in detail the scattering in our system. In Sec. IV we expose the time-dependent formalism and we argue the conjecture that usual scattering theory applies to our system. In Sec. V we expose our truncation method to make the numerical calculation of the diffusion eigenstates and of the S matrix. In Sec. VI we expose the method to account for the quark permutations; the two-Hilbert-space formalism introduced in Sec. IV allows to define the S matrix in this case. In Sec. VII we take an explicit meson-meson basis. In Sec. VIII we particularize to the harmonic potential: we show how the calculation of the T matrix, and hence the S matrix, reduces then to the inversion of a matrix of finite dimension. Finally in Sec. IX we make explicit calculations and show our numerical results. In Sec. X we conclude with a discussion on the scope and limits of the model.

II. THE CHROMOPOTENTIAL CONFINING MODEL

The model that we consider is the chromopotential confining model, i.e.,

$$H=T+\mathcal{V}\ ,$$
$$T=\sum_i \frac{\mathbf{p}_i^2}{2m_i}\ , \tag{2.1}$$
$$\mathcal{V}=-\frac{1}{2}\frac{N}{N^2-1}\sum_{i<j}\left[\sum_a \lambda_i^a\lambda_j^a\right]V(\mathbf{r}_{ij})\ ,$$

where λ_i^a $(a=1,\ldots,N^2-1)$ are the matrices λ^a $[-(\lambda^a)^t]$ of the generators of SU(N), acting on the color of the quark i [antiquark i], normalized by $\mathrm{tr}\lambda^a\lambda^b=2\delta_{ab}$. The coefficient $-N/2(N^2-1)$ in $\mathcal{V}$ is chosen in such a way that $V(\mathbf{r})$ is the potential between the quark and the antiquark in a meson, $q\bar{q}$ in a color singlet.

If we consider $q\bar{q}$ in the color state $\{N^2-1\}$, we have $\mathcal{V}=-[1/(N^2-1)]V(\mathbf{r})$, giving a long-range repulsive force between q and $\bar{q}$, a kind of anticonfinement. This effect does not seem to have a physical meaning. Moreover, the Hamiltonian is not bounded from below; i.e., the energy spectrum extends to $-\infty$.

This difficulty is completely solved if we *postulate* that the Hilbert space $\mathcal{H}$ where the Hamiltonian H is defined is the subspace of the color-singlet states. This postulate is legitimate since (2.1) is invariant under SU(N). In this subspace, for a confining potential of the form $V(\mathbf{r})=|\mathbf{r}|^\alpha$, $0<\alpha\leq 2$, the potential $\mathcal{V}$ is *positive* (see Appendix A), and H will in general be bounded from below.

We then will define the chromopotential model by the Hamiltonian H given by (2.1), restricted to the space $\mathcal{H}$ of the neutral color states.

The model gives confinement between color clusters coupled in a color singlet, and, on the contrary, absence of confinement between neutral clusters. However, there is a residual van der Waals force,[4] physically too strong, but decreasing fast enough to allow one to apply the standard scattering theory to hadrons.

We will below consider the simplest model of this type, taking equal masses and the harmonic-oscillator potential

$$V(r)=\frac{m\omega^2}{4}\mathbf{r}^2\ . \tag{2.2}$$

For the meson channel, the Hamiltonian is then the harmonic-oscillator Hamiltonian $H=\mathbf{p}^2/m+m\omega^2\mathbf{r}^2/4$, with excitation energy ω.

We study here the *qualitative* properties of the system $qq\bar{q}\,\bar{q}$ (diquonium) in the chromopotential model. We do not consider the spin-dependent part of the interaction. The symmetries of spin isospin come out by the Pauli principle, when we consider states symmetric or antisymmetric under space and color. However, as we will see, the effect turns out to be small: we obtain similar results for both cases.

The calculation method can be summarized as follows.[7] Let us take as an example the states that are symmetric under quark exchange that write $\Psi=\psi C_1+(P\psi)C_1'$, with the color wave functions $C_1=(q\bar{q})_1(q'\bar{q}')_1$ and $C_1'=(q\bar{q}')_1(q'\bar{q})_1$, and P is the spatial operator of quark permutations. We will take the functions ψ with a limited number of meson excitations $(q\bar{q})$ and $(q'\bar{q}')$, but with arbitrary functions of the relative coordinate. The truncated Schrödinger equation is

then obtained by the condition $\langle \delta\Psi|H-E|\Psi\rangle=0$, where $\delta\Psi$ varies in our subspace. We look then for the solutions having the form of an ingoing wave plus a diffused wave; the diffused wave gives then the S matrix for the different processes.

III. DIQUONIUM

We note the two quarks q,q' and the two antiquarks $\bar{q},\bar{q}'$. In color space, of dimension N^4, the subspace of the *neutral states* [singlets under SU(N)] is of *dimension 2*. A quark-antiquark system decomposes into two irreducible representations $\{N\}\otimes\{\bar{N}\}=\{1\}\oplus\{N^2-1\}$, and then the coupling of two quark-antiquark pairs gives the singlet $\{1\}\otimes\{1\}$, plus a singlet contained in $\{N^2-1\}\otimes\{N^2-1\}$.

We will call C_1, C_8 ($8=N^2-1$ for $N=3$) the color base thus obtained:

$$C_1=(q\bar{q})_1(q'\bar{q}')_1,\quad C_8=[(q\bar{q})_8(q'\bar{q}')_8]_1 . \tag{3.1}$$

Explicitly, we have, calling α a color index,

$$C_1(\alpha_q,\alpha_{\bar{q}},\alpha_{q'},\alpha_{\bar{q}'})=\frac{1}{N}\delta_{\alpha_q\alpha_{\bar{q}}}\delta_{\alpha_{q'}\alpha_{\bar{q}'}} ,$$
$$C_8(\alpha_q,\alpha_{\bar{q}},\alpha_{q'},\alpha_{\bar{q}'}) \tag{3.2}$$
$$=\frac{1}{\sqrt{N^2-1}}\left[\delta_{\alpha_q\alpha_{\bar{q}'}}\delta_{\alpha_{q'}\alpha_{\bar{q}}}-\frac{1}{N}\delta_{\alpha_q\alpha_{\bar{q}}}\delta_{\alpha_{q'}\alpha_{\bar{q}'}}\right] .$$

Instead of the coupling scheme $(q\bar{q})(q'\bar{q}')$, we could take $(q\bar{q}')(q'\bar{q})$, or the diquark-antidiquark $(qq')(\bar{q}\,\bar{q}')$. Although a single basis is enough, another color state plays an important role in the calculations: namely,

$$C_1'=(q\bar{q}')_1(q'\bar{q})_1 . \tag{3.3}$$

C_1 and C_1' have the property of factorizing into two singlets (meson-meson states) between which the interaction is not confining (we assume $N\geq 3$; if $N=2$ there is a third state $(qq')_1(\bar{q}\,\bar{q}')_1$ of this type, of baryon-antibaryon structure). In our basis, C_1' is given by

$$C_1'=\frac{1}{N}C_1+\frac{\sqrt{N^2-1}}{N}C_8 . \tag{3.4}$$

The matrix elements of the color operators entering in $\mathcal{V}$ are the following, in the basis (C_1,C_8):

$\sum\lambda_q\lambda_{q'}=\sum\lambda_{\bar{q}}\lambda_{\bar{q}'}$

	C_1	C_8
C_1	0	$2\frac{\sqrt{N^2-1}}{N}$
C_8	$2\frac{\sqrt{N^2-1}}{N}$	$-\frac{4}{N}$

$\sum\lambda_q\lambda_{\bar{q}}=\sum\lambda_{q'}\lambda_{\bar{q}'}$

	C_1	C_8
C_1	$-2\frac{N^2-1}{N}$	0
C_8	0	$\frac{2}{N}$

$\sum\lambda_q\lambda_{\bar{q}'}=\sum\lambda_{q'}\lambda_{\bar{q}}$

	C_1	C_8
C_1	0	$-2\frac{\sqrt{N^2-1}}{N}$
C_8	$-2\frac{\sqrt{N^2-1}}{N}$	$-2\frac{N^2-2}{N}$

We can now write down the Hamiltonian in the basis C_1,C_8. It is useful to introduce the intercluster potentials

$$V_M=V(\mathbf{r}_{q\bar{q}})+V(\mathbf{r}_{q'\bar{q}'}),\quad V_{M'}=V(\mathbf{r}_{q\bar{q}'})+V(\mathbf{r}_{q'\bar{q}}) ,$$
$$V_D=V(\mathbf{r}_{qq'})+V(\mathbf{r}_{\bar{q}\,\bar{q}'}) , \tag{3.5}$$

where M and M' design the two meson-meson clusterings $(q\bar{q})(q'\bar{q}')$ and $(q\bar{q}')(q'\bar{q})$, and D designs the diquark-antidiquark clustering $(qq')(\bar{q}\,\bar{q}')$.

The color matrix elements of $\mathcal{V}$ are

$$\langle C_1|\mathcal{V}|C_1\rangle=V_M ,$$
$$\langle C_8|\mathcal{V}|C_8\rangle=\frac{2}{N^2-1}V_D-\frac{1}{N^2-1}V_M+\frac{N^2-2}{N^2-1}V_{M'} , \tag{3.6}$$
$$\langle C_1|\mathcal{V}|C_8\rangle=\frac{1}{\sqrt{N^2-1}}(-V_D+V_{M'}) .$$

Writing down the wave function $\Psi=\Psi_1C_1+\Psi_8C_8$ under the matrix form

$$\begin{pmatrix}\Psi_1\\ \Psi_8\end{pmatrix} ,$$

we have the following expression for the Hamiltonian:

$$H=\sum_i\frac{\mathbf{p}_i^2}{2m_i}+\begin{pmatrix} V_M & \frac{V_{M'}-V_D}{\sqrt{N^2-1}} \\ \frac{V_{M'}-V_D}{\sqrt{N^2-1}} & \frac{2V_D}{N^2-1}-\frac{V_M}{N^2-1}+\frac{N^2-2}{N^2-1}V_{M'}\end{pmatrix} . \tag{3.7}$$

We see that if we restrict to the color states $C_1=(q\bar{q})_1(q'\bar{q}')_1$, we have the Hamiltonian of two mesons *without interaction*. One can estimate,[4] by the effective potential method, the effect of the coupling to the states C_8. One finds, at large distance, the following meson-meson potential (van der Waals force):

$$V_{\text{eff}}(\mathbf{R}) \sim -\frac{c}{2}\frac{\alpha^2}{N^2}[(\boldsymbol{\rho}\cdot\boldsymbol{\rho}')-(2-\alpha)(\hat{\mathbf{R}}\cdot\boldsymbol{\rho})(\hat{\mathbf{R}}\cdot\boldsymbol{\rho}')]^2|\mathbf{R}|^{\alpha-4}\ , \tag{3.8}$$

where $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ are the meson internal coordinates, and $\mathbf{R}$ is the distance between the mesons. We have assumed $V(\mathbf{r}) \sim c|\mathbf{r}|^{\alpha}$. Thus, the residual potential falls a power 4 relative to the confining potential.

One can rewrite the Hamiltonian under the interesting form

$$H = \sum_i \frac{\mathbf{p}_i^2}{2m_i} + \frac{1}{2}\left[\frac{N^2}{N^2-1}\begin{bmatrix}0\\1\end{bmatrix}(0,1)W_M + \frac{N^2}{N^2-1}\begin{bmatrix}\frac{\sqrt{N^2-1}}{N}\\ -\frac{1}{N}\end{bmatrix}\left(\frac{\sqrt{N^2-1}}{N}, -\frac{1}{N}\right)W_{M'} + \begin{bmatrix}1 & \frac{1}{\sqrt{N^2-1}}\\ \frac{1}{\sqrt{N^2-1}} & \frac{N^2-3}{N^2-1}\end{bmatrix}W_D\right], \tag{3.9}$$

where the combinations of potentials

$$W_M = -V_M + V_{M'} + V_D,\quad W_{M'} = V_M - V_{M'} + V_D,\quad W_D = V_M + V_{M'} - V_D \tag{3.10}$$

correspond each of them to effective U(1) interactions, for example W_M corresponds to a positive charge for q and $\bar{q}$ and a negative charge for q' and $\bar{q}'$, etc.

Form (3.9) is useful to make clear the *positivity*, since the W's are positive [for instance, for $V(\mathbf{r})=|\mathbf{r}|^{\alpha}$, $0<\alpha\leq 2$, see Appendix B], and to make explicit the *asymptotic states*.

We expect as asymptotic states the cluster decompositions such that each cluster is in a color-singlet representation, i.e., the meson-meson states $(q\bar{q})_1(q'\bar{q}')_1$, and $(q\bar{q}')_1(q'\bar{q})_1$. We can see this from (3.9) and (3.10) in the following way: in the channel $(q\bar{q})(q'\bar{q}')W_M$ is large and $W_{M'}$ and W_D remain bounded. Since the energy is finite, the coefficient of W_M in (3.9) must vanish: this imposes the color state $\binom{1}{0}=C_1$. In this color state, the mean value of the brackets is $(W_{M'}+W_D)/2=V_M$, and the Hamiltonian reduces to the one of two *noninteracting* mesons:

$$H_M = \sum_i \frac{\mathbf{p}_i^2}{2m_i} + V(\mathbf{r}_{q\bar{q}}) + V(\mathbf{r}_{q'\bar{q}'})\ . \tag{3.11}$$

In the same way, in the channel $(q\bar{q}')(q'\bar{q})$, if $W_{M'}$ is large, and W_M, W_D bounded, the color state must be

$$\begin{bmatrix}\frac{1}{N}\\ \frac{\sqrt{N^2-1}}{N}\end{bmatrix} = C_1'$$

(C_1' is obtained from C_1 by transposition of the quarks or the antiquarks). The mean value of the brackets is then $(W_M+W_D)/2=V_{M'}$, and we have again the Hamiltonian of two noninteracting mesons:

$$H_{M'} = \sum_i \frac{\mathbf{p}_i^2}{2m_i} + V(\mathbf{r}_{q\bar{q}'}) + V(\mathbf{r}_{q'\bar{q}})\ . \tag{3.12}$$

Finally, when $N>2$, the matrix multiplying W_D is of rank 2 and positive definite, so that the clusters $(qq')(\bar{q}\bar{q}')$ remain confined for any color state. However, if $N=2$, we have then a new asymptotic baryon-antibaryon channel (baryon $=N$ quarks in a color-singlet state).

IV. SCATTERING

The aim of this section is to convince the reader that our scattering problem is correctly set, in spite of the fact that the interaction we start from is confining. Indeed, the physical mechanism at work here is more subtle than in usual scattering theory, where the existence of the wave operators $\Omega^{\pm}$ follows simply from the fact that wave packets escape the region of influence of the potential.

We will now (1) describe the scattering in a qualitative way, (2) state a mathematical conjecture, which we believe to hold, that allows us to apply the usual scattering theory to our problem,[8] and (3) give intuitive physical arguments that support our conjecture.

(1) From the preceding remarks on the asymptotic meson-meson states, we see that, for $t\to\pm\infty$, the evolution becomes asymptotically close to an evolution without interaction, in the sense that a state $e^{-iHt}\Psi$ becomes close to $(e^{-i(T+V_M)t}\psi)C_1+(e^{-i(T+V_{M'})t}\psi')C_1'$ for certain functions $\psi,\psi'\in L^2(\mathbb{R}^{12})$, where $\mathbb{R}^{12}$ corresponds to the spatial coordinates of the four quarks involved. One can then describe the scattering process as follows. We have in the past a state of two mesons $(q\bar{q})$ and $(q'\bar{q}')$ far away from one another, described by $(e^{-i(T+V_M)t}\psi_{\text{in}})C_1$ at $t\to-\infty$ which, by the evolution operator e^{-iHt} becomes $(e^{-i(T+V_M)t}\psi_{\text{out}})C_1+(e^{-i(T+V_{M'})t}\psi'_{\text{out}})C_1'$ at $t\to+\infty$. The function ψ_{out} represents a final state that results from *elastic scattering* and from *internal excitations* of the two mesons we have started from, while ψ'_{out} represents a final state with *quark rearrangement*.

The two-Hilbert-space formalism[9] is well adapted to this situation, mostly when one wants to take into account the symmetry between $(q\bar{q})(q'\bar{q}')$ and $(q\bar{q}')(q'\bar{q})$, as we will see in Sec. VI. We have the physical space

$\mathcal{H}=L^2(\mathbb{R}^{12})\otimes\mathbb{C}^2$ of the wave functions $qq\bar{q}\,\bar{q}$ with *two* color components (hence $\mathbb{C}^2$), with the Hamiltonian H defined above. We have the asymptotic space $\mathcal{H}_0=L^2(\mathbb{R}^{12})\oplus L^2(\mathbb{R}^{12})$, corresponding to the Hamiltonian

$$H_0=(T+V_M)\oplus(T+V_{M'}) \tag{4.1}$$

direct sum of (3.11) and (3.12). And we have the application to identify a state in $\mathcal{H}_0$ to some state in $\mathcal{H}$:

$$J:\ (\psi,\psi')\in\mathcal{H}_0\rightarrow\psi C_1+\psi' C_1'\in\mathcal{H}\ . \tag{4.2}$$

The wave operators $\Omega^{\pm}$: $\mathcal{H}_0\rightarrow\mathcal{H}$, defined by

$$\Omega^{\pm}=\lim_{t\rightarrow\mp\infty} e^{iHt}Je^{-iH_0t} \tag{4.3}$$

will transform a (generalized) basis of eigenvectors of H_0 in a basis (up to the bound states) of eigenvectors of H. The isometry of the operators $\Omega^{\pm}$ is ensured by the asymptotic unitarity of the operator J:

$$\lim_{t\rightarrow\pm\infty}\|Je^{-iH_0t}\Psi\|=\|\Psi\|_0\quad(\Psi\in\mathcal{H}_0)\ . \tag{4.4}$$

And the S matrix $\mathcal{H}_0\rightarrow\mathcal{H}_0$ is given by

$$S=\Omega^{-*}\Omega^{+}\ . \tag{4.5}$$

(2) The preceding description is based on the following mathematical conjecture, that we think to hold, that allows us to apply the scattering theory to our problem:

Conjecture: For $N\geq 3$, and for a class of confining potentials that include $V(r)=|r|^{\alpha}$, $0<\alpha\leq 2$, (i) the limits (4.3) exist in the strong sense, (ii) formula (4.4) holds, and (iii) $\Omega^{+}\mathcal{H}_0=\Omega^{-}\mathcal{H}_0=\mathcal{H}_c$, where $\mathcal{H}_c$ is the orthogonal subspace in $\mathcal{H}$ of the subspace of the eigenstates of H.

(Note that for $N=2$, one would have to add the baryon-antibaryon channels.)

(3) Let us now give physical arguments that support the three parts of our conjecture.

(i) Let us first see why the limit (4.3) should exist. In usual scattering theory, this limit exists because the wave packets asymptotically escape the region of influence of the potential. In our case the interaction is confining and we cannot invoke this argument. However, all we need here is that, when *two mesons* $(q\bar{q})$ and $(q'\bar{q}')$ of spatial wave function ψ *are far enough from one another,* the complete evolution $e^{-iHt}(\psi C_1)$ should be close enough to the free evolution $(e^{-i(T+V_M)t}\psi)C_1$ [the case of the two mesons $(q\bar{q}')$ and $(q'\bar{q})$ would be similar]. However, the nondiagonal terms of H in the basis (C_1,C_8) [Eq. (3.7)] will generate a component C_8 in $e^{-iHt}(\psi C_1)$, and these terms decrease slowly (or do not decrease at all in the harmonic-oscillator model) with the meson-meson distance R. However, because of the high energy due to the confining force between the colored states $(q\bar{q})$ and $(q'\bar{q}')$, this component C_8 will have a high-frequency dependence on time, a frequency that will increase with the distance R because of the confining character of the interaction. The final result will be an incoherence of phase in the building up of the component C_8 from the main term $(e^{-i(T+V_M)t}\psi)C_1$, that will keep this component all the more small as R is large.

More precisely, one finds that, for the component C_1 of $e^{-iHt}(\psi C_1)$, the free evolution at large R is corrected by a potential decreasing like $R^{\alpha-4}$ [cf. Eq. (3.8)] or R^{-2} in the harmonic model. However, in potential scattering theory, the wave operators exist[8] for potentials decreasing faster than the Coulomb potential R^{-1}.

(ii) On the other hand, formula (4.4) could seem surprising, since the components C_1 and C_1' are not orthogonal. However, (4.4) holds because the *spatial* wave functions ψ and ψ', describing meson-meson states $(q\bar{q})(q'\bar{q}')$ and $(q\bar{q}')(q'\bar{q})$, become orthogonal when the mesons become far apart, so that in $\psi C_1+\psi' C_1'$ the nonorthogonality of C_1 and C_1' does not prevent the two terms from becoming orthogonal.

(iii) Concerning this part of the conjecture, let us simply note that it does not exclude the presence of bound states in the continuum spectrum, as it is desirable.[3,10] Let us recall that such states appear in the harmonic model,[3] as we have pointed out in Sec. I and in Appendix A.

V. TRUNCATION METHOD

After these considerations to convince the reader (and ourselves) that our scattering problem is correctly set, we now will give a method to solve it.

First, we will have to replace the Hilbert space $\mathcal{H}$ by a truncated Hilbert space $\mathcal{H}'$, and the Hamiltonian H by its projection H' in $\mathcal{H}'$. We will take for $\mathcal{H}'$ the set of states $\psi C_1+\psi' C_1'$, where ψ contains *a limited number of excitations of the mesons* $(q\bar{q})$ and $(q'\bar{q}')$, but with *arbitrary wave functions in the relative coordinates,* and we will do the same for ψ' for the mesons $(q\bar{q}')$ and $(q'\bar{q})$

This choice seems the simplest, but it has for us another fundamental advantage. The most interesting result of our calculations will be to establish the existence of resonances in our system. *Our choice of test functions in the variational calculation ensures us that the resonances that we find are not artifacts of the method.*

Indeed, a dangerous method (not to use) to look for resonances in a system would be to introduce in $\mathcal{H}'$ a set of normalized wave functions able to describe these resonances in the vanishing width limit, plus some states in which they can decay. With such a method, one cannot be sure that the resonances found are not an artifact of the calculation method. We are *not* using such a method: *we only introduce meson-meson states in which the system* $qq\bar{q}\,\bar{q}$ *can apparently decay without any problem.*

After these words of caution, we write down the truncated Schrödinger equation

$$H'\Psi=E\Psi \tag{5.1}$$

by imposing the condition

$$\langle\delta\Psi'|H-E|\Psi\rangle=0\ , \tag{5.2}$$

where $\Psi\in\mathcal{H}'$ and $\delta\Psi'$ varies in $\mathcal{H}'$.

These equations are complicated by the *nonorthogonality* of ψC_1 and $\psi' C_1'$, and this reflects in the fact that the *orthogonal* projection operator

$$\Pi:\ \mathcal{H}\to\mathcal{H}' \tag{5.3}$$

is complicated, and so is $H'=\Pi H\Pi$. In the truncation method we parametrize $\Psi\in\mathcal{H}'$ by $\Psi=J\Phi$ [see Eq. (4.2)], where $\Phi=(\psi,\psi')$, and ψ and ψ' are described above, form the truncated asymptotic space $\mathcal{H}'_0$. The orthogonal projector

$$\Pi_0:\ \mathcal{H}_0\to\mathcal{H}'_0 \tag{5.4}$$

is very simple by construction, and commutes with H_0. By the variational method (or directly using $\Pi_0 J^*\Pi=\Pi_0 J^*$), we obtain the Schrödinger equation under the form

$$\Pi_0(J^*HJ-EJ^*J)\Phi=0,\quad \Phi\in\mathcal{H}'_0\ . \tag{5.5}$$

Thus, the equation

$$(J^*HJ-EJ^*J)\Phi=0,\quad \Phi\in\mathcal{H}_0\ , \tag{5.6}$$

where J^*HJ and J^*J are operators in $\mathcal{H}_0$ is the best adapted to the truncation method. A solution $\Phi=(\psi,\psi')$ of Eq. (5.6) gives a solution $\Psi=J\Phi=\psi C_1+\psi' C'_1$ of $H\Psi=E\Psi$.

Among the solutions of the truncated Schrödinger equation we will now look for the ingoing and outgoing solutions by a Lippmann-Schwinger equation

$$\Phi_\pm=\Phi_0-[H_0-(E\pm i0)]^{-1}V(E)\Phi_\pm\ , \tag{5.7}$$

where

$$V(E)=J^*HJ-H_0-(J^*J-1)E \tag{5.8}$$

and Φ_0 is an eigenstate of H_0, $H_0\Phi_0=E\Phi_0$. Note that $J^*J\neq 1$ because C_1 and C'_1 are not orthogonal.

Finally, we will use a T-matrix formalism, defining $T(z)$ by the equation

$$T(z)=V(z)-V(z)\frac{1}{H_0-z}T(z) \tag{5.9}$$

with

$$V(z)=(J^*HJ-H_0)-(J^*J-1)z\ . \tag{5.10}$$

As we will see below, *in the harmonic-oscillator case, V(z) is, after truncation, an operator of finite rank* and we will be able to compute $T(z)$ from Eq. (5.9) by the inversion of a matrix of finite dimension.

When we have $T(z)$, the (generalized) wave functions of H

$$\Psi_i^\pm=\Omega^\pm\Phi_i,\quad H_0\Phi_i=E_i\Phi_i\ , \tag{5.11}$$

are given by

$$\Psi_i^\pm=J\Phi_i-J\frac{1}{H_0-(E_i\pm i0)}T(E_i\pm i0)\Phi_i \tag{5.12}$$

and the S matrix is given by

$$\langle\Phi_j|S|\Phi_i\rangle=\delta_{j,i}-2i\pi\delta(E_j-E_i)\langle\Phi_j|T(E_i+i0)|\Phi_i\rangle\ . \tag{5.13}$$

From (5.10) we see that we need the expressions J^*J and J^*HJ. We have $J^*\Psi=(\langle C_1|\Psi\rangle,\langle C'_1|\Psi\rangle)$. J^*J is the overlap matrix of C_1 and C'_1:

$$J^*J=\begin{pmatrix}1 & \frac{1}{N}\\ \frac{1}{N} & 1\end{pmatrix}\ . \tag{5.14}$$

The calculation of J^*HJ reduces then to the calculation of the color matrix elements of H between the states C_1 and C'_1. Using (3.4) and (3.7) we obtain

$$J^*HJ=\begin{pmatrix}1 & \frac{1}{N}\\ \frac{1}{N} & 1\end{pmatrix}\sum_i\frac{\mathbf{p}_i^2}{2m_i}+\begin{pmatrix}V_M & \frac{1}{N}W_D\\ \frac{1}{N}W_D & V_{M'}\end{pmatrix}, \tag{5.15}$$

where $W_D=V_M+V_{M'}-V_D$.

Of course, H_0 is given by

$$H_0=\sum_i\frac{\mathbf{p}_i^2}{2m_i}+\begin{pmatrix}V_M & 0\\ 0 & V_{M'}\end{pmatrix}\ . \tag{5.16}$$

VI. PERMUTATION SYMMETRY

We will make all the calculations for equal masses $m_q=m_{q'}$. This is an important simplification, since then the Hamiltonian H is invariant by quark permutations, and one can treat separately the symmetric and antisymmetric sectors. This is the reason why we have introduced the two-Hilbert-space formalism, as it allows us to easily do this job.[9] Let us call $\mathcal{P}$ the permutation operator in $\mathcal{H}$, and P the permutation operator acting only on the *spatial coordinates*. Since the permutation of color interchanges C_1 and C'_1, we have

$$\mathcal{P}(\psi C_1+\psi' C'_1)=(P\psi')C_1+(P\psi)C'_1\ . \tag{6.1}$$

Let us consider the evolution

$$e^{-iHt}[\psi C_1+(P\psi)C'_1] \tag{6.2}$$

of a symmetric (or antisymmetric) state. Asymptotically, we have two states

$$(e^{-i(T+V_M)t}\phi)C_1+(e^{-i(T+V_{M'})t}P\phi)C'_1 \tag{6.3}$$

that evolve according to *different* free Hamiltonians. We face then a difficulty to define the S matrix: in which space should it act? The problem is solved, in second quantization, by the introduction of asymptotic creation and annihilation operators.

In the two-Hilbert-space formalism, it will be enough to introduce the permutation operator $\mathcal{P}_0$ in the asymptotic space $\mathcal{H}_0$, defined by $\mathcal{P}_0(\psi,\psi')=(P\psi',P\psi)$. One has, manifestly, $\mathcal{P}J=J\mathcal{P}_0$, and $\mathcal{P}_0$ commutes with H_0, since $PV_MP=V_{M'}$. Then, in (4.3), the operator $e^{iHt}Je^{-iH_0t}$ respects the symmetry. All the formulas from (4.3) to (5.13) remain valid replacing $\mathcal{H}_0$ and $\mathcal{H}$ by their anti(symmetric) subspaces. In particular, Eq. (4.5) defines the S matrix as an operator in the (anti)symmetric subspace of $\mathcal{H}_0$.

We can identity the subspace of $\mathcal{H}_0$ of symmetry $\eta=\pm 1$ to the space $L^2(\mathbb{R}^{12})$ by the isomorphism

$$\psi \leftrightarrow \frac{1}{\sqrt{2}}(\psi, \eta P \psi) \ . \tag{6.4}$$

Then, the asymptotic evolution of states of a given symmetry can be identified to the evolution of states $(q\bar{q})(q'\bar{q}')$ with the nonsymmetric Hamiltonian $T+V_M$. The S matrix acts on such states $(q\bar{q})(q'\bar{q}')$ with conservation of the energy $T+V_M$.

Using (6.4) we can now rewrite Eqs. (4.2) and (5.14)–(5.16):

$$J\psi=\frac{1}{\sqrt{2}}[\psi C_1+\eta(P\psi)C_1'] \ , \tag{6.5}$$

$$J^*J=1+\frac{\eta P}{N} \ , \tag{6.6}$$

$$J^*HJ=\left[1+\frac{\eta P}{N}\right]\sum_i \frac{\mathbf{p}_i^2}{2m_i}+V_M+\frac{\eta P}{N}W_D \ , \tag{6.7}$$

$$H_0=\sum_i \frac{\mathbf{p}_i^2}{2m_i}+V_M \ . \tag{6.8}$$

One can then compute the scattering eigenstates of H and the S matrix by the formulas (5.9)–(5.13). Explicitly, the interaction operator $V(z)$ defined by (5.10) is given by

$$V(z)=\frac{\eta P}{N}\left[\sum_i \frac{\mathbf{p}_i^2}{2m_i}+W_D-z\right] \ . \tag{6.9}$$

VII. MESON-MESON BASIS

After these generalities we will now make explicit calculations. First, to be more explicit, we have seen that the symmetric and antisymmetric states $\Psi_{(s,a)}$ under the exchange $q\leftrightarrow q'$,

$$\Psi_{(s,a)}=\psi(\mathbf{r}_M,\mathbf{r}_{M'},\mathbf{r}_D)C_1\pm\psi(\mathbf{r}_{M'},\mathbf{r}_M,\mathbf{r}_D)C_1' \tag{7.1}$$

decouple, and that, by the identification (6.4), that allows us to write (6.5)–(6.9) in each separate sector, it is enough to consider the wave function $\psi(\mathbf{r}_M,\mathbf{r}_{M'},\mathbf{r}_D)$, since $\psi'(\mathbf{r}_M,\mathbf{r}_{M'},\mathbf{r}_D)=\psi(\mathbf{r}_{M'},\mathbf{r}_M,\mathbf{r}_D)$, as we read from (7.1).

We now introduce an orthonormal basis ϕ_a of eigenstates of the internal Hamiltonian of two mesons:

$$\left[\frac{\mathbf{p}_1^2}{m}+\frac{\mathbf{p}_2^2}{m}+V(\mathbf{r}_1)+V(\mathbf{r}_2)\right]\phi_a(\mathbf{r}_1,\mathbf{r}_2)=E_a\phi_a(\mathbf{r}_1,\mathbf{r}_2) \tag{7.2}$$

and it will be convenient to introduce the intercluster coordinates

$$\begin{aligned}\mathbf{r}_M&=\frac{\mathbf{r}_q+\mathbf{r}_{\bar{q}}-\mathbf{r}_{q'}-\mathbf{r}_{\bar{q}'}}{2} \ ,\\ \mathbf{r}_{M'}&=\frac{\mathbf{r}_q+\mathbf{r}_{\bar{q}'}-\mathbf{r}_{q'}-\mathbf{r}_{\bar{q}}}{2}=\frac{\mathbf{r}_{q\bar{q}}-\mathbf{r}_{q'\bar{q}'}}{2} \ ,\\ \mathbf{r}_D&=\frac{\mathbf{r}_q+\mathbf{r}_{q'}-\mathbf{r}_{\bar{q}}-\mathbf{r}_{\bar{q}'}}{2}=\frac{\mathbf{r}_{q\bar{q}}+\mathbf{r}_{q'\bar{q}'}}{2} \ ,\end{aligned} \tag{7.3}$$

that correspond to the two types of meson-meson states plus the $(qq')(\bar{q}\,\bar{q}')$ clustering.

The states $|a,\mathbf{r}\rangle$ of wave functions $\phi_a(\mathbf{r}_{q\bar{q}},\mathbf{r}_{q'\bar{q}'})\delta(\mathbf{r}_M-\mathbf{r})$, where $\mathbf{r}_M$ is the relative coordinate between the mesons, form an orthonormal basis in the space of the *spatial* functions of the $qq\bar{q}\,\bar{q}$ system. We will represent these functions Ψ by the series $\Psi_a(\mathbf{r})=\langle a,\mathbf{r}|\Psi\rangle$ of functions in $L^2(\mathbb{R}^3)$. We have, therefore,

$$\psi(\mathbf{r}_M,\mathbf{r}_{M'},\mathbf{r}_D)=\sum_a \Psi_a(\mathbf{r}_M)\phi_a(\mathbf{r}_{q\bar{q}},\mathbf{r}_{q'\bar{q}'}) \ . \tag{7.4}$$

Any operator A takes the form

$$(A\Psi)_a=\sum_{a'} A_{aa'}\Psi_{a'} \ , \tag{7.5}$$

where the $A_{aa'}$ are one-particle operators, acting on $L^2(\mathbb{R}^3)$.

In particular, the truncated Schrödinger equation obtained from (5.1) will write

$$\left[\frac{\mathbf{p}^2}{2m}+E_a\right]\Psi_a(\mathbf{r})+\sum_a V_{aa'}(E)\Psi_{a'}(\mathbf{r})=E\Psi_a(\mathbf{r}) \ , \tag{7.6}$$

where the operator $V(E)$ is given by (6.9).

To compute the T matrix and hence the S matrix from (5.9), we need to compute $V(z)$, given by (5.10), with (6.6)–(6.8) or more explicitly by (6.9).

The Hamiltonian without interaction (6.8) (i.e., for two free mesons in the color state C_1) will be

$$H_0=\sum_i \frac{\mathbf{p}_i^2}{2m_i}+V_M \tag{7.7}$$

and we will have simply

$$(H_0\Psi)_a=\left[\frac{\mathbf{p}^2}{2m}+E_a\right]\Psi_a \tag{7.8}$$

since m is the reduced mass of the two mesons and E_a is their internal excitation energy.

The other operators of the problem, (6.6), (6.7), and (6.9), involve the permutation operator of the spatial coordinates P. This operator exchanges r_M and $\mathbf{r}_{M'}$, and leaves $\mathbf{r}_D$ invariant:

$$P\mathbf{r}_MP=\mathbf{r}_{M'}, \quad P\mathbf{r}_DP=\mathbf{r}_D \tag{7.9}$$

(to make economy of signs we take for P the exchange of antiquarks). On the other hand, the arguments $\mathbf{r}_{q\bar{q}}$ and $\mathbf{r}_{q'\bar{q}'}$ of ϕ_a are independent combinations of $\mathbf{r}_{M'}$ and $\mathbf{r}_D$. Writing ϕ_a with $\mathbf{r}_{M'}$ and $\mathbf{r}_D$, we have

$$P\Psi_a(\mathbf{r}_M)\phi_a(\mathbf{r}_{M'},\mathbf{r}_D)=\Psi_a(\mathbf{r}_{M'})\phi_a(\mathbf{r}_M,\mathbf{r}_D) \tag{7.10}$$

and therefore

$$(P\Psi)_a=\sum_{a'}\langle\phi_a(\mathbf{r}_{M'},\mathbf{r}_D)\delta(\mathbf{r}_M-\mathbf{r})|\Psi_{a'}(\mathbf{r}_{M'})\phi_{a'}(\mathbf{r}_M,\mathbf{r}_D)\rangle \ . \tag{7.11}$$

The operator $P_{aa'}$ is then an integral operator whose kernel, taking into account

$$d\mathbf{r}_{q\bar{q}}d\mathbf{r}_{q'\bar{q}'}=8\,d\mathbf{r}_{M'}d\mathbf{r}_{D}\ , \tag{7.12}$$

is given by

$$P_{aa'}(\mathbf{r},\mathbf{r}')=8\int d\mathbf{r}_{D}\phi_{a}(\mathbf{r}',\mathbf{r}_{D})^{*}\phi_{a'}(\mathbf{r},\mathbf{r}_{D})\ . \tag{7.13}$$

VIII. HARMONIC-OSCILLATOR MODEL

We now chose the potential (2.2), so that, for a meson, we have the harmonic-oscillator Hamiltonian

$$H=\frac{\mathbf{p}^2}{m}+\frac{m\omega^2}{4}\mathbf{r}^2 \tag{8.1}$$

with excitation energy ω.

A. Meson-meson basis and permutation operator

Since $\phi_a(\mathbf{r}_M,\mathbf{r}_D)$ is then the product of a Gaussian (factorizable in $\mathbf{r}_M$ and $\mathbf{r}_D$) and a polynomial, *the kernel (7.13) is of finite rank.* This simplifies enormously the calculations. Following this advantage, we diminish the rank by taking factorized functions $\phi_a(\mathbf{r}_M,\mathbf{r}_D)$. This is indeed possible since (7.2) is also separable in two oscillators corresponding to the variables $(\mathbf{r}_1\pm\mathbf{r}_2)/2$. Finally, our functions ϕ_a will be defined as follows. We will call u_i an orthonormal basis of the three-dimensional harmonic oscillator,

$$\left[\frac{\mathbf{p}^2}{2m}+\frac{m\omega^2}{2}\mathbf{r}^2\right]u_i=E_iu_i\ , \tag{8.2}$$

and the functions $\phi_a(\mathbf{r}_{q\bar{q}},\mathbf{r}_{q'\bar{q}'})$ or $\phi_a(\mathbf{r}_{M'},\mathbf{r}_D)$ will be given by [the index a is now replaced by the double index (i,j)]

$$\begin{aligned}&\phi_{(i,j)}(\mathbf{r}_{M'},\mathbf{r}_D)=\frac{1}{\sqrt{8}}u_i(\mathbf{r}_{M'})u_j(\mathbf{r}_D)\ ,\\ &E_{(i,j)}=E_i+E_j\ .\end{aligned} \tag{8.3}$$

As pointed out above, we further make a truncation in the number of internal meson excitations below a certain given energy

$$E_a=E_i+E_j\le E_{\max}\ . \tag{8.4}$$

These particular combinations of different meson-meson internal states of fixed total excitation energy (note the high degenerescence of the harmonic oscillator) results in a partial diagonalization of the S matrix, and a reduction of the size of the matrices that enter in the calculation.

From (8.3) and (7.13), the permutation operator P, entering in (6.6), (6.7), (6.9), reduces to

$$P_{(ij)(i'j')}=\delta_{jj'}|u_{i'}\rangle\langle u_i|\ . \tag{8.5}$$

And we can now proceed to the calculation of the T matrix.

B. Calculation of the interaction operator $V(E)$

We will now compute the expression $V(z)$ [Eq. (6.9)], needed to compute the T matrix through (5.9) and (5.10). Let us first compute the interaction operator (6.9). Let us write down

$$X=\sum_i\frac{\mathbf{p}_i^2}{2m_i}+W_D=\frac{\mathbf{p}_M^2}{2m}+\frac{\mathbf{p}_{q'\bar{q}'}^2}{m}+\frac{\mathbf{p}_{q\bar{q}}^2}{m}+m\omega^2\mathbf{r}_D^2\ , \tag{8.6}$$

where we have taken out the center of mass. From (7.4) we obtain for the operators $X_{aa'}$:

$$X_{aa'}=\delta_{aa'}\frac{\mathbf{p}^2}{2m}+\left\langle\phi_a\left|\frac{\mathbf{p}_{q'\bar{q}'}^2}{m}+\frac{\mathbf{p}_{q\bar{q}}^2}{m}+m\omega^2\mathbf{r}_D^2\right|\phi_{a'}\right\rangle\ . \tag{8.7}$$

From this expression, with ϕ_a given by (8.3), and using

$$\frac{\mathbf{p}_{q'\bar{q}'}^2}{m}+\frac{\mathbf{p}_{q\bar{q}}^2}{m}=\frac{\mathbf{p}_{M'}^2}{2m}+\frac{\mathbf{p}_D^2}{2m} \tag{8.8}$$

we have

$$\begin{aligned}X_{(ij)(i'j')}=&\delta_{ii'}\delta_{jj'}\frac{\mathbf{p}^2}{2m}+\delta_{jj'}\left\langle u_i\left|\frac{\mathbf{p}^2}{2m}\right|u_{i'}\right\rangle\\&+\delta_{ii'}\left\langle u_j\left|\frac{\mathbf{p}^2}{2m}+m\omega^2\mathbf{r}^2\right|u_{j'}\right\rangle\ .\end{aligned} \tag{8.9}$$

We then multiply this expression by (8.5). We find the final result

$$\begin{aligned}V(z)_{(ij)(i'j')}=\frac{\eta}{N}\Bigg[&\delta_{jj'}\left(\frac{\mathbf{p}^2}{2m}|u_{i'}\rangle\langle u_i|+|u_{i'}\rangle\langle u_i|\frac{\mathbf{p}^2}{2m}\right)\\&+\left\langle u_j\left|\frac{\mathbf{p}^2}{2m}+m\omega^2\mathbf{r}^2-z\right|u_{j'}\right\rangle|u_{i'}\rangle\langle u_i|\Bigg]\ .\end{aligned} \tag{8.10}$$

C. Calculation of the T matrix: inversion of a finite dimension matrix

We need now to solve the equation for $T(z)$ [Eq. (5.9)] that, taking into account the expression (7.8) for H_0, takes the form

$$T(z)_{aa'}=V(z)_{aa'}-\sum_{a''}V(z)_{aa''}\frac{1}{\frac{\mathbf{p}^2}{2m}+E_{a''}-z}T(z)_{a''a'}\ . \tag{8.11}$$

We will first give here the general lines of the calculation to show that it reduces to *the inversion of a matrix of finite dimension.* We will then proceed to the explicit calculations in Sec. IX.

Let us write the operator $V(z)_{aa'}$, given by (8.10) under the general form

$$V(z)_{aa'}=\sum_{kk'}v(z)_{kk'}^{aa'}|u_k\rangle\langle u_{k'}|\ , \tag{8.12}$$

where the coefficients $v(z)_{kk'}^{aa'}$ are given by (8.10) by decomposing

$$\frac{\mathbf{p}^2}{2m}|u_i\rangle\ .$$

Looking for $T(z)_{aa'}$ under the form

$$T(z)_{aa'}=\sum_{kk'} t(z)^{aa'}_{kk'}|u_k\rangle\langle u_{k'}| \tag{8.13}$$

we see that (8.11) will be satisfied if the coefficients $t(z)^{aa'}_{kk'}$ satisfy the equations:

$$t(z)^{aa'}_{kk'}=v(z)^{aa'}_{kk'}-\sum_{bb'}\sum_{qq'} v(z)^{ab}_{kq} r(z)^{bb'}_{qq'} t(z)^{b'a'}_{q'k'} , \tag{8.14}$$

where

$$r(z)^{aa'}_{kk'}=\delta_{aa'}\left\langle u_k \left| \frac{1}{\frac{\mathbf{p}^2}{2m}+E_a-z} \right| u_{k'} \right\rangle . \tag{8.15}$$

The structure of Eq. (8.14) is very simple if we group the indices a (itself a double index) and k, and we consider $v(z)^{aa'}_{kk'}$, $t(z)^{aa'}_{kk'}$, and $r(z)^{aa'}_{kk'}$ as elements of some matrices $v(z)$, $t(z)$, and $r(z)$. We have the matrix equation

$$t(z)=v(z)-v(z)r(z)t(z) \tag{8.16}$$

whose solution is

$$t(z)=[1+v(z)r(z)]^{-1}v(z) . \tag{8.17}$$

From (8.10) and for a and a' fixed, the indices k and k' have only a finite number of values for which $v(z)^{aa'}_{kk'}\neq 0$. *The operator $V(z)_{aa'}$ is of finite rank.* We see that by our truncation, as described in Sec. V, to a finite number of meson excitations (limited E_a), *the calculation of t(z)* by Eq. (8.17) *reduces to the inversion of a matrix of finite dimension.*

D. S matrix

The eigenfunctions of the free Hamiltonian H_0 are, from (7.8),

$$\Phi_{a,\mathbf{k}}=e^{i\mathbf{k}\cdot\mathbf{r}_M}\phi_a \quad \left[E=\frac{\mathbf{k}^2}{2m}+E_a\right] . \tag{8.18}$$

From (5.12), (7.5), (7.8), and (8.13), the corresponding scattering waves are given by

$$\Psi^{\pm}_{a,\mathbf{k}}=J\left[\Phi_{a,\mathbf{k}}-\sum_{a'}\sum_{qq'}\frac{1}{\frac{\mathbf{p}_M^2}{2m}+E_{a'}-(E\pm i0)}|u_q\rangle t(E\pm i0)^{a'a}_{qq'}\langle u_{q'}|e^{i\mathbf{k}\cdot\mathbf{r}}\rangle\phi_{a'}\right] \tag{8.19}$$

and the S matrix is given by

$$\begin{aligned}\langle\Phi_{a',\mathbf{k}'}|S|\Phi_{a,\mathbf{k}}\rangle&=\delta_{a'a}(2\pi)^3\delta(\mathbf{k}'-\mathbf{k})-2i\pi\delta(E'-E)\langle e^{i\mathbf{k}'\cdot\mathbf{r}}|T(E+i0)_{a'a}|e^{i\mathbf{k}\cdot\mathbf{r}}\rangle\\&=\delta_{a'a}(2\pi)^3\delta(\mathbf{k}'-\mathbf{k})-2i\pi\delta(E'-E)\sum_{qq'}\langle e^{i\mathbf{k}'\cdot\mathbf{r}}|u_{q'}\rangle t(E+i0)^{a'a}_{q'q}\langle u_q|e^{i\mathbf{k}\cdot\mathbf{r}}\rangle ,\end{aligned} \tag{8.20}$$

IX. CALCULATIONS AND RESULTS

Let us now give the results. In the chromoharmonic model, the angular momenta between clusters L_M, $L_{M'}$, and L_D are separately conserved. We are going to study the sector $L_M=L_{M'}=L_D=0$, that contains the ground-state meson-meson states ($L_{M'}=L_D=0$) in the S wave ($L_M=0$). This is the simplest sector, and the one where the resonances, as we have discussed in the Introduction, are expected to be less probable. This leads one to consider the part of the matrix $V(z)_{(ij)(i'j')}$ with the indices referring to radial excitations of the harmonic oscillator. From now on, the indices $i,j,\ldots$ label the radial excitations

$$E_i=\tfrac{3}{2}\omega+2i\omega . \tag{9.1}$$

A. Calculation of V(z)

We simplify the formulas making $m=\frac{1}{2}$, to have $\mathbf{p}^2/2m=\mathbf{p}^2$. Equation (8.2) becomes $(\mathbf{p}^2+\mathbf{r}^2)u_i=E_iu_i$ taking $\omega=2$ as meson excitation. We use the functions u_i in momentum space, given by

$$u_i(\mathbf{p})=\frac{1}{\sqrt{4\pi}}\left[\frac{2i!}{\Gamma\left[i+\frac{3}{2}\right]}\right]^{1/2}L_i^{1/2}(\mathbf{p}^2)e^{-\mathbf{p}^2/2} \tag{9.2}$$

with the normalization $\int d^3\mathbf{p}\,u_i(\mathbf{p})u_j(\mathbf{p})=\delta_{ij}$.

Using Eq. (8.2), $m=\frac{1}{2}$ and $\omega=2$, Eq. (8.10) writes

$$\begin{aligned}V(z)_{(ij)(i'j')}=\frac{\eta}{N}\{&\delta_{jj'}[\mathbf{p}^2|u_{i'}\rangle\langle u_i|+|u_{i'}\rangle\langle u_i|\mathbf{p}^2\\&+(2E_j-z)|u_{i'}\rangle\langle u_i|]\\&-\langle u_j|\mathbf{p}^2|u_{j'}\rangle|u_{i'}\rangle\langle u_i|\} .\end{aligned} \tag{9.3}$$

One can expand $\mathbf{p}^2|u_{i'}\rangle$ and $\langle u_i|\mathbf{p}^2$ with the recurrence relations

$$\begin{aligned}\mathbf{p}^2u_n=&-[(n+1)(n+\tfrac{3}{2})]^{1/2}u_{n+1}\\&+(2n+\tfrac{3}{2})u_n-[n(n+\tfrac{1}{2})]^{1/2}u_{n-1}\end{aligned} \tag{9.4}$$

and obtain the matrix elements $v(z)^{(ij)(i'j')}_{kk'}$. The nonvanishing $v(z)^{(ij)(i'j')}_{kk'}$ (for given i, j, and i') correspond to the six following combinations for the indices j', k, and k':

$$j'=j,\quad \begin{cases}k=i', & k'=i ,\\ k=i'\pm 1, & k'=i ,\\ k=i', & k'=i\pm 1 ,\end{cases} \tag{9.5}$$

and

$$j'=j\pm 1,\quad k=i',\quad k'=i ,$$

where we have to ignore the negative values.

B. Calculation of $r(z)$

To compute the T matrix (8.13) and (8.14), we need the matrix elements $r(z)_{kk'}^{(ij)(i'j')}$, given by (8.15), that can be written as

$$r(z)_{kk'}^{(ij)(i'j')}=\delta_{ii'}\delta_{jj'}R(z-E_i-E_j)_{kk'}\ , \tag{9.6}$$

where $R(z)_{kk'}$ are the harmonic-oscillator matrix elements of the resolvent of $\mathbf{p}^2$:

$$R(z)_{kk'}=\left\langle u_k\left|\frac{1}{\mathbf{p}^2-z}\right|u_{k'}\right\rangle\ . \tag{9.7}$$

The recurrence relations (9.4) allow one to compute $u_i(\mathbf{p})$ [that we need for the S matrix (8.20)], starting from

$$u_0(p)=\frac{1}{\pi^{3/4}}e^{-\mathbf{p}^2/2}\ . \tag{9.8}$$

Moreover, one obtains, from (9.4) a recurrence relation for $R(z)_{kk'}$,

$$\begin{aligned}&[k(k+\tfrac{1}{2})]^{1/2}R(z)_{k,k'}\\&\quad=-\delta_{k-1,k'}+(2k-\tfrac{1}{2}-z)R(z)_{k-1,k'}\\&\qquad-[(k-1)(k-\tfrac{1}{2})]^{1/2}R(z)_{k-2,k'}\ .\end{aligned} \tag{9.9}$$

Using the analogous relation in $k'[R(z)_{kk'}=R(z)_{k'k}]$, we find

$$R(z)_{00}=\frac{4}{\sqrt{\pi}}\int_0^\infty dp\frac{p^2e^{-p^2}}{p^2-z}=\frac{2}{\sqrt{\pi}}\int_0^\infty du\frac{\sqrt{u}\,e^{-u}}{u-z}\ , \tag{9.10}$$

where the subindices 00 refer to the channel with no radial excitation within the asymptotic mesons.

The imaginary part of $R(E+i0)_{00}$ is simply given by

$$\mathrm{Im}R(E+i0)_{00}=2\sqrt{\pi}\theta(E)\sqrt{E}\,e^{-E} \tag{9.11}$$

and the real part is given by a principal part, that we can easily calculate by the formula

$$\begin{aligned}R(E+i0)_{00}=&-2e^{-E}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{2n-1}E^n\\&+\begin{cases}2i\sqrt{\pi}\sqrt{E}\,e^{-E}, & E>0\ ,\\ -2\sqrt{\pi}\sqrt{-E}\,e^{-E}, & E<0\ ,\end{cases}\end{aligned} \tag{9.12}$$

except for $E<0$ too large, since there are compensations in the series. In this case we can however use the integral (9.10), since it does not have singularities in this region.

C. Truncation and numerical results

We will finally truncate to a fixed number of meson excitations N_{exct}; i.e., we will restrain the indices by the conditions

$$0\le i+j\le N_{\mathrm{exct}},\quad 0\le i'+j'\le N_{\mathrm{exct}}\ . \tag{9.13}$$

From the matrix elements of $V(z)$ and $r(z)$ discussed above, in Secs. IX A and IX B, we can now obtain the T matrix from the matrix inversion (8.17).

For $N_{\mathrm{exct}}=0$, the channel is $a=(i,j)=(0,0)$, and the dimension of the matrix to be inverted is 2. For $N_{\mathrm{exct}}=1$, the channels are $(i,j)=(0,0),(0,1),(1,0)$, and the dimension of the matrix is 9. For $N_{\mathrm{exct}}=2$, the channels are $(i,j)=(0,0),(0,1),(1,0),(0,2),(1,1),(2,0)$ and the matrix dimension is 24.

We have made the calculation in these three approximations, in the symmetric and the antisymmetric cases (7.1), $\eta=\pm1$. The situation seems superficially similar in both cases, as we can see in Figs. 1 and 2, where we plot $\sin^2\delta_0$ and $\sin\delta_0\cos\delta_0$ as functions of the energy (the subindex 0 indicates the channel with ground-state asymptotic mesons, i.e., no radial excitations). In Figs. 3–5 we plot the Argand diagrams, of which Figs. 1 and 2 are projections.

For $N_{\mathrm{exct}}=0$, we do not find any resonance. For $N_{\mathrm{exct}}=1$, we find a narrow resonance just below 2ω, i.e. below the first radial excitation level $N=1$. The energy scale is such that we have substracted two ground-state meson masses. The resonance is quite narrow: for a meson level spacing ω of about 500 MeV, Γ is of the order of 10 MeV. For $N_{\mathrm{exct}}=2$ this resonance remains, slightly shifted, and a new inelastic resonance appears just below 4ω, the energy of the $N=2$ radial excitation. This seems to indicate that the first resonance is a bound state of $(q\bar{q})_{N=1}$ with $(q\bar{q})_{N=0}$, a kind of "molecular state." Similarly, the second resonance will be a bound state of $(q\bar{q})_{N=2}$ with $(q\bar{q})_{N=0}$ and of $(q\bar{q})_{N=1}$ with $(q\bar{q})_{N=1}$.

Note that besides these resonances we find a broad ground-state threshold effect (the threshold corresponding to the ground-state mesons channel) on the phase shifts, corresponding to attraction ($\eta=+1$) or repulsion ($\eta=-1$), as we see in Figs. 1 and 2.

We find however an important difference between the cases $\eta=+1$ and $\eta=-1$. Indeed, for $\eta=+1$, the region of the second peak shows in fact two resonances, as we see by studying the total phase shift. In Figs. 1–5 we have plotted the elastic amplitude. To exhibit more clearly all the resonances, even the inelastic ones, we have plotted in Figs. 6–8 the total phase shift $\frac{1}{2}$ phase $\det(S)$. We see that near 4ω, for $\eta=-$, the phase shift takes the value $\pi/2$ and we have a single resonance, while for $\eta=+$ we have two resonances, since the phase shift takes the values $3\pi/2$ and $\pi/2$. It is interesting to discuss the falling down of the total phase shift after the threshold in the light of Levinson's theorem: in potential theory, the fall down of $n\pi$ is related to the presence of n bound states. This is evidence of the fact that these resonances can be understood as bound states of the particles defining the threshold.

We give in Table I the parameters of the resonances with $\omega=400$ MeV as unit of energy. For the isolated resonances ($\eta=\pm1$ at $E=1.9$), the parameters E_r, Γ, and ϕ are fitted to reproduce the phase $\delta(E)$ of the $\det(S)$ by the formula

$$\det(S)=e^{i\delta(E)}=e^{i\phi(E)}\frac{E_r-E+i\Gamma/2}{E_r-E-i\Gamma/2}\ . \tag{9.14}$$

For the two close resonances ($\eta=+1$ at $E=3.9$) we

TABLE I. Results for the resonance masses and widths in the symmetric and antisymmetric cases from a fit to our results with the parametrizations (9.14) and (9.15).

η	E_r	Γ	ϕ (deg)	E (90°)	Γ (90°)
−1	1.9035	0.0192	12.1	1.9014	0.0201
−1	3.7419	0.0232	1.0	3.7417	0.0232
+1	1.9048	0.0173	−1.52	1.9050	0.0173
+1	3.8926	0.0125	5.02	3.8819	0.0317
+1	3.9023	0.0583	5.02	3.9101	0.0410

adopt

$$\det(S)=e^{i\delta(E)}=e^{i\phi(E)}\frac{E_1-E+i\Gamma_1/2}{E_1-E-i\Gamma_1/2}\frac{E_2-E+i\Gamma_2/2}{E_2-E-i\Gamma_2/2}\ . \tag{9.15}$$

We also give the values of E and Γ corresponding to $\delta(E)=90°$ and to the slope of $\delta(E)$ in this point.

X. DISCUSSION

One question that arises is the following: will these narrow resonances survive in a more realistic scheme? We can make some conjectures:

(1) The *departures from the harmonic-oscillator potential* will open new channels, since we will not have anymore the selection rules associated with the intercluster orbital angular momenta. This kind of effects can make the resonances somewhat wider.

(2) *Quark pair creation.* The decay of the $q\bar{q}q\bar{q}$ system by pair creation will also make the resonances wider. After pair creation, the system can rearrange itself into baryon-antibaryon or three mesons. For the lowest resonance, for which we expect a mass of about $4m_q+2\omega\simeq 2.0-2.4$ GeV, i.e., below $\Delta\bar{\Delta}$ threshold, we can expect partial widths of the same order as for ordinary mesons, smaller or of the order of 100 MeV.[11] Moreover, for a weakly bound state of, say $(q\bar{q})_{N=0}(q\bar{q})_{N=1}$, there will be channels in which each meson $(q\bar{q})_{N=0}$ or $(q\bar{q})_{N=1}$ will decay by pair creation, almost independently of the other one, with widths Γ_0

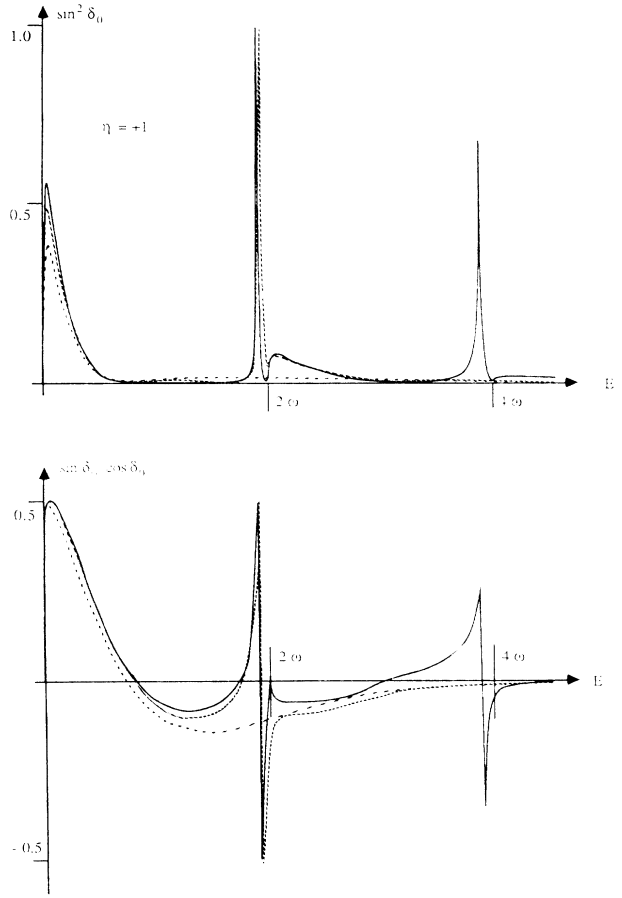

FIG. 1. The functions $\mathrm{Im}[(S_{00}-1)/2i]=\sin^2\delta_0(E)$ and $\mathrm{Re}[(S_{00}-1)/2i]=\sin\delta_0(E)\cos\delta_0(E)$ (δ_0 is the elastic phase shift) in the symmetric case. The energy scale is such that two ground-state meson masses are substracted. The dashed, dotted, and solid curves correspond, respectively, to the approximations $N=0$, 1, and 2.

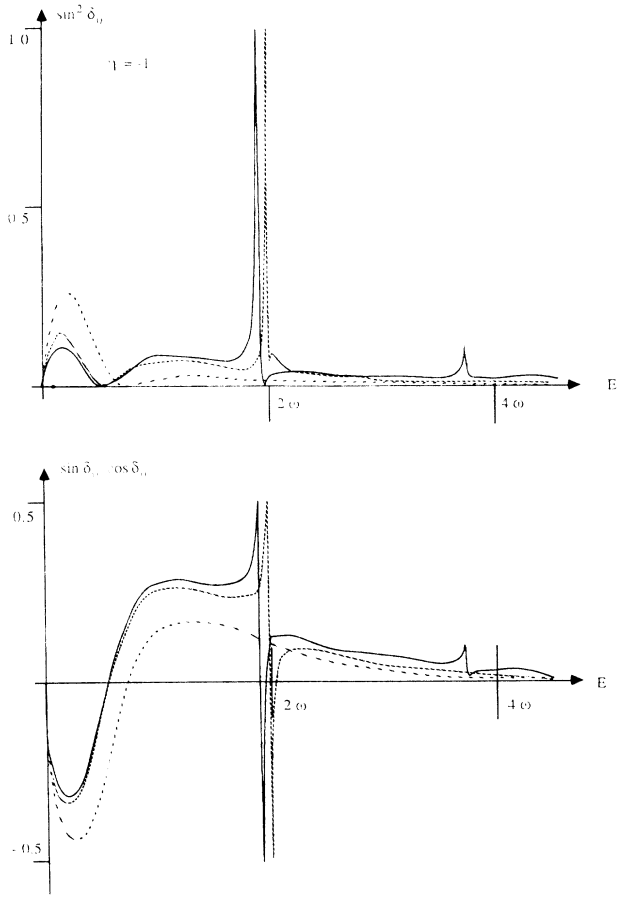

FIG. 2. The functions $\sin^2\delta_0(E)$ and $\sin\delta_0(E)\cos\delta_0(E)$ in the antisymmetric case. The dashed, dotted, and solid curves correspond, respectively, to the approximations $N=0$, 1, and 2.

and Γ_1. Then, the total width of our state will be at least of the order $\Gamma_0+\Gamma_1$. Then, these states could be as large as a few hundred MeV.

(3) *Spin-spin forces* can shift the states. We have seen that these diquonium resonances M_D appear as bound states of, say, a radially excited $N=1$ meson M_1 and a ground-state meson M_0. Considered alone, the spin-spin interaction *inside* the mesons M_1 and M_0 will not destroy the resonance, because the binding energy $M_D-M_1-M_0$ does not change. The reason is that we are not considering a change in the effective interaction between the mesons. However, the spin-spin interaction will also add to the effective interaction between M_1 and M_0. If this force is positive enough, it can destroy the binding between the mesons, and hence the resonance. But this will not be the case in all spin states and some resonances will remain. Also, if we consider orbital excitations, we know that the spin-spin force, being of short range, will not affect significantly these states.

(4) *van der Waals forces.* We know that the color-confining potential induces strong long-range van der Waals forces,[4] for which there is not empirical evidence. Could this long-range force be responsible for the states that we have found? As we have argued in the Introduction, the answer is likely to be no. The reason is that the truncated potential (8.10) and (9.3) falls exponentially. The van der Waals force would appear only in the untruncated limit, in which the number of considered states goes to infinity. The range of the effective potential is of some mesonic radii, and the resonances, being due to a weak binding effect, should have a wave function extending to larger distances, contrarily to what we should expect from binding due to a long-range potential.

Weinstein and Isgur have made interesting and exten-

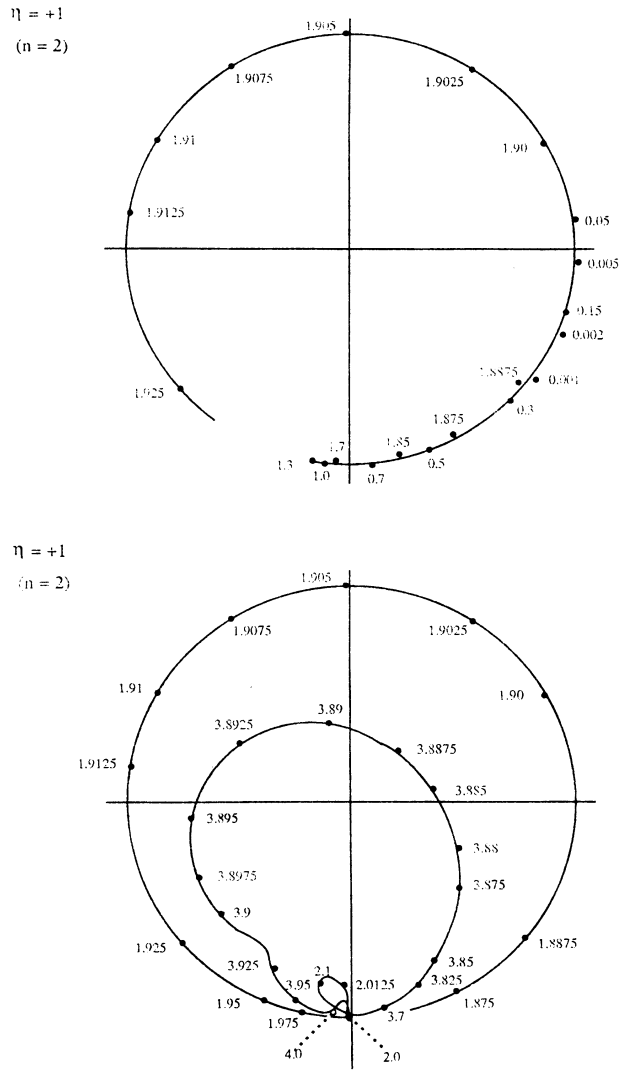

FIG. 3. Argand diagram in the symmetric case, in the $N=2$ approximation.

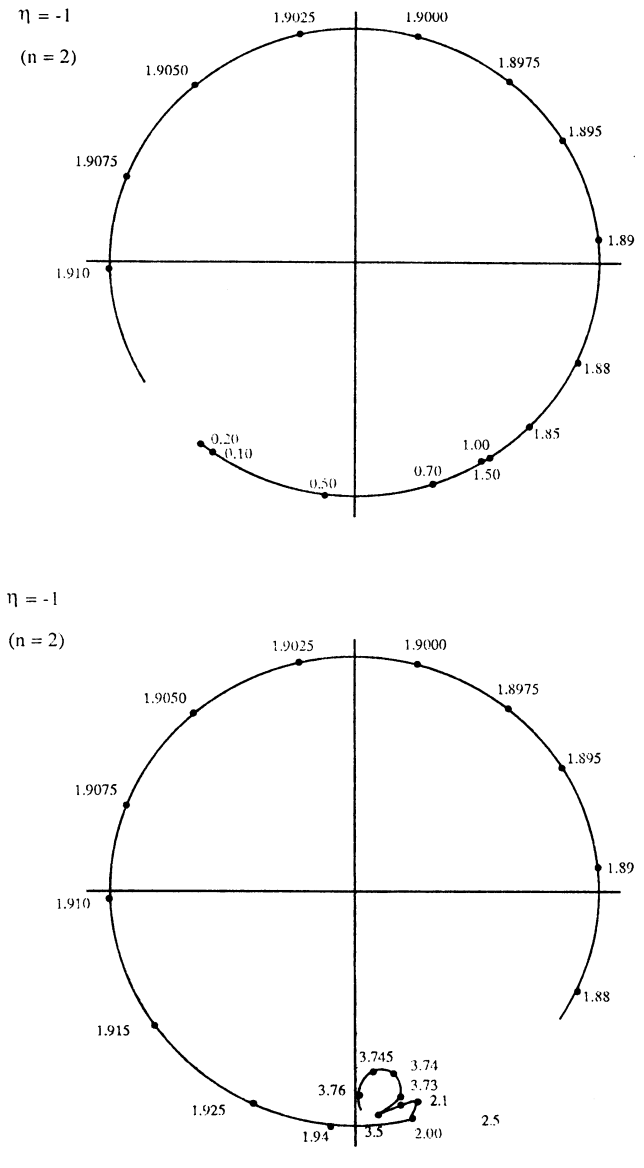

FIG. 4. Argand diagram in the antisymmetric case, in the $N=2$ approximation.

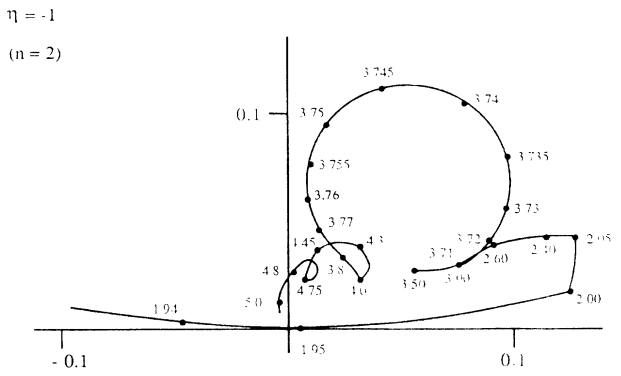

FIG. 5. Blow up of the inelastic resonance region of Fig. 4.

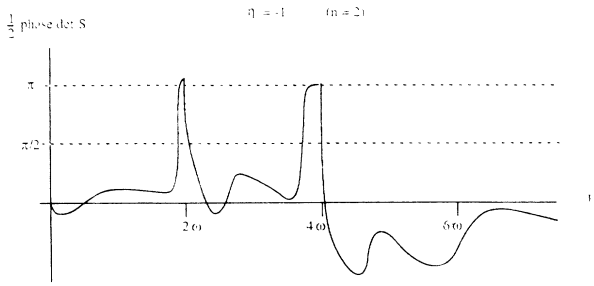

FIG. 7. Total phase shift, in the $N=2$ approximation, in the antisymmetric case.

sive calculations on $q\bar{q}q\bar{q}$ molecules.[2] The type of resonances that we find does not appear in their calculations in their present form. It would be interesting to see if they could appear. Their formalism is however different from ours in the sense that they write down an effective potential between *ground-state* mesons. It is not clear whether our states, that seem to involve the binding of ground-state mesons with radially excited ones, will not be lost in their formalism. This point deserves further research.

Let us now discuss the relation between our work and the flip-flop string models.[6] The aim of these models is to suppress the van der Waals force, and this is done at the price of introducing multibody potentials. In contrast to the simplicity of the two-body potential model of this paper, there is a multiplicity of flip-flop models, depending on new parameters, all of them reducing to the same model in the $q\bar{q}$ and qqq sectors, but with wide differences for the other multiquark states. In some cases, these models present these *resonances close to threshold*, and the authors propose the molecular interpretation.[6] The resonances are present or absent according to the version of the model. Also depending on the type of model, other resonances are found, called *hidden color resonances* (color-confined states weakly coupled to the asymptotic states). Both types of resonances are strongly dependent on the parameters and the symmetry of the states. For comparison, let us emphasize that our present work shows the presence of resonances close to threshold in the two-body potential model with color SU(3), in both the symmetric and the antisymmetric channels. On the contrary, we do not find these so-called hidden color resonances.

Masutani[12] has made a comparison between the flip-flop models and the two-body potential model. For the two-body potential model, he finds a result analogous to our results in the $N=0$ approximation, with a threshold effect, but no resonances. In our higher approximations ($N=1,2$), we find on the contrary threshold resonances. His approximation goes somewhat beyond our approximation for $N=0$, since it couples $[(q\bar{q})_8(q\bar{q})_8]_1$. This could reveal a hidden color resonance, but it is not appropriate to show up meson-meson molecules. Our method does not introduce explicitly the coupling $[(q\bar{q})_8(q\bar{q})_8]_1$ but couples $(q\bar{q})_1(q\bar{q})_1$ excited states.

A phenomenological discussion is out of the scope of the present paper. In particular, we would need to introduce spin to make a detailed comparison with experimental data. Let us simply quote two nice candidates for our phenomenon quoted in the Particle Data Group Tables.

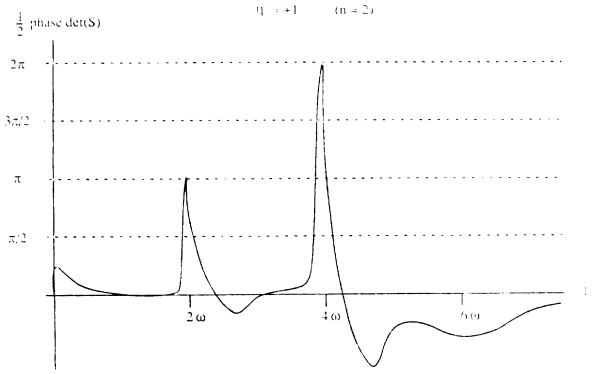

FIG. 6. Total phase shift in the symmetric case, in the $N=2$ approximation, showing two resonances close to 3.9ω.

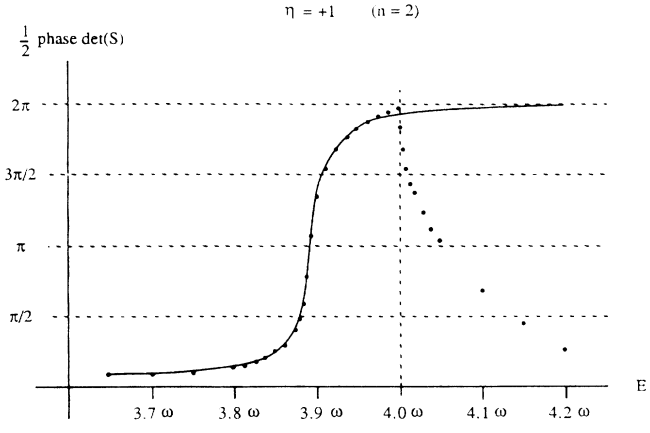

FIG. 8. Total phase shift in the symmetric case, in the second peak region. The solid line corresponds to the parametrization (9.15) and the dotted line to our calculation. We see the strong falloff after threshold.

One is the exotic narrow resonance of 3100 MeV seen in the modes $\Lambda\bar{p}\pi^+\pi^+$, $\Lambda\bar{p}\pi^+\pi^+\pi^-$, $\Lambda\bar{p}\pi^+\pi^-$ indicating a $I=\frac{3}{2}$ meson. The other one is nonexotic, X or $\xi(2220)$, with $I=0$ and $J^{PC}=\text{even}^{++}$. This state has been seen in $K\bar{K}$ and in $\eta\eta'$, suggesting strongly a $s\bar{s}q\bar{q}$ state. Moreover, this resonance is rather narrow, $\Gamma\sim 20$ MeV; being below the $\Lambda\bar{\Lambda}$ threshold, it cannot be enlarged by quark pair creation and decay into a baryon-antibaryon. However, its width seems small owing to the possible decay of the two weakly bound mesons in our interpretation.

ACKNOWLEDGMENTS

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APPENDIX A

Let us now consider the harmonic-oscillator model with equal masses, (2.1) and (8.1). The Hamiltonian depends then only on $\mathbf{r}_M^2$, $\mathbf{p}_M^2$, $\mathbf{r}_{M'}^2$, $\mathbf{p}_{M'}^2$, $\mathbf{r}_D^2$, $\mathbf{p}_D^2$ [$\mathbf{r}_M$, $\mathbf{r}_{M'}$, and $\mathbf{r}_D$ are the intercluster coordinates (7.3)] and we have the separate conservation of the associated orbital angular momenta $\mathbf{L}_M$, $\mathbf{L}_{M'}$, $\mathbf{L}_D$. Because of these conserving laws, there are interesting selection rules in this model. In particular, we did find,[3] using a variational calculation, bound states for $L_D>2$ in the meson-meson channel. Let us recall a simpler argument that establishes such bound states for $L_D>3$.

We project the Hamiltonian (2.1) and (8.1) on the $C_1=(q\bar{q})_1(q'\bar{q}')_1$ state:

$$H(C_1)=\frac{\mathbf{p}_M^2}{2m}+\left[\frac{\mathbf{p}_{M'}^2}{2m}+\frac{m\omega^2}{2}\mathbf{r}_{M'}^2\right]+\left[\frac{\mathbf{p}_D^2}{2m}+\frac{m\omega^2}{2}\mathbf{r}_D^2\right]. \tag{A1}$$

This will give us the meson-meson spectrum

$$E(\text{meson-meson})=3\omega+L_D\omega+L_{M'}\omega\ . \tag{A2}$$

We now project the Hamiltonian (2.1)–(8.1) on the state $C_3=\{(qq')\{\bar{3}\}(\bar{q}\,\bar{q}')\{3\}\}_1$ that corresponds to the color coupling corresponding to the "true" baryonium:

$$C_3=\left[\frac{N-1}{2N}\right]^{1/2}C_1-\left[\frac{N+1}{2N}\right]^{1/2}C_8\ . \tag{A3}$$

We find

$$H(C_3)=\left[\frac{\mathbf{p}_M^2}{2m}+\frac{N}{2(N-1)}\frac{m\omega^2}{2}\mathbf{r}_M^2\right]+\left[\frac{\mathbf{p}_{M'}^2}{2m}+\frac{N}{2(N-1)}\frac{m\omega^2}{2}\mathbf{r}_{M'}^2\right]+\left[\frac{\mathbf{p}_D^2}{2m}+\frac{N-2}{N-1}\frac{m\omega^2}{2}\mathbf{r}_D^2\right] \tag{A4}$$

that gives the baryonium spectrum

$$E(\text{baryonium}\{\bar{3}3\}_1)=\frac{3}{2}\left[\frac{1}{\sqrt{2}}+\sqrt{3}\right]\omega+L_D\frac{1}{\sqrt{2}}\omega+(L_M+L_{M'})\frac{\sqrt{3}}{2}\omega\ . \tag{A5}$$

We will have a baryonium bound state, if

$$E(\text{baryonium}\{\bar{3}3\}_1)<E(\text{meson-meson}) \tag{A6}$$

given, respectively, by (A5) and (A2). From (A5) we see that this happens indeed if $L_D>3$. This is the phenomenon that we found in Ref. 4: *there are baryonium bound states in the meson-meson continuum if the angular momentum is high enough.* This selection rule is specific to the harmonic-oscillator model, due to the separate conservation in this model of the orbital angular momenta $\mathbf{L}_M$, $\mathbf{L}_{M'}$, $\mathbf{L}_D$. These states could appear as resonances in a more realistic potential model. This phenomenon seemed to us at the moment related to high angular momenta. *The present study shows however that narrow resonances appear even in the S wave.*

APPENDIX B

Proposition. One considers arbitrary representations of a Lie group and their infinitesimal generators $T_1^a,\ldots,T_n^a$. We note by $\langle\ \rangle_0$ the mean value taken on any singlet state contained in their tensor product. Then, we have, for $0<\alpha\le 2$,

$$\sum_{i,j}\left\langle\sum_a T_i^a T_j^a\right\rangle_0|\mathbf{r}_i-\mathbf{r}_j|^\alpha\le 0\ , \tag{B1}$$

where $\mathbf{r}_1,\ldots,\mathbf{r}_n$ are arbitrary points of $\mathbb{R}^d$.

Proof. Let us consider $V(\mathbf{r})=|\mathbf{r}|^\alpha$ and $\tilde{V}(\mathbf{k})=\int d^d\mathbf{r}\,V(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}$. By a Fourier transformation we have

$$\sum_{i,j}\left[\sum_a T_i^a T_j^a\right]V(\mathbf{r}_i-\mathbf{r}_j)=\int\frac{d^d\mathbf{k}}{(2\pi)^d}\sum_a\left[\sum_i T_i^a e^{i\mathbf{k}\cdot\mathbf{r}_i}\right]\left[\sum_j T_j^a e^{-i\mathbf{k}\cdot\mathbf{r}_j}\right]\tilde{V}(\mathbf{k})\ . \tag{B2}$$

$\tilde{V}(\mathbf{k})$ is given by

$$\tilde{V}(\mathbf{k})=c_\alpha|\mathbf{k}|^{-d-\alpha},\quad c_\alpha=\pi^{d/2}2^{d+\alpha}\frac{\Gamma\left[\frac{d+\alpha}{2}\right]}{\Gamma(-\alpha/2)}\ . \tag{B3}$$

The function $\tilde{V}(\mathbf{k})$ is *negative*, since $c_\alpha<0$ for $0<\alpha<2$. Since the two matrices in the large parentheses in (B2) are the Hermitian conjugate of each other, the mean value of their product is positive. The inequality (B1) results then from the integral expression (B2).

Actually, this reasoning is not sufficient. It could apply to any mean value, on a singlet or a nonsinglet, and the conclusion would be then clearly wrong, since $V(\mathbf{r})\ge 0$.

One must take into account that $V(\mathbf{r})$ is not integrable, so that $\tilde{V}(\mathbf{k})$ is a distribution, defined by the Fourier transform of temperated distributions. The behavior of

$V(\mathbf{r})$ for $r \to \infty$ reflects in the singularity of $\tilde{V}(\mathbf{k})$ for $\mathbf{k}=0$, and the integral (B2) diverges for $\mathbf{k}=0$. The integral (B2) must be considered as a formal expression of the action of the distribution $\tilde{V}(\mathbf{k})$. This distribution can be defined cutting a domain around $\mathbf{k}=0$, adding a counterterm, and taking the limit of the domain to zero.[13] In general, the counterterm will change the sign of the formal integral.

To make the demonstration complete, it will be enough to see that, for the mean value on a singlet state, the integral (B2) converges in fact at $\mathbf{k}=0$ (the counterterm will vanish). For $\mathbf{k}=0$, the operator $\sum_i T_i^a e^{i\mathbf{k}\cdot\mathbf{r}_i}$ reduces to the generator $\sum_i T_i^a$ of the tensorial product representation, that acting on a singlet gives zero. It follows that $(\sum_i T_i^a e^{i\mathbf{k}\cdot\mathbf{r}_i})(\sum_j T_j^a e^{-i\mathbf{k}\cdot\mathbf{r}_j})$ vanishes as $|\mathbf{k}|^2$, and the singularity in the integrant is integrable, $|\mathbf{k}|^{-d-\alpha+2}$ $(0<\alpha<2)$.

This ends the demonstration for $0<\alpha<2$. For $\alpha=2$, we have

$$\sum_{i,j}\left\langle \sum_a T_i^a T_j^a \right\rangle_0 |\mathbf{r}_i-\mathbf{r}_j|^2 = -2\sum_a \left\langle \left| \sum_i T_i^a \mathbf{r}_i \right|^2 \right\rangle_0 \le 0 \ . \tag{B4}$$

Note. The argument holds for U(1), i.e., for ordinary charges

$$\sum_i e_i = 0 \Longrightarrow \sum_{i,j} e_i e_j |\mathbf{r}_i-\mathbf{r}_j|^\alpha \le 0 \quad (\text{for } 0<\alpha\le 2) \ . \tag{B5}$$

For $\alpha>2$, the proposition does not hold, as proved by Greenberg and Lipkin,[14] that have given a counter example.

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