Quantum decoherence and classical correlation in quantum mechanics

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We study two conditions for the quantum system to behave classically: decoherence in the quantum interference and the establishment of the classical trajectory in phase space. We show, despite the fact that these two conditions partially conflict with each other, the upside-down harmonic oscillator with a diffusion term satisfies them simultaneously. The implications for quantum cosmology and the measurement theory of quantum mechanics are given.

I. INTRODUCTION

Historically, quantum mechanics was derived from classical mechanics through the process of quantization. However, once the theory of quantum mechanics has been established in a general framework, classical mechanics should be retrieved in quantum mechanics. Since the classical objects appear automatically in the evolution of the Universe, the retrieval should be an autonomous physical process and not merely an approximation by hand.

This quantum-to-classical transition is particularly important in quantum cosmology. We have to derive the classical space-time and the classical density fluctuations from quantum mechanics, say the wave function of the Universe, in the course of cosmic evolution. Though this problem has been analyzed in many papers in the past, their proposals and conclusions are diverse and sometimes conflict with each other.

The problem is not restricted to quantum cosmology. In the measurement theory of quantum mechanics, there has been confusion for many years. The aim of the theory is to describe the measurement process as a dynamical phenomenon. In this case the information gained by the interaction with an entirely quantum system is transformed by a detector into classical information which can be definitely stored.

The confusion seems to originate from the lack of a comprehensive and quantitative definition of classicality. We clarify this in the present paper. In general, there are two indispensable characteristics for the system to behave classically:

Quantum decoherence (QD). Vanishing of the quantum interference. Because of the quantum interference, quantum states are not mutually exclusive, contrary to classical states.

Classical correlation (CC). Firm correlation in phase space. This condition is needed because the classical dynamics is described by a well-defined trajectory in phase space.

Operationally, the conditions QD and CC are defined as follows.

QD. Interference effects between two wave packets are easily measured by the present experimental techniques. An example is the incomplete measurement experiment.[1]

CC. It will be sufficient to check the existence of the position-measurement precision Δx such that a sequence of the position measurement in the precision Δx: $x(t_1), x(t_2), x(t_3), \ldots$ and the derived momentum from that: $p(t_1) \equiv m[x(t_2)-x(t_1)]/(t_2-t_1), p(t_2), \ldots$ determines the well-defined trajectory in phase space. An example is an alpha-particle trajectory in a bubble chamber.

The need of both conditions is clearly stated in Halliwell's paper.[2] Though QD (Refs. 2, 4, 7–9) and CC (Refs. 3, 5, and 6) have independently been studied so far, they have never been considered simultaneously. However, both of them are essential for the system to behave classically.

Only QD is not sufficient for the system to become classical. The degree of decoherence is ordinarily measured by the damping of the off-diagonal element of the density matrix of the system, which guarantees the reduction of the quantum interference. The extreme limit of this will be $\rho(x,x',t)=f(x_c)\delta(x_\Delta)$, where $2x_c \equiv x+x'$, $x_\Delta \equiv x-x'$, with f as some function. However, the corresponding Wigner function $W(x_c,p,t)$ (Fourier transformation of the density matrix with respect to the variable x_Δ into p), which represents the distribution in the phase space, becomes completely independent of the momentum: $W(x_c,p,t)=f(x_c)$. This distribution has nothing to do with the ordinary classical system.

On the other hand, only CC is not sufficient for the system to become classical. Though the Wigner function makes a sharp correlation in the phase space in the extreme limit of CC, the quantum interference never vanishes for unitary evolution.

We study, in this paper, the quantitative definition of QD and CC and their simultaneous realization in the Gaussian and non-Gaussian states. Implications of the argument for quantum cosmology and the measurement theory of quantum mechanics are given.

II. CLASSICALITY CONDITION FOR A GAUSSIAN STATE

Let us consider the general Gaussian state in order to obtain concrete conditions for QD and CC:

$$\rho(x,x',t)=\exp(-\alpha^2 x_c^2+i\beta x_c x_\Delta -\gamma^2 x_\Delta^2+\mu x_c+i\nu x_\Delta+\lambda)\ , \quad (1)$$

where the time-dependent coefficients $\alpha,\beta,\ldots$ are all real (for the Hermiticity) and $\alpha,\gamma>0$ (for the normalizability).

As a measure of the degree of quantum decoherence, we take the ratio of the dispersions $(\sqrt{2}\gamma)^{-1}$ in x_Δ and $(\sqrt{2}\alpha)^{-1}$ in x_c:

$$\delta_{\rm QD}\equiv\alpha/\gamma\ . \quad (2)$$

This is also the ratio of the quantum coherence length $l_{\rm QC}\equiv(\sqrt{2}\gamma)^{-1}$ and the typical system size Δx defined from the dispersion in x_c. Then, the QD condition becomes $\delta_{\rm QD}\ll 1$. In the literature, some authors neglect Δx in the QD measure. However, a dimensional measure is meaningless. The Wigner transform of the density matrix becomes

$$W(x_c,p,t)=\frac{\sqrt{\pi}}{\gamma}\exp\left[-\frac{(\beta x_c+\nu-p)^2}{4\gamma^2}-\alpha^2x_c^2+\mu x_c+\lambda\right]\ . \quad (3)$$

As a measure of the degree of classical correlation, we take the relative sharpness of the classical trajectory in the phase space determined from the dispersion $\sqrt{2}\gamma$ in p and the magnitude of the average of p ($p_0\equiv\beta x_c+\nu$): $\delta_{\rm CC}\equiv\sqrt{2}\gamma/|p_0|$. This becomes

$$\delta_{\rm CC}=2\alpha\gamma/|\beta|\ , \quad (4)$$

if we identify x_c as the dispersion in x_c. Then the CC condition becomes $\delta_{\rm CC}\ll 1$. Note that there are basically no independent dimensionless quantities other than the above. The extreme limit of QD ($\gamma\to\infty$) is incompatible with CC, and that of CC ($\gamma\to 0$) is incompatible with QD. Therefore, conditions QD and CC are mutually exclusive in part. We note that the CC condition reduces to those used in Ref. 3 for a pure state.

Though the above argument is based on the x representation, the same argument in the p representation leads to the same conditions for QD and CC as above. In fact, $\delta_{\rm QD}$ is shown to be the twice the linear entropy of the system ($\mathrm{Tr}\rho^2$): $\delta_{\rm QD}=2\,\mathrm{Tr}\rho^2$. Therefore, the measure $\delta_{\rm QD}$ and the QD condition are independent of a representation. This is also understood from the fact that $\delta_{\rm QD}$ is related with the area of the half-maximum value of the Wigner function in the phase space. Actually, in the coordinate x_c and $p'\equiv p-\beta x_c$, $(\sqrt{2}\alpha)^{-1}$ and $\sqrt{2}\gamma$ are the lengths of the longer and the shorter semimajor axes, respectively. The multiplication of them gives the area of the ellipsis.

On the other hand, the CC condition does depend on the coordinate system we choose. However, it is a good measure of the "squeezing" of the Wigner function in the phase space. Actually, in the coordinate βx_c and p', $\beta(\sqrt{2}\alpha)^{-1}$ and $\sqrt{2}\gamma$ are the lengths of the longer and the shorter semimajor axes. The ratio of them gives $\delta_{\rm CC}$. Note that we take the coordinate βx_c instead of x_c because it is meaningless to compare the quantities which have different dimensions.

III. EXAMPLE OF A GAUSSIAN STATE

Let us consider whether the conditions QD and CC are achieved or not in the course of evolution by using a general Gaussian state of Eq. (1) (with $\mu=\nu=0$). As an equation of motion for the density matrix, we take the most general phenomenological Liouville equation in quantum mechanics for a harmonic oscillator with a friction and a diffusion:

$$\frac{\partial\rho}{\partial t}=\left[\frac{i}{2m}\left[\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial x'^2}\right]-i\frac{m}{2}\omega^2(x^2-x'^2)-\epsilon x_\Delta\left[\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right]-\Lambda x_\Delta^2\right]\rho\ , \quad (5)$$

where m, ω, ϵ, and Λ are the mass of the system, the frequency, the friction strength, and the diffusion strength, respectively. In order to reduce the number of derivatives in the evolution equation, we study the Fourier transform of the density matrix with respect to the variable x_c:

$$P(x_\Delta,q,t)=\exp(-x_i A_{ij} x_j - D)\ , \quad (6)$$

where

$$A_{ij}\equiv\begin{bmatrix}A & B\\ B & C\end{bmatrix},\ \text{with}$$

$$A=\gamma^2+\frac{\beta^2}{4\alpha^2},\ B=\frac{\beta}{4\alpha^2},\ C=\frac{1}{4\alpha^2},\ D=-\lambda-\ln\frac{\sqrt{\pi}}{\alpha}\ . \quad (7)$$

The evolution equation for P becomes, from Eq. (5),

$$\frac{\partial A_{ij}}{\partial t}=\gamma_{ik}A_{kj}+\gamma_{jk}A_{ki}+\mu_{ij},\quad \frac{\partial D}{\partial t}=0\ , \quad (8)$$

where $(x_i)=(x_\Delta,q)$ and

$$\gamma_{ij}\equiv\begin{bmatrix}-2\epsilon & -m\omega^2\\ m^{-1} & 0\end{bmatrix},\quad \mu_{ij}\equiv\begin{bmatrix}\Lambda & 0\\ 0 & 0\end{bmatrix}\ . \quad (9)$$

The solution can be given explicitly, but is useless here. We just mention the general evolution equation for the classicality measures:

$$\begin{aligned}\frac{\partial\delta_{\rm QD}}{\partial t}&=-2\Lambda C\delta_{\rm QD}^3+2\epsilon\delta_{\rm QD}\ ,\\ \frac{\partial\delta_{\rm CC}}{\partial t}&=4\delta_{\rm CC}^3\delta_{\rm QD}^2 B(2BC\Lambda\delta_{\rm QD}^2+m\omega^2C-m^{-1}A)\ .\end{aligned} \quad (10)$$

We notice that the QD (reduction of $\delta_{\rm QD}$) is promoted by the diffusion, but is reduced by friction. On the other hand, CC (reduction of $\delta_{\rm CC}$) is reduced by the diffusion, in general for the Gaussian state.

We shall study individual cases with a pure state initial condition.

(i) The case $\Lambda=0$, $\epsilon=0$, $\omega^2=0$.

From Eq. (10), it is obvious that $\delta_{QD}=2$. For CC, we get

$$\delta_{CC}=\frac{m\alpha_0^2}{\alpha_0^4+\beta_0^2}t^{-1}, \quad (11)$$

for the large-t limit. The subscript 0 means the initial values for the pure state ($\alpha_0=2\gamma_0$). In this case, only the CC condition is satisfied. If $\omega^2<0$, then the reduction of δ_{CC} is exponential in the time scale $|\omega|^{-1}$. If $\omega^2>0$, then δ_{CC} remains finite and just oscillates. Therefore even CC cannot be expected.

(ii) The case $\Lambda>0$, $\epsilon=0$, $\omega^2=0$.

We get

$$\delta_{QD}=\frac{\sqrt{3}m}{\Lambda t^2}, \quad \delta_{CC}=1/\sqrt{3}. \quad (12)$$

Note that by the effect of diffusion, CC is destroyed but instead, QD is achieved.

(iii) The case $\Lambda>0$, $\epsilon>0$, $\omega^2>0$.

In this case, there is a stationary state, where

$$\delta_{QD}=\frac{2m\omega\epsilon}{\Lambda}, \quad \delta_{CC}=\infty. \quad (13)$$

The condition CC is apparently not achieved and QD depends on the values of the parameters.

(iv) The case $\Lambda>0$, $\epsilon=0$, $\omega^2(\equiv-\mu^2)<0$.

We get

$$\delta_{QD}=\text{const}\times\Lambda^{-1/2}\exp(-\lambda_+t),$$
$$\delta_{CC}=\text{const}\times\Lambda^{1/2}\exp(-\lambda_+t), \quad (14)$$

where $\lambda_+=-\epsilon+(\epsilon^2-\omega^2)^{1/2}$. In this case, both QD and CC are achieved. Another example where both QD and CC are achieved is a linear potential case with diffusion.

From the above examples, we learn that the simultaneous achievement of QD and CC is not a trivial problem since the two conditions are not independent conditions. In general for QD and CC, some kind of instability of the potential as well as the diffusion seem to be necessary.

IV. CLASSICALITY CONDITION FOR A NON-GAUSSIAN STATE

Our consideration was restricted, so far, within the Gaussian state. Let us consider the generalization to non-Gaussian states. First, we consider a superposition of two generalized coherent states ϕ_1 and ϕ_2:

$$\Psi(x)=\phi_1(x)+\phi_2(x),$$
$$\phi_1(x)=N\exp\left[-\frac{(x-x_1)^2}{4\sigma^2}+ip_0x\right], \quad (15)$$
$$\phi_2(x)=N\exp\left[-\frac{(x-x_2)^2}{4\sigma^2}+ip_0x\right].$$

Let us consider the interference term (ρ_{int}) in the density matrix $\rho(x,x')=\Psi^*(x')\Psi(x)$. For the element $x_\Delta=0$,

$$\rho_{\text{int}}(x,x)=2N^2\exp\left[-\frac{(x-x_1)^2+(x-x_2)^2}{4\sigma^2}\right]$$
$$\le 2N^2\exp\left[-\frac{(x_1-x_2)^2}{8\sigma^2}\right], \quad (16)$$

which is exponentially small if $|x_1-x_2|\gg\sigma$. Among the elements $x_\Delta\neq0$, large ones are such that $x\approx x_1$, $x'\approx x_2$ or $x\approx x_2$, $x'\approx x_1$. However, according to Eq. (5), they eventually decay in a time scale $\Lambda^{-1}(x_1-x_2)^{-2}$. Therefore in general, if we consider the larger separation of individual wave packets compared with their de Broglie wavelengths, then the quantum interference among them eventually decays and the problem reduces to QD and CC within the individual blobs.

Next, we consider how the previous criteria for QD and CC are generalized to those individual non-Gaussian states. The previous QD condition is directly applicable to a general state since it appeared in a coordinate-invariant way ($\delta_{QD}=2\,\text{Tr}\rho^2$). For a general non-Gaussian state, we pick up one virtual classical trajectory whose action is given by $S(x)$. We need the CC condition along this trajectory. We notice the previous correspondence between the mean value of the momentum p_0 and the action $p_0=\partial S/\partial x\equiv S'$. The parameter β in the linear Gaussian case corresponds to S''. If we replace α, β, and γ by S', S'', and $\text{Tr}\rho^2$ in the expression for δ_{CC}, we get

$$\delta_{CC}=\frac{1}{2\,\text{Tr}\rho^2}\frac{|S''|}{S'^2}. \quad (17)$$

Note that the second factor on the right-hand side in the above is the ratio of the two scales: scale of change of the de Broglie wavelength $\lambda\equiv\hbar/S'$ and the typical system size Δx:

$$\frac{|S''|}{S'^2}=\left|\frac{\partial}{\partial x}\left[\frac{\hbar}{S'}\right]\right|=\frac{\Delta\lambda}{\Delta x}. \quad (18)$$

In terms of $\Delta\lambda$, the quantum coherence length l_c and Δx, the conditions are rephrased as

$$l_c\ll\Delta x\Longrightarrow\text{QD}, \quad \Delta\lambda\ll l_c\Longrightarrow\text{CC}. \quad (19)$$

Note that if we neglect the QD condition, then this is equivalent to that for the WKB approximation to work.

V. IMPLICATIONS FOR QUANTUM COSMOLOGY AND THE MEASUREMENT THEORY OF QUANTUM MECHANICS

In the literature of quantum cosmology, QD (Ref. 2) and CC (Ref. 3) conditions have been considered separately, except a brief calculation in the last paper of Ref. 2. However, the individual QD or CC limits are far from classical in the sense we saw at the beginning of the paper. Though the individual conditions may be trivially realized, their simultaneous realization is nontrivial since QD and CC are partially conflicting conditions. Since the CC condition does depend on the variable we choose (cf. the paper by Sasaki of Ref. 3), we have to consider which variable is respected by the actual measurement process of the Universe. However, not the measuring apparatus but the variables of the Universe themselves should become classical for the stability of the Universe.

The measurement theory of quantum mechanics will be clarified from our point of view. The essential process of measurement seems to be at first, (a) the establishment of the firm correlation between the system and the detector through their interaction, then by (b) the classicalization of the detector variables hence storing precise information about the system.

The first process (a) is rather easy to describe. In the latter process (b), we expect QD and CC conditions are required for the detector. The essential part of the measurement ends when QD and CC are achieved. For example, let us consider the gedanken experiment of Schrödinger's cat in a box: the state of the cat (dead or alive) has firm correlation with the microscopic system such as the state of the alpha particle (not decayed or decayed) through a Geiger counter. Here the flow of information seems to achieve QD and CC far before it reaches the front end the cat, presumably at the level of the Geiger counter. The cat, which also should satisfy QD and CC conditions, is just a memory storage. The system inside the box constitutes a complete measurement and no observer outside is essential. We will give a full argument of the measurement theory based on the present study elsewhere. Here, we just mention how other measurement theories are related with our argument.

(i) Machida-Namiki.[4] They claim that the reduction of the wave function originates from the averaging over the unobservable microscopic degrees of freedom of a detector. This averaging process destroys the interference terms and leads to QD. Actually, their measure for the reduction of a wave function (the ratio of the averaging width and de Broglie wavelength of the system) is also a good measure for $\mathrm{Tr}\rho^2$. However, they disregard CC of a detector. Though a thermal irreversible amplification process may not be necessary, as they claim, some kind of instability will be needed for CC.

(ii) Fukuda;[5] Maki.[6] They claim the vanishing of the quantum fluctuations in the detector variables (class-I operator) for the limit of large N (number of degrees of freedom) or that of large volume. Since the classical trajectory of the detector is established in phase space, this leads to CC. Though they further claim that also the interference term vanishes in the same limit, it cannot be borne out. The interference term, in fact, becomes more oscillatory in this limit, but it never vanishes. This is related to the fact that they cannot derive a finite time scale for the vanishing of the interference term. In their framework, neither the nonzero entropy, mixed state property, or loss of information can be obtained.

(iii) Zurek.[7] If we replace the "unobservable microscopic degrees of freedom" in the specific case of Machida-Namiki by the "environment" in general, the argument is the same as that of them. However, the environment is not at all "universal," but is strongly dependent on systems.

(iv) Joos and Zeh;[8] Unruh and Zurek.[9] They also consider the "environment" and use the similar equation as Eq. (5). Though they study QD by using the ordinary entropy as a measure, they disregard CC.

(v) van Kampen.[10] He introduces the metastable detector and identifies the decay of it as the measurement process. This kind of instability may derive CC, but we cannot expect QD directly.

We think that perpetual fluctuations, which will be guaranteed by some kind of instability (v) of a detector in a realistic case, will realize at large but finite N. Then the relative fluctuation of certain operators reduces as $1/\sqrt{N}$ and CC will be achieved (ii). On the other hand, the absolute fluctuation of the remaining operators increases as $\sqrt{N}$ and QD will be achieved [(i), (iii)]. However, real dynamical fluctuations will give a time-dependent diffusion (iv).

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[1]H. Rauch, in Proceedings of the 3rd International Symposium on the Foundations of Quantum Mechanics, 1989 (unpublished), p. 3.

[2]C. Kiefer, Class. Quantum Grav. **4**, 1369 (1987); J. J. Halliwell, Phys. Rev. D **39**, 2912 (1989); T. Padmanabhan, *ibid.* **39**, 2924 (1989); T. Fukuyama and M. Morikawa, *ibid.* **39**, 462 (1989); M. Morikawa, *ibid.* **40**, 4023 (1989).

[3]D. H. Lyth, Phys. Rev. D **31**, 1792 (1985); A. H. Guth and S. Y. Pi, *ibid.* **32**, 1899 (1985); M. Sasaki, Prog. Theor. Phys. **76**, 1036 (1986); J. J. Halliwell, Phys. Rev. D **36**, 3626 (1987); Phys. Lett. B **196**, 444 (1987).

[4]S. Machida and M. Namiki, Prog. Theor. Phys. **63**, 1457 (1980); **63**, 1833 (1980).

[5]R. Fukuda, Phys. Rev. A **35**, 8 (1987); **36**, 3023 (1987); Prog. Theor. Phys. **81**, 34 (1989).

[6]Z. Maki, Prog. Theor. Phys. **79**, 313 (1988); **82**, 638 (1989).

[7]W. H. Zurek, Phys. Rev. D **24**, 1516 (1981); **26**, 1862 (1982).

[8]E. Joos and H. D. Zeh, Z. Phys. B **59**, 223 (1985).

[9]W. H. Zurek, Institute for Theoretical Physics Report No. NSF-ITP-88-197, 1988 (unpublished); W. G. Unruh and W. H. Zurek, Phys. Rev. D **40**, 1071 (1989).

[10]N. G. van Kampen, in Proceedings of the 3rd International Symposium on the Foundations of Quantum Mechanics (Ref. 1), p. 107.