On-shell expansion of the effective action. II. Coherent state and S matrix

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(Received 19 June 1990)

The meaning of the on-shell variation of the effective action is clarified. The higher orders in the on-shell expansion can be summed up into the coherent state. From these observations, the effective action is shown to have really the property of a generating functional of the S-matrix elements.

I. INTRODUCTION

The effective action is a convenient tool and is widely used in various fields, especially in particle physics, for the study of the symmetry-breaking solution. It has two important properties: (i) Its stationary solution determines the vacuum; (ii) it is the generating functional of the whole set of *off-shell* one-particle-irreducible (1PI) Green's functions.

Recently the on-shell properties of the effective action have been investigated[1-3] in the form of the on-shell expansion. This extracts all the observable information out of the effective action.

In order to make the subsequent discussions clear, let us recall the arguments raised in Ref. 1. Consider the classical mechanics in the time interval $t_i \le t \le t_f$. Suppose the system has the Lagrangian $L(\dot{q}_i, q_i)$ and the action functional $I[q]=\int_{t_i}^{t_f} dt\, L(\dot{q}_i,q_i)$ where q_i $(i=1,\ldots,N)$ signifies the particle coordinate and $\dot{q}_i \equiv dq_i/dt$. The Euler-Lagrange equation of motion is obtained by taking the functional derivative of $I[q]$:

$$\frac{\delta I[q]}{\delta q_i(t)}=0\ , \tag{1.1}$$

with the constraint $\delta q(t_f)=\delta q(t_i)=0$. Let one of the solutions to (1.1) be $q_i^{(0)}(t)$ with an appropriate boundary conditions at $t=t_f$ and t_i. We want to study the other solution $q_i(t)$ to (1.1) which is close to $q_i^{(0)}(t)$ so that we write

$$q_i(t)=q_i^{(0)}(t)+\Delta q_i(t) \tag{1.2}$$

and obtain the equation of motion satisfied by $\Delta q_i(t)$ which is assumed to be small:

$$\sum_j \int dt' I_{ij}^{(2)}(t,t')\Delta q_j(t')=0\ , \tag{1.3}$$

where $I_{ij}^{(2)}(t,t') \equiv [\delta I[q]/\delta q_i(t)\delta q_j(t')]_0$. The symbol $[\cdots]_0$ implies the value at $q_i(t)=q_i^{(0)}(t)$. The nontrivial solution to (1.3) exists only when

$$\mathrm{Det} I_{ij}^{(2)}(t,t')=0\ , \tag{1.4}$$

which determines the eigenmode of the oscillation around $q_i^{(0)}(t)$. Here the functional determinant is taken in the space specified by (i,t). However if we fix the boundary conditions by specifying $q_i(t_f)$ and $q_i(t_i)$ a unique solution is singled out so that the existence of a nonzero $\Delta q_i(t)$ implies that $q_i(t)$ and $q_i^{(0)}(t)$ obey the different boundary conditions as follows:

the variation Δq in (1.2)
corresponds to the change in the
boundary conditions . (1.5)

The coefficient matrix $I_{ij}^{(2)}(t,t')$ determines the stability of the solution $q_i^{(0)}(t)$.[1]

Now for the quantum system, the role of the classical action is played by the effective action. In Ref. 1 the equation corresponding to (1.3) has been studied and it has turned out that Eq. (1.3) is the general equation to determine the spectrum of the system. The higher orders in Δq_j have been investigated in Ref. 2 in a form which we call the on-shell expansion of the effective action. We have shown there that the expansion coefficients are related to the connected S-matrix elements of the scattering among the particle spectra determined in the lowest order. The purpose of this paper is to study the expansion scheme more closely by paying particular attention to the change of the boundary conditions which define the effective action. The precise meaning of Δq_j is clarified and we thereby obtain a novel compact formula for the effective action in terms of the *coherent states*. They are summarized in (2.29) and (2.34) below.

Our formalism is expected to have ample application to various fields of physics since it can be applied to any quantum system once the Hamiltonian of the system is given. The scope of this paper is therefore to present the formulation in general terms. Our formalism starts from the effective action which is the generating functional of all the necessary information of the theory. It is *complete* in the sense that it determines the following three objects successively: (1) the ground state; (2) the excitation spectrum above the ground state; (3) the scattering among the excitation spectra. These exhaust the whole observables contained in the theory. The nonperturbative ground state, if it exists, can automatically be picked up so that the spectra and the S-matrix elements *based on the nonperturbative condensed vacuum* can be calculated systematically. This is impossible by graphical considerations only. Note that through (1) and (2) above we know how the Fock space is constructed by the effective action.

In Sec. II, taking the simplest field-theoretical model, it is shown that the series of the on-shell expansion can be summed up into the form of the coherent state. Here the special role played by the change of the boundary states is essential. Another proof of our formula is given in Sec. III starting from the observation that the change of the boundary states is equivalent to the addition of the instantaneous source term at the initial or the final time. The generalization to other cases including the bound state is shown to be straightforward in Sec. IV. As the simplest example of the bound state, we discuss the Gross-Neveu model[4] in Sec. V, which is the N-component fermionic theory in two dimensions.

II. ON-SHELL EXPANSION AND THE COHERENT STATE

Let us concentrate on a single-component scalar field model described by the Lagrangian density $\mathcal{L}(\Phi)$. We first introduce the generating functional $W[J]$ of the connected Green's function by the functional integral as

$$\exp(iW[J]) = \int [d\Phi]\exp\left[i\int_{-\infty}^{\infty} d^4x\,[\mathcal{L}(\Phi)+J(x)\Phi(x)]\right], \tag{2.1}$$

where $J(x)$ denotes the external source used as a probe. The definition of the effective action $\Gamma[\phi]$ is then given by the Legendre transformation

$$\Gamma[\phi]\equiv W[J]-\int d^4x\,J(x)\phi(x), \tag{2.2}$$

$$\phi(x)\equiv\frac{\delta W[J]}{\delta J(x)}. \tag{2.3}$$

It is well known that the solution $\phi^{(0)}(x)$ of the stationary condition

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)}=-J(x)=0 \tag{2.4}$$

determines the vacuum expectation value of the field. We notice, however, this statement is based on the fact that certain boundary conditions have been assumed on the functional integral in (2.1). To clarify this point, let us consider the following generating functional $W_{\rm FI}[J]$ defined in the finite time interval between t_i and t_f:

$$\exp(iW_{\rm FI}[J])\equiv\left\langle F\left|T\exp\left[i\int_{t_i}^{t_f} d^4x\,J(x)\hat{\Phi}(x)\right]\right|I\right\rangle, \tag{2.5}$$

where $|I\rangle$ or $|F\rangle$ denotes the initial or final state, respectively, and $\hat{\Phi}(x)$ indicates the corresponding field operator of $\Phi(x)$. To get the generating functional introduced in (2.1) we must take the limit $t_i\to-\infty$ and $t_f\to+\infty$ in (2.5). This is achieved by the Feynman prescription;[5,6] the infinitesimal negative imaginary part is added to the mass squared, $m^2\to m^2-i\epsilon$, which automatically picks up the contributions of the vacuum state $|0\rangle$. By this method Eq. (2.5) becomes, apart from the irrelevant constant,

$$\exp(iW_{00}[J])\equiv\left\langle 0\left|T\exp\left[i\int_{-\infty}^{\infty} d^4x\,J(x)\hat{\Phi}(x)\right]\right|0\right\rangle. \tag{2.6}$$

Equation (2.6) defines the generating functional of the connected Green's function and, instead of writing it as $W[J]$, we have employed the notation $W_{00}[J]$ to clarify its boundary contributions. From this expression, $\phi^{(0)}(x)$ is obtained by relations (2.3) and (2.4) as $\phi^{(0)}(x)=\langle 0|\hat{\Phi}(x)|0\rangle$.

Based on the ground state characterized by $\phi(x)=\phi^{(0)}(x)$, we have studied in Ref. 1 another solution of (2.4) in the form of $\phi(x)=\phi^{(0)}(x)+\Delta\phi(x)$ and have discussed how we obtain the information concerning the excited states or the modes. In the following, it will be shown that the variation $\Delta\phi(x)$ from the vacuum solution can be reduced to the changes of the initial and final states on the right-hand side of (2.6) and, by this observation, we will make clear the relation between the effective action and the connected S-matrix element.

A. Basic formulation

Before starting our arguments, we want to summarize some results of our earlier paper.[2] Here we use $\Gamma_{00}[\phi]$ defined from $W_{00}[J]$ and assume that $\phi^{(0)}(x)$ is the solution of the stationary condition of $\Gamma_{00}[\phi]$; $\delta\Gamma_{00}[\phi]/\delta\phi=-J=0$. Looking for another solution on the trajectory defined by $J=0$, we set $\phi(x)=\phi^{(0)}(x)+\Delta\phi(x)$ and write $\Delta\phi(x)$ as

$$\Delta\phi(x)\equiv\Delta\phi^{(1)}(x)+\Delta\phi^{(2)}(x)+\Delta\phi^{(3)}(x)+\cdots, \tag{2.7}$$

by assuming $\Delta\phi^{(1)}(x)$ is small and $\Delta\phi^{(n)}(x)(n\geq 2)$ is of the order $[\Delta\phi^{(1)}(x)]^n$. Each $\Delta\phi^{(n)}$ is then determined successively by the equations

$$(\Gamma_{00}^{(2)})_{x,y}\Delta\phi^{(1)}(y)=0, \tag{2.8}$$

$$(\Gamma_{00}^{(2)})_{x,y}\Delta\phi^{(2)}(y)+\frac{1}{2!}(\Gamma_{00}^{(3)})_{x,y,z}\Delta\phi^{(1)}(y)\Delta\phi^{(1)}(z)=0, \tag{2.9}$$

$$(\Gamma_{00}^{(2)})_{x,y}\Delta\phi^{(3)}(y)+2\times\frac{1}{2!}(\Gamma_{00}^{(3)})_{x,y,z}\Delta\phi^{(1)}(y)\Delta\phi^{(2)}(z)+\frac{1}{3!}(\Gamma_{00}^{(4)})_{x,y,z,w}\Delta\phi^{(1)}(y)\Delta\phi^{(1)}(z)\Delta\phi^{(1)}(w)=0, \tag{2.10}$$

etc., where we have employed the notation

$$(\Gamma_{00}^{(n)})_{x,y,\ldots,z} \equiv \left[\frac{\delta^n \Gamma_{00}[\phi]}{\delta\phi(x)\delta\phi(y)\cdots\delta\phi(z)} \right]_{\phi=\phi^{(0)}} , \quad (2.11)$$

and four-dimensional integration over the repeated variables is implied. (We will use the similar notation $W_{00}^{(n)}$ for W_{00}, which is evaluated at $J=0$.)

We have called (2.8) the generalized on-shell condition[1,2] since, for the translationally invariant case, $\Delta\phi^{(1)}$ has its support only at the pole of the Green's function as is seen from the relation

$$(\Gamma_{00}^{(2)})_{x,y}(W_{00}^{(2)})_{y,z} = (W_{00}^{(2)})_{x,y}(\Gamma_{00}^{(2)})_{y,z} = -\delta^4(x-z) . \quad (2.12)$$

So we can write down $\Delta\phi^{(1)}$ in Fourier space with the arbitrary functions C^+ and C^- as

$$\Delta\phi^{(1)}(p) \equiv [C^+(\mathbf{p})\theta(p^0) + C^-(-\mathbf{p})\theta(-p^0)]\delta(p^2-m^2) . \quad (2.13)$$

In x-space, Eq. (2.13) takes the form

$$\Delta\phi^{(1)}(x) \equiv \int \frac{d^4k}{(2\pi)^4} \Delta\phi^{(1)}(k) e^{-ik\cdot x} = \int \frac{d^3k}{(2\pi)^4} \frac{1}{2k^0} [C^+(\mathbf{k})e^{-ik\cdot x} + C^-(\mathbf{k})e^{ik\cdot x}] \equiv \int d^3k [\tilde{C}^+(\mathbf{k})e^{-ik\cdot x} + \tilde{C}^-(\mathbf{k})e^{ik\cdot x}] , \quad (2.14)$$

with $k^0 = (\mathbf{k}^2+m^2)^{1/2}$. In these equations, $C^\pm$ are to be determined by the boundary condition on $\Delta\phi^{(1)}(x)$.

For the higher orders, $\Delta\phi^{(n)}(n \geq 2)$ can be summarized into the simple form

$$\Delta\phi^{(n)}(x) = \frac{1}{n!}[(W_{00}^{(n+1)})_{x,x_1,\ldots,x_n}(W_{00}^{(2)-1})_{x_1,x_1'} \cdots (W_{00}^{(2)-1})_{x_n,x_n'}]\Delta\phi^{(1)}(x_1') \cdots \Delta\phi^{(1)}(x_n') , \quad (2.15)$$

where the channel specified by x is off the mass shell, while the remaining n channels are projected onto the mass shell by $\Delta\phi^{(1)}$'s. In this equation the order of the integration is crucial but it is given automatically—since $\Delta\phi^{(n)}$'s are originally written by $\Gamma_{00}^{(n)}$'s, we first amputate the external legs of $W_{00}^{(n+1)}$ and then take the on-shell projections.

Note that, in (2.13) and (2.14), the existence of the nonvanishing $\Delta\phi^{(1)}$ ($C^+\neq 0$ and/or $C^-\neq 0$) is assumed. Strictly speaking, however, the above obtained solution $\phi^{(0)}(x)+\Delta\phi(x)$ which is different from $\phi^{(0)}(x)$ does not satisfy $\Gamma_{00}^{(1)}[\phi]=0$, because $C^\pm\neq 0$ does not satisfy the boundary conditions corresponding to the vacuum states at $t=\pm\infty$. Note that both $\phi^{(0)}(x)$ and $\phi^{(0)}(x)+\Delta\phi(x)$ satisfy the condition $J=0$ and therefore the only freedom left to be varied is the change in the boundary states. This is analogous to the classical mechanics as stated in (1.5). The "boundary conditions" in (1.5) are now replaced by the "boundary states." That this is indeed the case can be shown explicitly by using the reduction formula[7] and we can construct the Fock space by this method. The following discussions are devoted to these points.

B. The coherent state and connected S-matrix elements

Now we want to clarify the physical meaning of $\Delta\phi(x)$. For this purpose we use the asymptotic field

$$\hat{\Phi}_{\text{in(out)}}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} [\hat{a}_{\text{in(out)}}(\mathbf{k})e^{-ik\cdot x} + \hat{a}^\dagger_{\text{in(out)}}(\mathbf{k})e^{ik\cdot x}] , \quad (2.16)$$

where $k^0=(\mathbf{k}^2+m^2)^{1/2}$. This is defined under the condition of the weak limit:

$$\hat{\Phi}(x) \to Z^{1/2}\hat{\Phi}_{\text{in(out)}}(x) + \phi^{(0)}(x) \quad [t\to -\infty(+\infty)] , \quad (2.17)$$

with Z being the normalization factor. We utilize the Lehmann-Symanzik-Zimmermann original proof of their well-known reduction formula.[7] The following steps just correspond to the inverse process of their approach.

For instance, let us first examine $\Delta\phi^{(2)}(x)$. We assume here, for simplicity, the vanishing vacuum expectation value of the field ($\phi^{(0)}(x)=0$). However we notice that the arguments can directly be applied to the case for $\phi^{(0)}\neq 0$ by employing the shifted operator $\hat{\Phi}(x)-\phi^{(0)}(x)$ instead of $\hat{\Phi}(x)$ in (2.6).

By using (2.15) with the relations

$$(W_{00}^{(n)})_{x_1,\ldots,x_n} = i^{n-1}\langle 0|T\hat{\Phi}(x_1)\cdots\hat{\Phi}(x_n)|0\rangle_c , \quad (2.18)$$

$$(W_{00}^{(2)-1})_{x,y} = Z^{-1}(\Box_x + m^2)\delta^4(x-y) \quad \text{(on the mass shell)} , \quad (2.19)$$

where the subscript c implies the connected part and Z is the residue of the mass-shell pole, we can write $\Delta\phi^{(2)}(x)$ in the form

$$\Delta\phi^{(2)}(x)=\frac{1}{2!}\int d^4x_1 d^4x_2 iZ^{-1}\Delta\phi^{(1)}(x_1)(\overleftarrow{\Box}_{x_1}+m^2)iZ^{-1}\Delta\phi^{(1)}(x_2)(\overleftarrow{\Box}_{x_2}+m^2)\langle 0|T\hat{\Phi}(x)\hat{\Phi}(x_1)\hat{\Phi}(x_2)|0\rangle_c \ . \quad (2.20)$$

In order to see that $\Delta\phi$ indeed corresponds to the variation of the boundary states, Eq. (2.20) is transformed in the following way. First we carry out the integration over x_1 as follows:

$$\begin{aligned}
iZ^{-1}&\int d^4x_1\Delta\phi^{(1)}(x_1)(\overleftarrow{\Box}_{x_1}+m^2)\langle 0|T\hat{\Phi}(x)\hat{\Phi}(x_1)\hat{\Phi}(x_2)|0\rangle_c\\
&=iZ^{-1}\sum_{\delta=\pm}\int d^3k\int d^4x_1\tilde{C}^{\delta}(\mathbf{k})\partial_{x_1^0}[e^{-i\delta k\cdot x_1}\overleftrightarrow{\partial}_{x_1^0}\langle 0|T\hat{\Phi}(x)\hat{\Phi}(x_1)\hat{\Phi}(x_2)|0\rangle_c]\\
&=iZ^{-1}\sum_{\delta=\pm}\int d^3k\int d^3x_1\{\lim_{x_1^0\to\infty}\tilde{C}^{\delta}(\mathbf{k})e^{-i\delta k\cdot x_1}\overleftrightarrow{\partial}_{x_1^0}\langle 0|\hat{\Phi}(x_1)T[\hat{\Phi}(x)\hat{\Phi}(x_2)]|0\rangle_c\\
&\qquad-\lim_{x_1^0\to-\infty}\tilde{C}^{\delta}(\mathbf{k})e^{-i\delta k\cdot x_1}\overleftrightarrow{\partial}_{x_1^0}\langle 0|T[\hat{\Phi}(x)\hat{\Phi}(x_2)]\hat{\Phi}(x_1)|0\rangle_c\}\\
&=Z^{-1/2}\int d^3k\{\tilde{C}^{-}(\mathbf{k})\langle 0|\hat{a}_{\text{out}}(\mathbf{k})T[\hat{\Phi}(x)\hat{\Phi}(x_2)]|0\rangle+\tilde{C}^{+}(\mathbf{k})\langle 0|T[\hat{\Phi}(x)\hat{\Phi}(x_2)]\hat{a}^{\dagger}_{\text{in}}(\mathbf{k})|0\rangle\}_c \ ,
\end{aligned} \quad (2.21)$$

where we have used (2.14), (2.16), (2.17), and the notation $f\overleftrightarrow{\partial}g\equiv f(\partial g)-(\partial f)g$. The second integration for x_2 can be done in the same way and we get, as the result,

$$\begin{aligned}
\Delta\phi^{(2)}(x)&=\frac{1}{2!}(Z^{-1/2})^2\int d^3k\, d^3l[\tilde{C}^{-}(\mathbf{k})\tilde{C}^{-}(l)\langle 0|\hat{a}_{\text{out}}(\mathbf{k})\hat{a}_{\text{out}}(l)\hat{\Phi}(x)|0\rangle+2\tilde{C}^{-}(\mathbf{k})\tilde{C}^{+}(l)\langle 0|\hat{a}_{\text{out}}(\mathbf{k})\hat{\Phi}(x)\hat{a}^{\dagger}_{\text{in}}(l)|0\rangle\\
&\qquad+\tilde{C}^{+}(\mathbf{k})\tilde{C}^{+}(l)\langle 0|\hat{\Phi}(x)\hat{a}^{\dagger}_{\text{in}}(\mathbf{k})\hat{a}^{\dagger}_{\text{in}}(l)|0\rangle]_c \quad (2.22)\\
&\equiv\langle 2^-|\hat{\Phi}(x)|0\rangle_c+\langle 1^-|\hat{\Phi}(x)|1^+\rangle_c+\langle 0|\hat{\Phi}(x)|2^+\rangle_c \ , \quad (2.23)
\end{aligned}$$

where we have introduced the new states

$$|1^{+(-)}\rangle\equiv Z^{-1/2}\int d^3k\,\tilde{a}^{\dagger}_{\text{in(out)}}(\mathbf{k})|0\rangle \ , \quad (2.24)$$

$$|2^{+(-)}\rangle\equiv\frac{1}{2!}(Z^{-1/2})^2\int d^3k\int d^3l\,\tilde{a}^{\dagger}_{\text{in(out)}}(\mathbf{k})\tilde{a}^{\dagger}_{\text{in(out)}}(l)|0\rangle \ , \quad (2.25)$$

with the notation $\tilde{a}_{\text{out}}(\mathbf{k})\equiv\tilde{C}^{-}(\mathbf{k})\hat{a}_{\text{out}}(\mathbf{k})$ and $\tilde{a}^{\dagger}_{\text{in}}(\mathbf{k})\equiv\tilde{C}^{+}(\mathbf{k})\hat{a}^{\dagger}_{\text{in}}(\mathbf{k})$. In these equations we have assumed the uniqueness of the vacuum; i.e., the relative phase factors among $|0,\text{in}\rangle$, $|0,\text{out}\rangle$, and $|0\rangle$ have been set equal to one.

In the same way, for the one-particle channel, we can easily check the following relation by using (2.24):

$$\Delta\phi^{(1)}(x)=\langle 1^-|\hat{\Phi}(x)|0\rangle_c+\langle 0|\hat{\Phi}(x)|1^+\rangle_c \ . \quad (2.26)$$

This exhibits the reason why the on-shell condition (2.8) generally takes the form of the wave equation.

From (2.23) and (2.26), we see that the variational process Δ changes the initial and final states and that, in particular, the particle number is changed. More generally, we introduce the states

$$|n^{+(-)}\rangle\equiv\frac{1}{n!}(Z^{-1/2})^n\int d^3k_1\cdots d^3k_n\tilde{a}^{\dagger}_{\text{in(out)}}(\mathbf{k}_1)\cdots\tilde{a}^{\dagger}_{\text{in(out)}}(\mathbf{k}_n)|0\rangle \ , \quad (2.27)$$

for $n=0,1,2,\ldots$. The general term $\Delta\phi^{(n)}(x)$ is obtained by a mathematical induction, which can be summed up to get the total form of the nonzero solution $\Delta\phi(x)$. The result is expressed by the coherent state as

$$\Delta\phi(x)=[\langle\theta^-|\hat{\Phi}(x)|\theta^+\rangle+(\text{disconnected terms})]_c \ , \quad (2.28)$$

$$|\theta^{+(-)}\rangle\equiv\sum_{n=0}^{\infty}|n^{+(-)}\rangle=\exp\left[Z^{-1/2}\int d^3k\,\tilde{a}^{\dagger}_{\text{in(out)}}(\mathbf{k})\right]|0\rangle \ . \quad (2.29)$$

The "disconnected terms" in (2.28) represent the situations where at least one particle is not affected by the collision process (except for the one referring to the channel x that is still off the mass shell). These contributions have appeared from the terms which include the product $\hat{a}_{\text{out}}\hat{a}^{\dagger}_{\text{out}}$ or $\hat{a}_{\text{in}}\hat{a}^{\dagger}_{\text{in}}$. [The corresponding terms are absent in (2.22) because of the condition $\phi^{(0)}(x)=\langle 0|\hat{\Phi}(x)|0\rangle=0$.] On the other hand, when we calculate $\langle\theta^-|\hat{\Phi}(x)|\theta^+\rangle$ by using the LSZ reduction formula, the same disconnected contributions appear with the minus sign. So the "disconnected terms" in (2.28) play the role of canceling them. In this sense we write the right-hand side of (2.28) as $\langle\theta^-|\hat{\Phi}(x)|\theta^+\rangle_{(c)}$. We notice that the meaning "(c)" coincides with the one which is used in the usual definition of the connected S-matrix elements.[8]

Let us study the relation between (2.28) and (2.6). For this purpose we further introduce the new generating functional $W_{\theta^-\theta^+}[J]$,

$$\exp(iW_{\theta^-\theta^+}[J]) = \left\langle \theta^- \left| T \exp\left[i\int_{-\infty}^{+\infty} d^4x\, J(x)\hat{\Phi}(x) \right] \right| \theta^+ \right\rangle , \tag{2.30}$$

whose boundary states are selected to satisfy the equation

$$\left[\frac{\delta W_{\theta^-\theta^+}[J]}{\delta J(x)} \right]_{J=0} = \langle \theta^- | \hat{\Phi}(x) | \theta^+ \rangle_{(c)} . \tag{2.31}$$

Since $W_{\theta^-\theta^+}$ is in the exponential we can forget the restriction (c) in (2.30). From this expression we find that looking for another solution of (2.4) around the vacuum solution can be interpreted as changing the initial and final states into the form of the coherent states (2.29). In this case, of course, the Feynman prescription is not adopted to obtain (2.30) from (2.5).

We can also define the following effective action by using $W_{\theta^-\theta^+}[J]$:

$$\Gamma_{\theta^-\theta^+}[\phi^*] \equiv W_{\theta^-\theta^+}[J] - \int d^4x\, J(x)\phi^*(x) , \tag{2.32}$$

$$\phi^*(x) \equiv \frac{\delta W_{\theta^-\theta^+}[J]}{\delta J(x)} . \tag{2.33}$$

We remark that ϕ^* satisfies $\phi^*_{J=0} = [\delta W_{\theta^-\theta^+}/\delta J]_{J=0} = \phi^{(0)} + \Delta\phi$ with $\phi^{(0)}=0$. If we set $J=0$ in (2.32), then we find the relation

$$\begin{aligned} \Gamma_{\theta^-\theta^+}[\Delta\phi] &= W_{\theta^-\theta^+}[J=0] \\ &= -i\langle \theta^- | \theta^+ \rangle_{(c)} \\ &= \Gamma_{00}[\phi^{(0)}=0] - i\langle 1^- | 1^+ \rangle \\ &\quad + \sum_{n=3}^{\infty} \frac{1}{n!} (\tilde{W}_{00}^{(n)})_{x_1,\ldots,x_n} \Delta\phi^{(1)}(x_1) \cdots \Delta\phi^{(1)}(x_n) , \end{aligned} \tag{2.34}$$

where $\tilde{W}_{00}^{(n)}$ denotes $W_{00}^{(n)}$ with its external legs amputated by $W_{00}^{(2)-1}$. In the final step of (2.34), the reduction formula has been used together with the fact $\langle 1^- | 0 \rangle = \langle 0 | 1^+ \rangle = 0$. We have written $\Gamma_{00}[\phi^{(0)}=0] = -i \ln\langle 0 | 0 \rangle = 0$ in (2.34) for the comparison with our previous result in I. Because of the definition of $\Gamma_{\theta^-\theta^+}$, the second term $-i\langle 1^- | 1^+ \rangle$ appears in (2.34), which is not present in I. By assuming the stability of the one-particle state we get $\langle 1^- | 1^+ \rangle = Z^{-1}\int d^3k\, 2k^0 (2\pi)^3 \tilde{C}^-(\mathbf{k})\tilde{C}^+(\mathbf{k})$.

Equation (2.34) is the on-shell expansion of the effective action in the sense that all the terms in (2.34) are projected onto the mass shell by $\Delta\phi^{(1)}$'s. It is $\Gamma_{\theta^-\theta^+}[\Delta\phi]$ that corresponds to the generating functional of the connected S-matrix elements, $S^{(c)}(\tilde{C}^-, \tilde{C}^+)$.[8] This fact is summarized again into the simple relation

$$\begin{aligned} i\Gamma_{\theta^-\theta^+}[\Delta\phi] &= Z^{-1}\int d^3k\, 2k^0 (2\pi)^3 \tilde{C}^-(\mathbf{k})\tilde{C}^+(\mathbf{k}) \\ &\quad + iW_{00}[J \equiv Z^{-1}\Delta\phi^{(1)}(\overleftarrow{\Box} + m^2)] \\ &= S^{(c)}(\tilde{C}^-, \tilde{C}^+) . \end{aligned} \tag{2.35}$$

The S-matrix element is generated by

$$(Z^{1/2})^{n+m} \left[\frac{\delta^{n+m} i\Gamma_{\theta^-\theta^+}[\Delta\phi]}{\delta\tilde{C}^-(\mathbf{p}_1)\cdots\delta\tilde{C}^-(\mathbf{p}_n)\delta\tilde{C}^+(\mathbf{q}_1)\cdots\delta\tilde{C}^+(\mathbf{q}_m)} \right]_{\tilde{C}^+=\tilde{C}^-=0} = \langle 0 | \hat{a}_{\mathrm{out}}(\mathbf{p}_1)\cdots\hat{a}_{\mathrm{out}}(\mathbf{p}_n)\hat{a}^\dagger_{\mathrm{in}}(\mathbf{q}_1)\cdots\hat{a}^\dagger_{\mathrm{in}}(\mathbf{q}_m) | 0 \rangle , \tag{2.36}$$

where the normalization factor Z can be absorbed by the proper renormalization of the field; $\hat{\Phi}' \equiv Z^{-1/2}\hat{\Phi}$.

In the next section our previous approach in I is discussed again from another standpoint and the relation between $\Gamma_{\theta^-\theta^+}$ and Γ_{00} is clarified. This type of discussion is useful when we calculate the connected S-matrix elements in terms of the usual effective action, i.e., Γ_{00}.

III. THE FORMULATION IN TERMS OF THE SOURCE AT THE BOUNDARY

As has been stated (Sec. II), in the functional-integral formulation for the infinite time interval $t_i = -\infty$, $t_f = \infty$, the boundary states are naturally taken as the vacuum by the use of the Feynman prescription. In this section we show that our conclusions in the previous section can be obtained by sticking to the *vacuum* as the boundary state. For this purpose we define the *new* effective action whose boundary states are the vacuum, not the coherent states. The essential point is that we introduce the δ-functionlike source at the initial ($t=-T_1$) and the final time ($t=T_2$) which plays the role of changing the vacuum state into the desired coherently excited state. The effective action is calculated under the presence of this type of source while the boundary state is the vacuum. By using this effective action we will follow our previous approach in I.

First we note that we can interpret the solution $\phi(x) = \phi^{(0)}(x) + \Delta\phi(x)$ of the equation of motion (2.4) in a different way. It has been understood in Sec. II as the expectation value of the field operator with the two different coherent states characterized by the function $C^{\pm}(\mathbf{k})$. But we now want to regard it as the *vacuum* expectation value of $\hat{\Phi}$ under the influence of some external source. This source is introduced in the action as $\int d^4x\, K(x)\hat{\Phi}(x)$, where

$$K(x) = -\int_{-T_1}^{T_2} d^4x'\, \Delta\phi^{(1)}(x')(\Gamma_{00}^{(2)})_{x',x} . \tag{3.1}$$

The way to take the limit $T_{1,2}\to\infty$ will be shown below.

Now we have to solve the following set of equations:

$$\left[\frac{\delta\Gamma_{00}[\phi]}{\delta\phi(x)}\right]_{\phi(x)=\phi^{(0)}(x)}=0\ , \tag{3.2a}$$

$$\left[\frac{\delta\Gamma_{00}[\phi]}{\delta\phi(x)}\right]_{\phi(x)=\phi^{(0)}(x)+\Delta\phi(x)}=-K(x)\ . \tag{3.2b}$$

Here $\Delta\phi(x)$ is expanded into the series $\Delta\phi(x)=\sum_{n=1}^{\infty}\Delta\tilde{\phi}^{(n)}(x)$ and $\Delta\tilde{\phi}^{(n)}$ is determined successively by assuming that $\Delta\tilde{\phi}^{(1)}$ is of the same order as $\Delta\phi^{(1)}$ and that $\Delta\tilde{\phi}^{(n)}$ is $O((\Delta\phi^{(1)})^n)$. Substituting this series into (3.2b) we get the lowest-order equation

$$(\Gamma_{00}^{(2)})_{x,x'}\Delta\tilde{\phi}^{(1)}(x')=-K(x)\ . \tag{3.3}$$

The equations of higher orders are the same as (2.9), (2.10), etc. We are assuming that the state at $t=\pm\infty$ is the vacuum; therefore, for $t<-T_1$ and $t>T_2$, the solution ϕ becomes $\phi=\phi^{(0)}$. The limit $T_{1,2}\to\infty$ is taken in the end of the calculation. The solution of (3.3) is obtained as

$$\Delta\tilde{\phi}^{(1)}(x)=(W_{00}^{(2)})_{x,x'}K(x') \tag{3.4a}$$

$$=\Delta\phi^{(1)}(x)\quad(-T_1\le t\le T_2)\ . \tag{3.4b}$$

For $-T_1<t<T_2$, the solutions of higher orders are the same as in the previous case (2.15). Here we notice that, owing to the above boundary condition, (3.4a) does not include the solution of the homogeneous equation. However, it is easily seen from (3.4b) that the solution of the homogeneous equation is recovered after $T_{1,2}\to\infty$ and that the solution $\Delta\phi(x)=\sum_{n=1}^{\infty}\Delta\tilde{\phi}^{(n)}(x)$ coincides with that in Sec. II. Therefore the introduction of $K(x)$ is equivalent to setting the boundary condition which leads to the coherent states.

More comments on the source $K(x)$. $\Delta\phi^{(1)}(x)$ is the function on mass shell so that $(\Gamma_{00}^{(2)})_{x',x}$ can be replaced by the operator $-Z^{-1}(\Box_{x'}+m^2)\delta^4(x'-x)$. Then the definition (3.1) is rewritten as

$$K(x)=Z^{-1}\int_{-T_1}^{T_2}d^4x'\Delta\phi^{(1)}(x')(\overleftarrow{\Box}_{x'}+m^2)\delta^4(x'-x)\ . \tag{3.5}$$

If we integrate (3.5) by parts we find that $K(x)$ has nonzero value only on the boundary $t=-T_1,T_2$. In this sense $K(x)$ has a different nature from $J(x)$, which is usually assumed to be a smooth function of x and the support extends all over the space-time. Then the meaning of $T_{1,2}\to\infty$ is now obvious. Namely, in order that $K(x)$ have real physical effects, it must have its support *inside* the time interval which defines the theory. So we must keep $T_{1,2}$ finite during the calculation. After taking the limit $T_{1,2}\to\infty$, the effect of the artificially introduced source term $K(x)$ becomes to excite the vacuum state at the boundary $t=\pm\infty$.

On the basis of the above discussion we define the effective action which includes $K(x)$. We have now two kinds of source, $K(x)$ and $J(x)$, the latter of which serves as the one for the usual Legendre transformation in the definition of the effective action Γ. We start with the generating functional $W_{00}[J,K]$ defined as

$$\exp(iW_{00}[J+K])\equiv\int_{-\infty}^{\infty}[d\Phi]\exp\left[i\int d^4x\{\mathcal{L}(\Phi)+[J(x)+K(x)][\Phi(x)-\phi^{(0)}(x)]\}\right]\ , \tag{3.6}$$

where the sources are coupled to the fluctuating part $\Phi(x)-\phi^{(0)}(x)$ of the field $\Phi(x)$. The Legendre transformation with respect to $J(x)$ gives the new effective action $\Gamma_{00}[\phi,K]$:

$$\Gamma_{00}[\phi,K]\equiv W_{00}[J+K]-\int d^4x\ \phi(x)J(x)\ , \tag{3.7}$$

$$\phi(x)\equiv\frac{\delta W_{00}[J+K]}{\delta J(x)}\ . \tag{3.8}$$

The difference between $\Gamma_{00}[\phi,K]$ and the previous effective action $\Gamma_{00}[\phi]$ in I (defined using the shifted field $\Phi-\phi^{(0)}$), is the term $K(x)\phi(x)$:

$$\Gamma_{00}[\phi,K]=\Gamma_{00}[\phi]+\int d^4x\ K(x)\phi(x)\ . \tag{3.9}$$

Equation (3.2b) is given as the stationary condition of $\Gamma_{00}[\phi,K]$ where $K(x)$ is fixed by the definition (3.1) or (3.5). So we can regard $\Gamma_{00}[\phi,K]$ as the effective action accompanied with the boundary condition implied by $K(x)$.

The use of $\Gamma_{00}[\phi,K]$ gives a clear derivation of the on-shell expansion which is more transparent than that given in I where only $\Gamma_{00}[\phi]$ has been utilized.

The on-shell expansion of $\Gamma_{00}[\phi,K]$ is followed by the substitution of $\phi(x)=\Delta\phi(x)$ and $K(x)=-\Delta\phi^{(1)}(x')(\Gamma_{00}^{(2)})_{x',x}$. From (3.1) and (3.9), we get, in the limit $T_{1,2}\to\infty$,

$$\begin{aligned}\Gamma_{00}[\phi(x)=\Delta\phi(x),K(x)=-\Delta\phi^{(1)}(x')(\Gamma_{00}^{(2)})_{x',x}]\\&=\Gamma_{00}[\phi(x)=\Delta\phi(x)]-\Delta\phi^{(1)}(x')(\Gamma_{00}^{(2)})_{x',x}\Delta\phi(x)\\&=\Gamma_{00}[\phi(x)=0]+\sum_{n=2}^{\infty}\frac{1}{n!}(\Gamma_{00}^{(n)})_{x_1,x_2,\dots,x_n}\prod_{k=1}^{n}\Delta\phi(x_k)-\Delta\phi^{(1)}(x')(\Gamma_{00}^{(2)})_{x',x}\Delta\phi(x)\end{aligned}$$

$$= \Gamma_{00}[\phi(x)=0] - \tfrac{1}{2}\Delta\phi^{(1)}(x')(\Gamma_{00}^{(2)})_{x',x}\Delta\phi^{(1)}(x) + \sum_{n=3}^{\infty}\frac{1}{n!}(\Gamma_{00}^{(n)})_{x_1,x_2,\ldots,x_n}\prod_{k=1}^{n}\Delta\phi(x_k)$$
$$+ \tfrac{1}{2}[\Delta\phi(x') - \Delta\phi^{(1)}(x')](\Gamma_{00}^{(2)})_{x',x}[\Delta\phi(x) - \Delta\phi^{(1)}(x)]$$
$$= \Gamma_{00}[\phi(x)=0] + \sum_{n=3}^{\infty}\frac{1}{n!}(\tilde{W}_{00}^{(n)})_{x_1,x_2,\ldots,x_n}\prod_{k=1}^{n}\Delta\phi^{(1)}(x_k)\,, \tag{3.10}$$

where $\tilde{W}_{00}^{(n)}$ is defined in (2.34) and we have used $\Delta\phi^{(1)}\Gamma^{(2)}\Delta\phi^{(1)}=0$ because of the on-shell property of $\Delta\phi^{(1)}$.

The last equality is based on $-\Gamma_{00}^{(2)} = W_{00}^{(2)-1} = \tilde{W}_{00}^{(2)}$ and on the crucial identity

$$Q \equiv \sum_{n=3}^{\infty}\frac{1}{n!}(\tilde{W}_{00}^{(n)})_{x_1,x_2,\ldots,x_n}\prod_{k=1}^{n}\Delta\phi^{(1)}(x_k) = \sum_{n=3}^{\infty}\frac{1}{n!}(\Gamma_{00}^{(n)})_{x_1,x_2,\ldots,x_n}\prod_{k=1}^{n}\Delta\phi(x_k)$$
$$+ \tfrac{1}{2}[\Delta\phi(x') - \Delta\phi^{(1)}(x')](\Gamma_{00}^{(2)})_{x',x}[\Delta\phi(x) - \Delta\phi^{(1)}(x)]\,. \tag{3.11}$$

This is proved as follows. Let us introduce the graphical representations shown in Fig. 1. By the notation in Fig. 1, $\Delta\phi(x)$ and $K(x)$ are illustrated in Fig. 2. Recall that Q is the sum of all the connected Green's functions whose external legs are replaced by the wave function $\Delta\phi^{(1)}$. Therefore the general term in the sum is expressed as the aggregation of the topologically different tree diagrams constructed by means of the proper vertices $\Gamma_{00}^{(n\geq 3)}$, the propagator $W_{00}^{(2)}$, and $\Delta\phi^{(1)}$. Only the tree diagrams are allowed since $\tilde{W}_{00}^{(n)}$, $\Gamma_{00}^{(n\geq 3)}$, and $W_{00}^{(2)}$ are all full-order expressions. In Fig. 3 we illustrate one of such diagrams as an example.

The tree diagrams have a topological relation

$$N_v - N_p = 1\,, \tag{3.12}$$

where N_v and N_p are the number of vertices and propagators, respectively, in each graph. This relation is easily proved by induction and leads us to the following three steps of the resummation of the series Q. Figures 2 and 3 help us to understand our procedure (which is based on the similar argument used in Ref. 9).

(1) Consider the diagram of Q which has N_v vertices. (See Fig. 3 for example.) Such a diagram appears "N_v times" in the sum

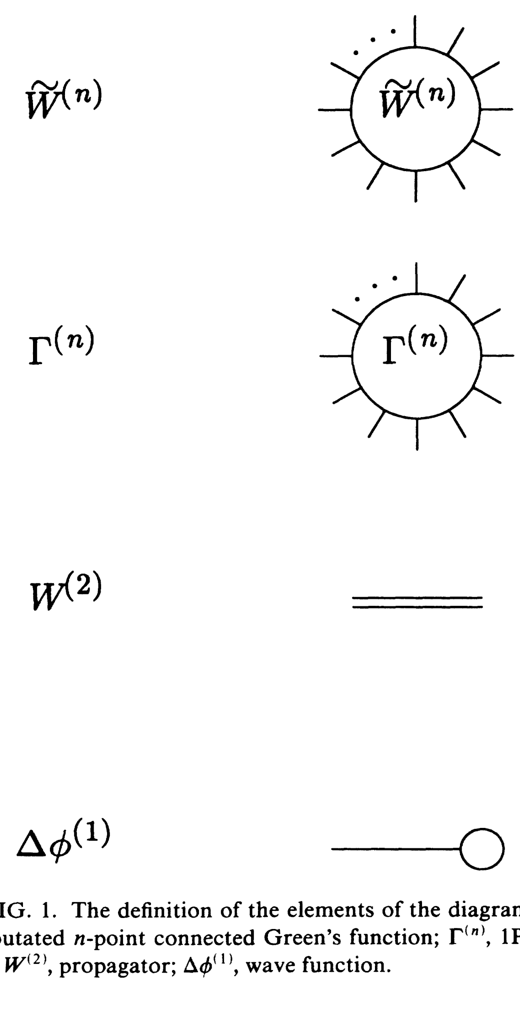

FIG. 1. The definition of the elements of the diagram. $\tilde{W}^{(n)}$, amputated n-point connected Green's function; $\Gamma^{(n)}$, 1PI n vertex; $W^{(2)}$, propagator; $\Delta\phi^{(1)}$, wave function.

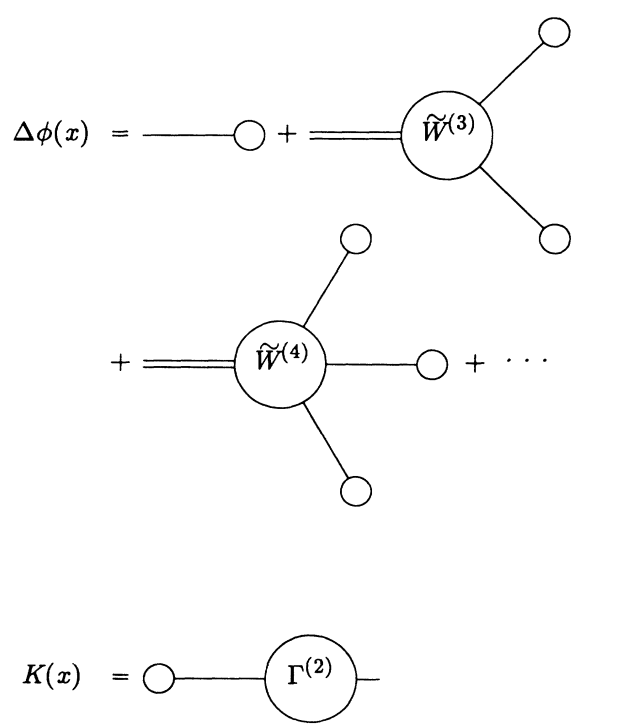

FIG. 2. The diagrammatic representation of $\Delta\phi(x)$ and $K(x)$.

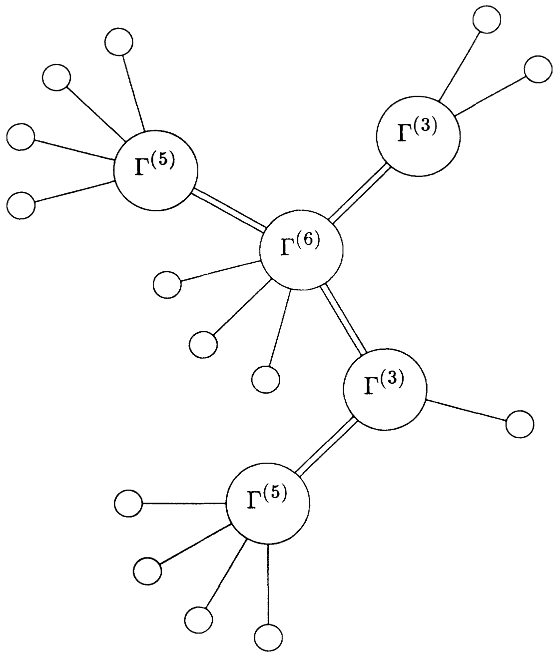

FIG. 3. An example of the graphs included in the sum Q. It has $N_v=5$ vertices and $N_p=4$ internal lines (propagators).

$$\sum_{n=3}^{\infty}\frac{1}{n!}(\Gamma_{00}^{(n)})_{x_1,x_2,\ldots,x_n}\prod_{k=1}^{n}\Delta\phi(x_k)\ . \tag{3.13}$$

This is because, by distinguishing one particular vertex $\Gamma_{00}^{(n)}$ in the diagram, we can regard $\Delta\phi^{(1)}$ and the other tree diagrams connected to it as the terms coming from

$$\Delta\phi=\Delta\phi^{(1)}+\sum_{n=2}^{\infty}\frac{1}{n!}W_{00}^{(2)}\tilde{W}_{00}^{(n+1)}\prod_{k=1}^{n}\Delta\phi^{(1)}\ .$$

(2) On the other hand, if we distinguish one propagator $W_{00}^{(2)}$, the same graph can be regarded as the graph in which the propagator connects two $(1/n!)\tilde{W}_{00}^{(n+1)}\prod_{k=1}^{n}\Delta\phi^{(1)}$ coming from $\Delta\phi$. So each graph in the sum appears "N_p times" in the term

$$\tfrac{1}{2}\{(W_{00}^{(2)-1})_{x,x'}[\Delta\phi(x')-\Delta\phi^{(1)}(x')]\}(W_{00}^{(2)})_{x,y}\times\{(W_{00}^{(2)-1})_{y,y'}[\Delta\phi(y')-\Delta\phi^{(1)}(y')]\}\ . \tag{3.14}$$

The reason we subtract $\Delta\phi^{(1)}$ from $\Delta\phi$ is that the propagator is an internal line which always connects two vertices.

(3) From (3.12), the subtraction of (3.14) from (3.13) gives the right weight in the sum. Therefore we have accomplished the proof of (3.11) using the identity $W_{00}^{(2)-1}=-\Gamma_{00}^{(2)}$.

Let us check the equivalence between (3.10) and the previous results (2.34). For this purpose we show the following relation between $\Gamma_{00}[\phi,K]$ and $\Gamma_{\theta^-\theta^+}[\phi^*]$:

$$\Gamma_{00}[\Delta\phi,K]=\Gamma_{\theta^-\theta^+}[\Delta\phi]+iZ^{-1}\int d^3k\,2k^0(2\pi)^3\tilde{C}^+(\mathbf{k})\tilde{C}^-(\mathbf{k})\ . \tag{3.15}$$

This is directly proved as follows: by using the definition (3.7) of $\Gamma_{00}[\phi,K]$ with $J=0$ and through the inverse reduction formula,

$$\begin{aligned}\exp(i\Gamma[\Delta\phi,K])&=\left\langle 0\left|T\exp\left[iZ^{-1}\int d^4x\,\Delta\phi^{(1)}(x)(\overleftarrow{\Box}+m^2)[\hat{\Phi}(x)-\phi^{(0)}(x)]\right]\right|0\right\rangle\\&=\left\langle 0\left|\exp\left[Z^{-1/2}\int d^3k[\tilde{C}^-(\mathbf{k})\hat{a}_{\rm out}(\mathbf{k})-\tilde{C}^+(\mathbf{k})\hat{a}^\dagger_{\rm out}(\mathbf{k})]\right]\right.\right.\\&\qquad\left.\left.\times\exp\left[Z^{-1/2}\int d^3k[\tilde{C}^+(\mathbf{k})\hat{a}^\dagger_{\rm in}(\mathbf{k})-\tilde{C}^-(\mathbf{k})\hat{a}_{\rm in}(\mathbf{k})]\right]\right|0\right\rangle\\&=\exp\left[-Z^{-1}\int d^3k\,2k^0(2\pi)^3\tilde{C}^-(\mathbf{k})\tilde{C}^+(\mathbf{k})\right]\\&\qquad\times\left\langle 0\left|\exp\left[Z^{-1/2}\int d^3k\,\tilde{C}^-(\mathbf{k})\hat{a}_{\rm out}(\mathbf{k})\right]\exp\left[Z^{-1/2}\int d^3p\,\tilde{C}^+(\mathbf{p})\hat{a}^\dagger_{\rm in}(\mathbf{p})\right]\right|0\right\rangle\\&=\exp\left[-Z^{-1}\int d^3k\,2k^0(2\pi)^3\tilde{C}^-(\mathbf{k})\tilde{C}^+(\mathbf{k})\right]\exp(i\Gamma_{\theta^-\theta^+}[\Delta\phi])\ .\end{aligned} \tag{3.16}$$

In (3.16) we have used the formula

$$\exp(\hat{X}+\hat{Y})=\exp(-\tfrac{1}{2}[\hat{X},\hat{Y}])\exp(\hat{X})\exp(\hat{Y})\ , \tag{3.17}$$

where $[\hat{X},\hat{Y}]$ commutes with $\hat{X}$ and $\hat{Y}$.

Note that Eq. (3.15) and the relation (3.9) enables us to write the generating functional $S^{(c)}(\tilde{C}^-,\tilde{C}^+)$ of the connected S-matrix element in terms of the usual effective action $\Gamma_{00}[\phi]$.

Now we want to summarize the approach of this section in connection with the results of Sec. II. Let us start with

$\Gamma_{\theta^-\theta^+}[\phi^*]$ introduced in (2.32). In the case of $\phi^{(0)}(x)\neq 0$, the corresponding generating functional $W_{\theta^-\theta^+}[J]$ is assumed to be defined with the operator $\hat{\Phi}-\phi^{(0)}$. Then, by using the relation

$$\exp(iW_{\theta^-\theta^+}[J]) \equiv \left\langle \theta^- \left| T\exp\left[i\int_{-\infty}^{\infty} d^4x\, J(x)[\hat{\Phi}(x)-\phi^{(0)}(x)] \right] \right| \theta^+ \right\rangle$$

$$= \exp\left[Z^{-1}\int d^3k\, 2k^0(2\pi)^3 \tilde{C}^-(\mathbf{k})\tilde{C}^+(\mathbf{k}) \right] \left\langle 0 \left| T\exp\left[i\int_{-\infty}^{\infty} d^4x [J(x)+K(x)][\hat{\Phi}(x)-\phi^{(0)}(x)] \right] \right| 0 \right\rangle , \tag{3.18}$$

which can be derived in the same way as in (3.16), we get the equation

$$\Gamma_{\theta^-\theta^+}[\phi^*] \equiv W_{\theta^-\theta^+}[J] - \int d^4x\, J(x)\phi^*(x) = W_{00}[J+K] - iZ^{-1}\int d^3k\, 2k^0(2\pi)^3 \tilde{C}^-(\mathbf{k})\tilde{C}^+(\mathbf{k}) - \int d^4x\, J(x)\phi^*(x) , \tag{3.19}$$

where $W_{00}[J+K]$ is defined by (3.6). Equation (3.19) corresponds to the extension of (2.34) for the case of $J\neq 0$ (and $\phi^{(0)}\neq 0$). From this equation we can get the connection between $\Gamma_{\theta^-\theta^+}[\phi^*]$ and $\Gamma_{00}[\phi]$ by the Legendre transformation of $W_{00}[J+K]$ as

$$\Gamma_{\theta^-\theta^+}[\phi^*] = \left[W_{00}[J+K] - \int d^4x\,(J+K)(x)\frac{\delta W_{00}[J+K]}{\delta(J+K)(x)} \right] - iZ^{-1}\int d^3k\, 2k^0(2\pi)^3 \tilde{C}^-(\mathbf{k})\tilde{C}^+(\mathbf{k}) + \int d^4x\, K(x)\phi^*(x)$$

$$= \Gamma_{00}[\phi^*] - iZ^{-1}\int d^3k\, 2k^0(2\pi)^3 \tilde{C}^-(\mathbf{k})\tilde{C}^+(\mathbf{k}) + Z^{-1}\int d^4x\, \Delta\phi^{(1)}(x)(\vec{\Box}+m^2)\phi^*(x) . \tag{3.20}$$

If we set $J=0$, $i\Gamma_{\theta^-\theta^+}[\phi^*]$ becomes the generating functional $S^{(c)}(\tilde{C}^-,\tilde{C}^+)$ of the connected S-matrix elements, see (2.35). In this case, Eq. (3.20) directly relates $S^{(c)}(\tilde{C}^-,\tilde{C}^+)$ with the usual effective action $\Gamma_{00}[\phi]$.

IV. GENERALIZATIONS TO THE OTHER CASES

In the previous sections we have studied the single-component scalar field case, however, our formalism is easily generalized to the other cases.

We first consider a model of multicomponent scalar field $\Phi_i(x)$ $(i=1,\ldots,n)$ since it can be a basis for the composite field case. In this case, the generalized on-shell condition (2.8) takes the form

$$(\Gamma_{00}^{(2)})_{ix,jy}\Delta\phi_j^{(1)}(y)=0 . \tag{4.1}$$

By assuming the space-time translational invariance, $(\Gamma_{00}^{(2)})_{ix,jy}$ becomes a function of $x-y$ and we get the Fourier representation of (4.1) as

$$\Gamma_{00ij}^{(2)}(p^2)\Delta\phi_j^{(1)}(p)=0 . \tag{4.2}$$

To investigate the eigenvalue of this equation we then diagonalize $\Gamma_{00ij}^{(2)}$ as is discussed in I:

$$U_{ii'}^{-1}(p^2)\Gamma_{00i'j'}^{(2)}(p^2)U_{j'j}(p^2) \equiv \bar{\Gamma}_{00ij}^{(2)}(p^2) \equiv \delta_{ij}\gamma_i(p^2) , \tag{4.3}$$

where $U(p^2)$ is the appropriate orthogonal matrix. We also introduce $\Delta\bar{\phi}_i^{(1)}(p)$ defined as

$$\Delta\bar{\phi}_i^{(1)}(p) \equiv U_{ij}^{-1}(p^2)\Delta\phi_j^{(1)}(p) = \Delta\phi_j^{(1)}(p)U_{ji}(p^2) . \tag{4.4}$$

Since it satisfies the equation $\bar{\Gamma}_{00ij}^{(2)}\Delta\bar{\phi}_j^{(1)}=0$ we can write it in the following form as in (2.13):

$$\Delta\bar{\phi}_i^{(1)}(p) = [C_i^+(\mathbf{p})\theta(p^0) + C_i^-(-\mathbf{p})\theta(-p^0)]\delta(p^2-m_i^2) , \tag{4.5}$$

where we have assumed that the solution of $\gamma_i(p^2)=0$ is $p^2=m_i^2$.

With these mechanisms, let us study the following identity, which corresponds to (2.26) in the single-component field case:

$$\Delta\phi_i^{(1)}(x) = -i\int d^4y\, d^4y' \Delta\phi_j^{(1)}(y)[\Gamma^{(2)}_{00jj'}(y-y')\langle 0|T\hat{\Phi}_i(x)\hat{\Phi}_{j'}(y')|0\rangle]$$
$$= -i\int d^4y\, d^4y' \Delta\bar{\phi}_j^{(1)}(y)[\bar{\Gamma}^{(2)}_{00jj'}(y-y')\langle 0|T\hat{\Phi}_i(x)\bar{\hat{\Phi}}_{j'}(y')|0\rangle] , \quad (4.6)$$

where we have further introduced $\bar{\hat{\Phi}}_i$ defined similarly as $\Delta\bar{\phi}_i^{(1)}$ in (4.4),

$$\bar{\hat{\Phi}}_i(p) \equiv U_{ij}^{-1}(p^2)\hat{\Phi}_j(p) = \hat{\Phi}_j(p)U_{ji}(p^2) . \quad (4.7)$$

The asymptotic form of $\bar{\hat{\Phi}}$ can be written for its each component as

$$\bar{\hat{\Phi}}_i(x)_{\text{in(out)}} \equiv \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k^0}[\hat{a}_{i\ \text{in(out)}}(\mathbf{k})e^{-ik\cdot x} + \hat{a}^\dagger_{i\ \text{in(out)}}(\mathbf{k})e^{ik\cdot x}] , \quad (4.8)$$

where $k^0 = (\mathbf{k}^2 + m_i^2)^{1/2}$. The right-hand side of (4.6) is then transformed into the following form by using the same approach as has been shown in Sec. II:

$$\Delta\phi_i^{(1)}(x) = \langle \bar{1}^- |\hat{\Phi}_i(x)|0\rangle_c + \langle 0|\hat{\Phi}_i(x)|\bar{1}^+\rangle_c , \quad (4.9)$$

$$|\bar{1}^{+(-)}\rangle \equiv Z^{-1/2}\sum_j \int d^3k\, \tilde{a}^\dagger_{j\ \text{in(out)}}(\mathbf{k})|0\rangle , \quad (4.10)$$

where the similar notation $\tilde{a}$ as in (2.24) has been used for each j. In general, we obtain $|\bar{\theta}^{+(-)}\rangle$ instead of (2.29) as

$$|\bar{\theta}^{+(-)}\rangle \equiv \exp\left[Z^{-1/2}\sum_j \int d^3k\, \tilde{a}^\dagger_{j\ \text{in(out)}}(\mathbf{k})\right]|0\rangle . \quad (4.11)$$

Namely, looking for the possible solution on the trajectory $J=0$ is equivalent to creating the physical particles which are given by the on-shell condition $\gamma_i(p^2)=0$.

Once the effective action $\Gamma_{\bar{\theta}^-\bar{\theta}^+}$ is defined through (4.11), it is straightforwardly related to the generating functional of the connected S-matrix elements as in (2.35). The relation between $\Gamma_{\bar{\theta}^-\bar{\theta}^+}$ and Γ_{00} is also obtained in the same way as in Sec. III.

Now we consider the bilocal composite field case. The generating functional $W_{00}[J]$ is introduced as

$$\exp(iW_{00}[J]) \equiv \left\langle 0\left|T\exp\left[i\int_{-\infty}^{\infty} d^4x\, d^4y\, J(x,y)\hat{G}(x,y)\right]\right|0\right\rangle , \quad (4.12)$$

with the shifted operator $\hat{G}(x,y) \equiv \hat{\Phi}(x)\hat{\Phi}(y) - \langle T\hat{\Phi}(x)\hat{\Phi}(y)\rangle_{J=0}$. Then we define the effective action

$$\Gamma_{00}[G] \equiv W_{00}[J] - \int d^4x\, d^4y\, J(x,y)G(x,y) , \quad (4.13)$$

$$G(x,y) \equiv \frac{\delta W_{00}[J]}{\delta J(x,y)} . \quad (4.14)$$

We notice, however, if we write $G(x,y)$ into the form $G_i(X)$ [$i \equiv x-y$ and $X \equiv (x+y)/2$, for example], we can make the parallel arguments with the ones in the multicomponent scalar field case—all we need is to write $G_i(X)$ and $\hat{G}_i(X)$ instead of $\phi_i(x)$ and $\hat{\Phi}_i(x)$, respectively. In this case, $\tilde{a}^\dagger_{j\ \text{in(out)}}$ in (4.11) is interpreted as the operator which creates a certain mode of the bound state.

Of course, other cases can also be studied by the straightforward extension of this approach. In the next section we consider the Gross-Neveu model[4] and present the derivation of the S-matrix element for composite fields as an example.

V. AN EXAMPLE

In this section we apply our formalism to the Gross-Neveu model[4] in the large-N limit. It is one of the examples of our general theory, and we will show that our method is applicable to the system including the bound state.

Many people have discussed the S-matrix elements among the bound states.[10–12] We will check by the model that our formalism reproduces the existing results.

Let us start with the action of the Gross-Neveu model:

$$I = \int d^2x\, [\bar{\psi}_i i\partial\!\!\!/ \psi_i + \tfrac{1}{2}g^2(\bar{\psi}_i\psi_i)^2] , \quad (5.1)$$

where $\psi(x)$ is the N-component massless fermion field and the coupling constant g^2 behaves as $1/N$ in the large-N limit (the summation over the repeated indices is understood).

Next, the source terms are introduced as

$$I_{\text{source}} = \int d^2x\, [\eta_i(x)\bar{\psi}_i(x) + \bar{\eta}_i(x)\psi_i(x)] + \int d^2x\, d^2y\, J_{ij}(x,y)\psi_i(x)\bar{\psi}_j(y) \quad (5.2)$$

(here it is more convenient to couple η, $\bar{\eta}$, and J to the nonshifted operator) and the generating functional as

$$\exp(iW[\eta,\bar{\eta},J]) \equiv \int [d\psi\, d\bar{\psi}]\exp[i(I + I_{\text{source}})] , \quad (5.3)$$

where $[d\psi d\bar{\psi}]$ denotes the functional integral with respect to ψ and $\bar{\psi}$. Here η and $\bar{\eta}$ are the Grassmann number sources. In (5.2) the vacuum expectation value has not been subtracted from each operator, but it causes no trouble. Through the Legendre transformation from $W[\eta,\bar{\eta},J]$, we define the effective action as

$$\Gamma[\bar{\psi},\psi,S] \equiv W - \int d^2x\, [\eta_i(x)\bar{\psi}_i(x) + \bar{\eta}_i(x)\psi_i(x)] - \int d^2x\, d^2y\, J_{ij}(x,y)S_{ij}(x,y) , \quad (5.4)$$

where

$$\bar{\psi}_i(x) \equiv \frac{\overrightarrow{\delta} W[\eta,\bar{\eta},J]}{\delta \eta_i(x)} , \quad (5.5)$$

$$\psi_i(x) \equiv \frac{\overrightarrow{\delta} W[\eta,\bar{\eta},J]}{\delta \bar{\eta}_i(x)} , \quad (5.6)$$

$$S_{ij}(x,y) \equiv \frac{\delta W[\eta,\bar{\eta},J]}{\delta J_{ij}(x,y)} . \quad (5.7)$$

We have used the same symbol $\psi_i(x)$ or $\bar{\psi}_i(x)$ in (5.5) or (5.6). In the above definitions, $\overrightarrow{\delta}/\delta\eta_i(x)$ denotes the left derivative, which is defined for the Grassmann number χ as

$$F[\chi+\delta\chi]-F[\chi]$$

$$\equiv \int d^2x\, \delta\chi(x) \frac{\overrightarrow{\delta} F[\chi]}{\delta\chi(x)} + O((\delta\chi)^2)$$

$$\equiv \int d^2x \frac{\overleftarrow{\delta} F[\chi]}{\delta\chi(x)} \delta\chi(x) + O((\delta\chi)^2) , \quad (5.8)$$

where the right derivative is also introduced.

In the large-N limit, it is easy to see that the effective action of the Gross-Neveu model is given by[13]

$$\Gamma[\bar{\psi},\psi,S]=\Gamma[\bar{\psi},\psi,\tilde{S}]=i\,\mathrm{Tr}(\ln\tilde{S}-S_0^{-1}\tilde{S})+i\int d^2x\, d^2y\, \bar{\psi}_i(x)S_{0ij}^{-1}(x,y)\psi_j(y)+\frac{g^2}{2}\int d^2x\,[\bar{\psi}_i(x)\psi_i(x)-\mathrm{tr}\tilde{S}(x,x)]^2 , \quad (5.9)$$

where $\tilde{S}_{ij}(x,y) \equiv S_{ij}(x,y)-\psi_i(x)\bar{\psi}_j(y)$ is the connected part of S and S_0 is the bare propagator. "Tr" indicates the trace operation in the functional sense; that is, the integration over the space-time is implied in addition to the matrix trace.

To the above effective action we apply our general formalism to get the S-matrix elements between the bound states.

The first stage is to determine the vacuum state through the vacuum expectation values of the operators. They are given by the solutions of the equations of motion for ψ, $\bar{\psi}$, and S or $\tilde{S}$, which are the stationary condition of the effective action. As for the propagator, it takes the form of the Schwinger-Dyson equation. They are given as

$$0=\frac{\overleftarrow{\delta}\Gamma[\bar{\psi},\psi,\tilde{S}]}{\delta\bar{\psi}_i(x)}=-i\int d^2y\, S_{0ij}^{-1}(x,y)\psi_j(y)-g^2[\bar{\psi}_j(x)\psi_j(x)-\mathrm{tr}\tilde{S}(x,x)]\psi_i(x) \quad (5.10)$$

(and the conjugate of this equation),

$$0=\frac{\delta\Gamma[\bar{\psi},\psi,\tilde{S}]}{\delta\tilde{S}_{ij}(x,y)}=i[\tilde{S}_{ji}^{-1}(y,x)-S_{0ji}^{-1}(y,x)]-g^2[\bar{\psi}_k(x)\psi_k(x)-\mathrm{tr}\tilde{S}(x,x)]\delta_{ij}\delta^2(x-y) . \quad (5.11)$$

We look for the solutions where the fermion-antifermion pairs are allowed to condense but fermions do not condense. Indeed we find the solutions

$$\bar{\psi}^{(0)}(x)=\psi^{(0)}(x)=0 \quad (5.12)$$

and

$$S_{ij}^{(0)}(p)=\tilde{S}_{ij}^{(0)}(p)=\frac{i\delta_{ij}}{\not{p}-m} , \quad (5.13)$$

in momentum space where m is determined by the self-consistent gap equation:

$$m=ig^2\int \frac{d^2p}{(2\pi)^2}\mathrm{tr}S^{(0)}(p) . \quad (5.14)$$

This equation is known to give two types of solutions: $m=0$ and $m=\mu\exp(-\pi/\lambda_r)$ where μ denotes the renormalization point ($\mu>0$) and $\lambda_r \equiv g_r^2N$ is the renormalized coupling.[4] In the following we will continue our arguments based on the stable vacuum, i.e., on the solution $m=\mu\exp(-\pi/\lambda_r)$.

The second stage is to give the on-shell conditions, which are the wave equations for $\Delta\psi^{(1)}$ and $\Delta\bar{\psi}^{(1)}$ and the Nambu-Bethe-Salpeter (NBS) equation[14] for ΔS. In general, they are coupled to each other, but the fermion number is conserved in the Gross-Neveu model, so that they decouple. We have the wave equation

$$0=\int d^2y \left. \frac{\delta^2\Gamma[\bar{\psi},\psi,S]}{\delta\psi_j(y)\delta\bar{\psi}_i(x)} \right|_0 \Delta\psi_j^{(1)}(y)$$

$$=-i\int d^2y\, S_{0ij}^{-1}(x,y)\Delta\psi_j^{(1)}(y)$$

$$+g^2\mathrm{tr}[S^{(0)}(x,x)]\Delta\psi_i^{(1)}(x) \quad (5.15)$$

and the charge-conjugate equation of it. The NBS equation is

$$0=\int d^2x_2 d^2y_2 \left[\frac{\delta^2\Gamma[\bar{\psi},\psi,S]}{\delta S_{i_2j_2}(x_2,y_2)\delta S_{i_1j_1}(x_1,y_1)}\right]_0 \Delta S^{(1)}_{i_2,j_2}(x_2,y_2)$$
$$=-i\int d^2x_2 d^2y_2 S^{(0)-1}_{j_1i_2}(y_1,x_2)\Delta S^{(1)}_{i_2,j_2}(x_2,y_2)S^{(0)-1}_{j_2i_1}(y_2,x_1)+g^2\delta_{i_1j_1}\delta^2(x_1-y_1)\mathrm{tr}\Delta S^{(1)}(x_1,x_1) . \quad (5.16)$$

Equation (5.15) is rewritten as

$$-\int S^{(0)-1}_{ij}(x,y)\Delta\psi^{(1)}_j(y)=0 , \quad (5.17)$$

by using the Schwinger-Dyson equation (5.11) with $\bar{\psi}^{(0)}=\psi^{(0)}=0$. Then we find that $\Delta\psi^{(1)}$ is on the mass-shell which is determined by the pole of the full propagator. As for the NBS equation (5.16) we easily get the solution in momentum space as

$$\Delta S^{(1)}_{ij}(P,q)=A(P)[S^{(0)}_{ik}(q+\tfrac{1}{2}P)S^{(0)}_{kj}(q-\tfrac{1}{2}P)] , \quad (5.18)$$

where P or q is the total or relative momentum, respectively, and

$$A(P)\equiv -ig^2\int \frac{d^2k}{(2\pi)^2}\mathrm{tr}\Delta S^{(1)}(P,k) . \quad (5.19)$$

Substituting (5.18) into this definition we obtain

$$A(P)=-i\lambda A(P)\int \frac{d^2k}{(2\pi)^2}\mathrm{tr}[S^{(0)}(k+\tfrac{1}{2}P)S^{(0)}(k-\tfrac{1}{2}P)] . \quad (5.20)$$

This determines P^2, which is equal to $4m^2$ with the large-N limit. It corresponds to the (threshold) bound state composed of fermion and antifermion.[4] Therefore, the solution becomes now

$$\Delta S^{(1)}_{ij}(X+\tfrac{1}{2}x,X-\tfrac{1}{2}x)$$
$$=\int dP^1\frac{d^2q}{(2\pi)^2}[(2\pi)^{-1/2}C^+(P^1)\phi_{ij}(q,P)e^{-iP\cdot X-iq\cdot x}+(2\pi)^{-1/2}C^-(P^1)\bar{\phi}_{ij}(q,P)e^{+iP\cdot X+iq\cdot x}] , \quad (5.21)$$

where X or x is the center of mass or the relative coordinate, respectively, and $P_0=[(P^1)^2+4m^2]^{1/2}$. We have introduced

$$\phi_{ij}(q,P)=B[S^{(0)}_{ik}(q+\tfrac{1}{2}P)S^{(0)}_{kj}(q-\tfrac{1}{2}P)] , \quad (5.22)$$

$$\bar{\phi}_{ij}(q,P)=B[S^{(0)}_{ik}(-q-\tfrac{1}{2}P)S^{(0)}_{kj}(-q+\tfrac{1}{2}P)] . \quad (5.23)$$

Here, B is a normalization constant which is determined by the well-known normalization condition of the NBS amplitude.[15] After some algebra it takes the form

$$i\frac{\partial m}{\partial g_r^2}\int \frac{d^2p}{(2\pi)^2}\mathrm{tr}[\bar{\phi}(-p,P)\phi(p,P)S^{(0)-1}(p-\tfrac{1}{2}P)+\bar{\phi}(-p,P)S^{(0)-1}(p+\tfrac{1}{2}P)\phi(p,P)]$$
$$+i\int\frac{d^2p}{(2\pi)^2}\mathrm{tr}[\bar{\phi}(p,P)]\int\frac{d^2q}{(2\pi)^2}\mathrm{tr}[\phi(q,P)]=i\frac{\partial(4m^2)}{\partial g_r^2} , \quad (5.24)$$

where $P^2=4m^2$. In the above normalization condition, however, the first integration shows the infrared divergence on the mass shell $P^2=4m^2$. This is due to the fact that the bound state in the large-N limit is actually the threshold bound state so that the wave function ϕ or $\bar{\phi}$ is not well localized as a function of the coordinate x. But this difficulty is known to be solved when we take into account the next-order terms in the large-N limit.

Finally the S-matrix elements are calculated. We concentrate on the matrix element of the scattering between two bound states as an example, but it includes all the essential points. Only the fourth-order term in (2.35) or (3.10) is needed, which is calculated as follows. First we draw all the topologically different diagrams with four amputated external legs in terms of the vertex $\Gamma^{(3)}$ and $\Gamma^{(4)}$, and the line $W^{(2)}$. The line or the leg for the composite particle has twice the number of indices and arguments compared with that for the elementary fermion. Next we sum up these graphs to get $\tilde{W}^{(4)}$, and put the bound-state wave function on each external leg. Then we get the expression

$$\frac{1}{4!}\tilde{W}^{(4)}_{x_1,y_1;\ldots;x_4,y_4}\Delta S^{(1)}(x_1,y_1)\cdots\Delta S^{(1)}(x_4,y_4)$$
$$=\frac{1}{4!}(\Gamma^{(4)}_{x_1,y_1;\ldots;x_4,y_4}+3\Gamma^{(3)}_{x_1,y_1;x_2,y_2;x_5,y_5}W^{(2)}_{x_5,y_5;x_6,y_6}\Gamma^{(3)}_{x_6,y_6;x_3,y_3;x_4,y_4})\Delta S^{(1)}(x_1,y_1)\cdots\Delta S^{(1)}(x_4,y_4) , \quad (5.25)$$

where the indices of the components are included in the arguments such as $x_1 \equiv (i_1, x_1)$. We have used the notation

$$\Gamma^{(n)}_{x_1,y_1;\ldots;x_n,y_n} = \left[\frac{\delta^n \Gamma[\bar{\psi},\psi,S]}{\delta S(x_1,y_1)\cdots\delta S(x_n,y_n)} \right]_0 \tag{5.26}$$

evaluated at solutions (5.12) and (5.13). $W^{(2)}_{x_1,y_1;x_2,y_2}$ is the two-particle propagator which satisfies

$$\Gamma^{(2)}_{x_1,y_1;x_2,y_2} W^{(2)}_{x_2,y_2;x_3,y_3} = W^{(2)}_{x_1,y_1;x_2,y_2} \Gamma^{(2)}_{x_2,y_2;x_3,y_3} = -\delta^2(x_1-x_3)\delta^2(y_1-y_3) . \tag{5.27}$$

In the large-N limit we can write down the explicit form of $W^{(2)}$ as

$$W^{(2)}_{x_1,y_1;x_2,y_2} = -i\{S^{(0)}(x_2,y_1)S^{(0)}(x_1,y_2) + \int d^2x\, d^2y\, [S^{(0)}(x_1,x)S^{(0)}(x,y_1)]K(x,y)[S^{(0)}(x_2,y)S^{(0)}(y,y_2)]\} , \tag{5.28}$$

where $K(x,y)$ is expressed in the momentum space as

$$K(P) = \frac{(-ig^2)}{1 + ig^2 \int \frac{d^2q}{(2\pi)^2} \mathrm{tr} \left[S^{(0)}\left(q + \frac{P}{2} \right) S^{(0)}\left(q - \frac{P}{2} \right) \right]} . \tag{5.29}$$

The above procedures are equivalent to the method to calculate the S-matrix elements of the bound states as formulated by Nishijima.[10] The specific point to our method is that it directly provides the amputated Green's function $\tilde{W}^{(4)}$ which is noted as σ in Ref. 10. There, the σ function for the composite particle has been defined from the τ function (the usual Green's function) by the process in which the external legs corresponding to the composite particles are grouped together, and the two-particle propagator is amputated from each pair of legs. However, our method does not need such a process, since $\tilde{W}^{(n)}$ is constructed as the sum of the tree diagrams with amputated external legs from the beginning. The effective action Γ from which we started automatically produces the correct diagrams.

Finally, we evaluate $\Gamma^{(3)}$ and $\Gamma^{(4)}$ from the effective action (5.9) and substitute them into (5.25) together with $\Delta S^{(1)}$ and $W^{(2)}$. As a result, we get the explicit expression (see Fig. 4)

$$\begin{aligned} \frac{1}{4!} \tilde{W}^{(4)}_{x_1,y_1;\ldots;x_4,y_4} \Delta S^{(1)}(x_1,y_1)\cdots\Delta S^{(1)}(x_4,y_4) \\ = \frac{i}{4!}\{ 6\,\mathrm{tr}[S^{(0)-1}(y_4,x_1)\Delta S^{(1)}(x_1,y_1)S^{(0)-1}(y_1,x_2)\Delta S^{(1)}(x_2,y_2) \\ \times S^{(0)-1}(y_2,x_3)\Delta S^{(1)}(x_3,y_3)S^{(0)-1}(y_3,x_4)\Delta S^{(1)}(x_4,y_4)] \\ + 12\,\mathrm{tr}[\Delta S^{(1)}(x,y_1)S^{(0)-1}(y_1,x_2)\Delta S^{(1)}(x_2,x)]K(x,y)\mathrm{tr}[\Delta S^{(1)}(y,y_3)S^{(0)-1}(y_3,x_4)\Delta S^{(1)}(x_4,y)]\} . \end{aligned} \tag{5.30}$$

The functional differentiation of this expression with respect to two C^+'s and two C^-'s contained in $\Delta S^{(1)}$ yields the S-matrix elements for which we are looking.

More general S-matrix elements such as the scattering of one elementary fermion and one composite particle are also calculable. In such a case, the order of the Grassmann numbers needs careful treatment, which may bring the extra minus sign in the expressions.

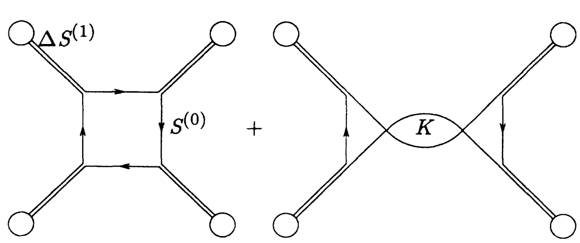

FIG. 4. Graphical representation of (5.30).

VI. DISCUSSIONS

We have presented the on-shell expansion of the effective action and have shown that the effective action generates all the physically observable quantities. Let us point out here several important issues about our investigations.

(1) $\delta\phi$ and $\Delta\phi$—the off-shell variation and the on-shell variation. As has been stated in the Introduction, the essential concept in our work is the on-shell variation $\Delta\phi$. The conventional variation $\delta\phi$ is taken for the purpose of obtaining the dynamical equation of motion. We get $\delta\Gamma[\phi]/\delta\phi(t) = -J(t) = 0$ by the free variation $\delta\phi(t)$ $(t_i < t < t_f)$ but without allowing the variation at the

boundary $\delta\phi(t_i)=\delta\phi(t_f)=0$. This is the off-shell variation since the meaning of $\delta\phi$ is the variation away from the physical trajectory.

The on-shell variation $\Delta\phi$ on the other hand stays on the physical trajectory therefore the only permissible variation is the one at the boundary; $-J(t)=0$ is always satisfied. In this way we get the different *physical* trajectories with different boundary conditions. In quantum theory, the boundary conditions are the conditions on the boundary states and indeed our on-shell condition (2.8) determines the possible boundary states which are the discrete excited spectra of the theory. The solutions $\Delta\phi$ turn out to be the wave functions of these states. The difference between the higher-order off-shell expansion and the on-shell expansion is that the former generates the 1PI *off-shell Green's functions* whereas the latter generates *the connected S-matrix elements.*

(2) Construction of Fock space. By our method the whole Fock space can be constructed starting from the effective action. In order to do that we have to couple the source J_i to some operator O_i for every channel i and calculate the effective action. Different channels are assumed to have different quantum numbers. The operator O_i can be anything as long as it has correct quantum number. The on-shell condition in each channel determines the one-particle spectrum in that channel and the entire Fock space is constructed according to our scheme of on-shell expansion.

(3) Continuous spectrum. We have assumed that the on-shell condition has only the discrete spectrum. In order to discuss the case of the continuous spectrum (scattering states) we first put the system in a large but finite box of the volume V. All the allowed spectra are now discrete and after all the calculation the limit $V\to\infty$ is taken. This is the usual process to discuss the continuum states. But in this paper we are interested in the real discrete states which remain discrete even after the limit $V\to\infty$. As has been studied in this paper the continuum region is more appropriately discussed by the scattering of these discrete bound states.

[1]R. Fukuda, Prog. Theor. Phys. **78**, 1487 (1987).
[2]R. Fukuda, M. Komachiya, and M. Ukita, Phys. Rev. D **38**, 3747 (1988). This paper is called I in the following.
[3]M. Komachiya, M. Ukita, and R. Fukuda, Phys. Rev. D **40**, 2654 (1989).
[4]D. J. Gross and A. Neveu, Phys. Rev. D **10**, 3235 (1974).
[5]See, for example, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
[6]H. Higurashi and R. Fukuda, Prog. Theor. Phys. **83**, 305 (1990).
[7]H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento **1**, 205 (1955).
[8]C. Izykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1985).
[9]C. De Dominicis and P. C. Martin, J. Math. Phys. **5**, 31 (1964).
[10]K. Nishijima, Prog. Theor. Phys. **10**, 549 (1953); **12**, 279 (1954); **13**, 305 (1955).
[11]S. Mandelstam, Proc. R. Soc. London **A233**, 248 (1955).
[12]K. Huang and H. A. Weldon, Phys. Rev. D **11**, 257 (1975); D. W. McKay and H. J. Munczek, *ibid.* **40**, 4151 (1989).
[13]J. M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D **10**, 31 (1964).
[14]Y. Nambu, Prog. Theor. Phys. **5**, 614 (1950); E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
[15]See, for a review, N. Nakanishi, Prog. Theor. Phys. Suppl. **43**, 1 (1969).