Classical adiabatic holonomy in field theory

E. Gozzi
CERN, Theory Division, Geneva, Switzerland

D. Rohrlich
School of Physics and Astronomy, Tel-Aviv University, Ramat Aviv, Tel-Aviv, Israel

W. D. Thacker
Department of Science and Mathematics, Parks College, Saint Louis University, Cahokia, Illinois 62206
(Received 9 February 1990)

In this paper we develop the notion of adiabatic holonomy in *classical* fermionic field theory and apply it to chiral gauge theory. In chiral gauge theory the *classical* adiabatic holonomy leads to a deformation of the Poisson algebra of translation generators in the space of gauge fields and to an additional term in the Poisson brackets among gauge-transformation generators. We study all this in detail in a $(1+1)$-dimensional model.

I. INTRODUCTION

An anomaly in quantum field theory is a case of a symmetry of the classical Lagrangian that cannot be maintained at the quantum level. Several authors[1-7] have recently discussed the connection between anomalies and the quantum holonomy effect known as the Berry phase.[8] The Berry phase, for quantum systems depending on external parameters, consists of a geometric phase acquired by the wave function under an adiabatic transport around a closed loop in parameter space. This phenomenon has been associated with the SU(2),[1] local gauge,[1-5] parity,[6] and conformal[7] anomalies. We shall be primarily concerned with the chiral gauge anomaly.

In chiral gauge theory we consider Weyl fermions $\psi(x)$ interacting with gauge fields $A(x)$ in space-time of dimension D. Although we ultimately want to quantize both fermions and gauge fields, for considerations regarding the anomaly it is sufficient to treat the gauge fields as classical and quantize the fermions. The gauge fields then play the role of parameters in the fermion Hamiltonian

$$H=\int d^{D-1}x\ \psi^\dagger(x) i[\nabla+\mathbf{A}(x)]\cdot\boldsymbol{\sigma}\psi(x) \qquad [A^0(x)=0]\ , \quad (1.1)$$

where σ are the Pauli matrices in $(D-1)$-dimensional space. The parameter space is $\mathcal{M}^{D-1}$, the space of all static gauge field configurations with $A^0=0$. At each point in this parameter space we can solve Dirac's equation for the first-quantized energy eigenstates, and construct the vacuum $|\Phi_0(\mathbf{A})\rangle$ by filling the negative-energy sea. In this way we obtain a Hilbert bundle of Fock-space vacua over $\mathcal{M}^{D-1}$. Gauge invariance means that the real configuration space is the manifold of gauge orbits $\mathcal{M}^{D-1}/\mathcal{G}$ where $\mathcal{G}$ is the group of gauge transformations, and hence the vacuum bundle should reduce to a bundle over orbit space. The anomaly can be viewed as an obstruction to this bundle reduction.

According to the adiabatic assumption, which will be justified in Sec. IV, the gauge fields change slowly enough so as not to induce transitions between the vacuum and excited Fock states. This provides a natural connection

$$\mathcal{A}_{\rm Berry}=\int d^{D-1}x\left\langle\Phi_0(A)\left|i\frac{\delta}{\delta A_j^a}\right|\Phi_0(A)\right\rangle\delta A_j^a \quad (1.2)$$

which allows us to probe the structure of the vacuum bundle. The integral of $\mathcal{A}_{\rm Berry}$ around a closed circuit in parameter space yields the Berry phase. A nonzero Berry phase for a loop generated by gauge transformations indicates the presence of an anomaly which prevents the reduction of the vacuum bundle to a bundle over orbit space. Stated more precisely, the Berry phase for an infinitesimal loop of gauge transformations is exactly equal to the Schwinger term in the commutator among gauge generators. If the Schwinger term (or Berry phase) cannot be removed by a redefinition of the generators, then the group of gauge transformations must be realized projectively[9] and the Gauss law cannot be maintained in quantum theory. The quantum adiabatic holonomy of the vacuum can therefore be viewed as the signal for the gauge anomaly.

It is now well known that the Berry phase has a classical counterpart, known as the Hannay angle,[10,11] which consists, for integrable systems, of a shift $\Delta\Theta(\mathcal{C})$ in the angle variables after adiabatic transport around a closed circuit $\mathcal{C}$ in parameter space. In previous work[11,12] the connection between these two phenomena was studied. They are related by

$$\frac{\partial\gamma_n(\mathcal{C})}{\partial n}=\Delta\Theta(\mathcal{C})+O(\hbar^2)\ . \quad (1.3)$$

Therefore, a *classical* Hannay angle is a sure sign of a *quantum* Berry phase different from zero. Moreover for quadratic Hamiltonians the relationship between the classical and quantum holonomies is exact, without the $O(\hbar^2)$ term.

Then a gauge theory with a nonzero Hannay angle should also have a Berry phase different from zero at the quantum level and the two should be related as in (1.3). [Moreover, since the Hamiltonian (1.1) is quadratic, the relation (1.3) should be exact without $O(\hbar^2)$ corrections, and this is in line with the Adler-Bardeen nonrenormalization theorem for chiral anomalies.] Thus we conjecture that a nonzero Hannay angle should be a *sufficient* condition for an anomaly at the quantum level. This condition would allow us to diagnose a classical theory for a propensity to develop an anomaly when quantized. For more speculations about this connection we refer the reader to the conclusions of Ref. 13.

Our present goal is more modest. We will argue that a classical theory of chiral fermions, interacting with gauge fields, exhibits a Hannay angle corresponding to the anomaly. In brief, the argument goes as follows: Ref. 1 showed that anomalies can be understood as being related to a Berry phase that arises from an *isolated* degeneracy of the vacuum. Because the degeneracy is *isolated,* a theorem, proved in Ref. 12, leads to the conclusion that different Fock states [indexed by n in Eq. (1.3)] will get different Berry phases and Eq. (1.3) will then imply a Hannay angle different from zero. The purpose of this paper is to develop the notion of Hannay's angle for a classical chiral field theory, with some explicit examples, and to investigate in detail the argument just outlined.

This paper is organized as follows. In Sec. II we establish the formalism by treating a toy model with a finite number of degrees of freedom. We show how to calculate the Hannay angles for this model and demonstrate that, in the adiabatic limit, the holonomy is responsible for a deformation of the Poisson brackets among generators of translations in parameter space. In Sec. III we show how the deformation of this Poisson algebra can modify the realization of symmetries in a special case of the toy model. Section IV describes the application of this formalism to chiral gauge theory. We derive an anomalous symplectic form, defined on the space of gauge field configurations, whose coefficients are given by the Poisson brackets among translation generators in this space. The anomalous symplectic form, restricted to a two-dimensional surface generated by gauge transformations, is related to an extra contribution to the Poisson brackets between gauge transformation generators, *reminiscent* of the Schwinger term. A semiclassical quantization rule allows us to establish a connection with quantum results.[3] In Sec. V we calculate the anomalous symplectic form explicitly for a specific model. We end with some concluding remarks.

II. A TOY MODEL IN 0+1 DIMENSIONS

Our goal is to calculate the holonomy for the classical version of the Hamiltonian (1.1). This means that, rather than being an operator, $\psi(x)$ is to be treated as a classical Grassmann-valued spinor field.

In order to get an idea of how to treat this system, let us first consider a (0+1)-dimensional toy model with a finite number of degrees of freedom. This model is a generalization of the SU(2) model of Stone.[14] We introduce N complex Grassmann variables[15] $(\psi_1 \cdots \psi_N)$ satisfying $\psi_j\psi_k+\psi_k\psi_j=0$. The Lagrangian and Hamiltonian for this system are

$$L=i\psi^\dagger\dot{\psi}-\psi^\dagger\hat{H}(B)\psi \ , \tag{2.1}$$

$$H=\psi^\dagger\hat{H}(B)\psi \ , \tag{2.2}$$

where $\psi=(\psi_1\cdots\psi_N)^T$ is a Grassmann vector, $B=(B_1\cdots B_M)$ is a set of external parameters, and $\hat{H}(B)$ is a Hermitian $N\times N$ matrix which we consider to be the first-quantized Hamiltonian. We choose the energy scale so that $\mathrm{tr}[\hat{H}(B)]=0$. Then $\hat{H}(B)$ is an element of the algebra SU(N) in one of the fundamental representations and can be written in the most general case as

$$\hat{H}(B)=\sum_{a=1}^{N^2-1}B^aT^a \ , \tag{2.3}$$

where T^a are generators of SU(N). In previous work[12] we studied the Hannay angle in a classical mechanical model, containing real Grassmann variables, of the quantum-mechanical Hamiltonian (2.3). Now we want to treat the "field-theory" Hamiltonian (2.2), obtained by inserting the first-quantized Hamiltonian $\hat{H}$ between complex Grassmann fields. We will show that the Hannay angles for H are equal to the Berry phases for $\hat{H}$.

We define, for polynomials $f(\psi^*,\psi)$ in the Grassmann fields, the Poisson brackets

$$\{f(\psi^*,\psi),g(\psi^*,\psi)\}=i\left[f\frac{\overleftarrow{\partial}}{\partial\psi_j^*}\frac{\overrightarrow{\partial}}{\psi_j}g+f\frac{\overleftarrow{\partial}}{\partial\psi_j}\frac{\overrightarrow{\partial}}{\partial\psi_j^*}g\right] , \tag{2.4}$$

where j is summed from 1 to N. In particular, we have

$$\begin{aligned}\{\psi_j,\psi_k\}&=\{\psi_j^*,\psi_k^*\}=0 \ ,\\ \{\psi_j^*,\psi_k\}&=\{\psi_k,\psi_j^*\}=i\delta_{jk} \ .\end{aligned} \tag{2.5}$$

Hamilton's equations of motion then yield

$$\begin{aligned}\dot{\psi}_j&=\{H,\psi_j\}=-i\hat{H}_{jk}\psi_k \ ,\\ \dot{\psi}_j^*&=\{H,\psi_j^*\}=i\psi_k^*\hat{H}_{kj} \ .\end{aligned} \tag{2.6}$$

Equations (2.6) correspond to the Euler-Lagrange equations derivable from (2.1) using right derivatives with respect to the Grassmann variables.

To derive the Hannay angles we hold the parameters fixed and expand the fields ψ in terms of normal modes $\tilde{\psi}$,

$$\psi_j=U_{jk}\tilde{\psi}_k; \quad \psi_j^*=\tilde{\psi}_k^*U_{kj}^\dagger \ , \tag{2.7}$$

which diagonalize the Hamiltonian and Lagrangian:

$$\begin{aligned}H&=\psi^\dagger\hat{H}\psi\\ &=\tilde{\psi}^\dagger U^\dagger\hat{H}U\tilde{\psi}\\ &=E_k\tilde{\psi}_k^*\tilde{\psi}_k \ ,\\ L&=i\tilde{\psi}_k^*\dot{\tilde{\psi}}_k-E_k\tilde{\psi}_k^*\tilde{\psi}_k \ .\end{aligned} \tag{2.8}$$

Here $E_k=U_{ki}^\dagger\hat{H}_{ij}U_{jk}$ (i and j summed) are the eigenvalues of the first-quantized Hamiltonian $\hat{H}(B)$ while the columns of the unitary matrix $U(B)$ are its eigenvectors $|k;B\rangle$. The Poisson brackets (2.4) and (2.5) are preserved by the transformation to normal modes and we find the equation of motion

$$\dot{\tilde{\psi}}_k=\{H,\tilde{\psi}_k\}=-iE_k\tilde{\psi}_k\ ,\qquad \dot{\tilde{\psi}}^*_k=\{H,\tilde{\psi}^*_k\}=iE_k\tilde{\psi}^*_k\ , \tag{2.9}$$

which have the solutions

$$\tilde{\psi}_k(t)=e^{-iE_kt}\tilde{\psi}_k(0)\ ,\qquad \tilde{\psi}^*_k(t)=e^{iE_kt}\tilde{\psi}^*_k(0)\ . \tag{2.10}$$

The normal modes have the canonical momenta

$$\tilde{\Pi}_k=\frac{\partial L}{\partial\dot{\tilde{\psi}}_k}=i\tilde{\psi}^*_k\ . \tag{2.11}$$

As a consequence, the action variables are

$$I_k=\frac{1}{2\pi}\oint dt\,\tilde{\Pi}_k(t)\dot{\tilde{\psi}}_k(t)=\tilde{\psi}^*_k\tilde{\psi}_k \tag{2.12}$$

in terms of which the instantaneous Hamiltonian is

$$H=E_kI_k\ . \tag{2.13}$$

The angle variables are given by the equations of motion

$$\dot{\Theta}_k=\frac{\partial H}{\partial I_k}=E_k \tag{2.14}$$

with solutions [we have made the choice $\Theta(0)=0$]

$$\Theta_k(t)=E_kt\ . \tag{2.15}$$

Therefore the normal modes (2.10) are related to the angle variables by

$$\tilde{\psi}_k(\Theta)=e^{-i\Theta_k}\tilde{\psi}_k(0),\quad \tilde{\psi}^*_k(\Theta)=e^{i\Theta_k}\tilde{\psi}^*_k(0)\ . \tag{2.16}$$

Now suppose we allow the parameters to change, tracing out a trajectory $\mathbf{B}(t)$ in parameter space. The transformation (2.7) then becomes time dependent and, expressing the Lagrangian (2.1) in terms of normal modes, yields

$$L=i\tilde{\psi}^*_k\dot{\tilde{\psi}}_k-E_k\tilde{\psi}^*_k\tilde{\psi}_k+\mathcal{A}\cdot\dot{\mathcal{B}} \tag{2.17}$$

which contains, in addition to (2.8), the term $\mathcal{A}\cdot\dot{\mathcal{B}}$, where

$$\begin{aligned}\mathcal{A}&\equiv i\tilde{\psi}^*_jU^\dagger_{jl}\nabla_B(U_{lk}\tilde{\psi}_k)\\ &=\tilde{\psi}^*_j\tilde{\psi}_k(iU^\dagger_{jl}\nabla_BU_{lk}-\delta_{jk}\nabla_B\Theta_k)\equiv\mathcal{A}_{jk}\tilde{\psi}^*_j\tilde{\psi}_k\end{aligned} \tag{2.18}$$

will be responsible for the holonomy. We are using Berry's approach[11] to the Hannay angle, in which the action-angle coordinates and the normal modes $(\tilde{\psi}_k(0),\tilde{\psi}^*_k(0))$ are fixed while the (ψ_k,ψ^*_k) coordinates depend on the parameters. The only dependence of Θ on B comes from the freedom to choose the origin of the angle variable independently at each point in parameter space.

Corresponding to the Lagrangian (2.17) we have the Hamiltonian

$$H=\tilde{\Pi}_j\dot{\tilde{\psi}}_j-L=E_j\tilde{\psi}^*_j\tilde{\psi}_j-\mathcal{A}\cdot\dot{\mathcal{B}}=E_jI_j-\mathcal{A}\cdot\dot{\mathcal{B}}\ . \tag{2.19}$$

The first term in H is just (2.2) reexpressed in action-angle variables while the second term comes from the time dependence of the canonical transformation to normal modes. In the adiabatic approximation we insert the instantaneous solutions (2.16) for the normal modes into (2.19) and average over the fast (angle) variables to obtain the effective Hamiltonian

$$\begin{aligned}\overline{H}&=E_jI_j-\mathcal{A}_{jk}\cdot\dot{\mathbf{B}}\int_0^{2\pi}\frac{d\Theta_1}{2\pi}\cdots\frac{d\Theta_N}{2\pi}e^{i(\Theta_j-\Theta_k)}\times\tilde{\psi}^*_j(0)\tilde{\psi}_k(0)\\ &=E_jI_j-\mathcal{A}_{jj}I_j\cdot\dot{\mathbf{B}}\\ &=E_jI_j-\overline{\mathcal{A}}\cdot\dot{\mathbf{B}}\end{aligned} \tag{2.20}$$

(with the overbar we indicate, from now on, angle-averaged quantities; for the reader not familiar with all this, we refer to the third paper of Ref. 15). The angle average has removed the off-diagonal terms from $\mathcal{A}$ so that there will be no transitions among normal modes. Now we can integrate the equation of motion

$$\dot{\Theta}_j=\frac{\partial\overline{H}}{\partial I_j}=E_j-\mathcal{A}_{jj}\cdot\dot{\mathbf{B}} \tag{2.21}$$

around a closed loop $\mathcal{C}$ in parameter space, to obtain

$$\Theta_j(T)=\Theta_j(0)+\int_0^TE_j(B(t))dt+\Delta\Theta_j(\mathcal{C})\ , \tag{2.22}$$

where (we indicate $\partial\mathcal{S}=\mathcal{C}$)

$$\Delta\Theta_j(\mathcal{C})=\oint\overline{\mathcal{A}}_j=\int\int_{\mathcal{S}}W_j \tag{2.23}$$

is the Hannay angle and

$$\overline{\mathcal{A}}_j=iU^\dagger_{jk}d_BU_{kj}+d_B\Theta_j \tag{2.24}$$

is the Hannay connection with curvature

$$W_j=id_BU^\dagger_{jk}\wedge d_BU_{kj}\ . \tag{2.25}$$

The term $d_B\Theta_j$ in $\overline{\mathcal{A}}$ does not contribute to the Hannay angle but we keep it in order to exhibit the Hannay-gauge transformation

$$\Theta_j(B)\to\Theta_j(B)+\alpha_j(B) \tag{2.26}$$

under which $\overline{\mathcal{A}}_j$ transforms like a U(1) gauge field:

$$\overline{\mathcal{A}}_j(B)\to\overline{\mathcal{A}}_j(B)-d\alpha_j(B)\ . \tag{2.27}$$

Recalling that the columns of $U(B)$ are the eigenvectors $|j;B\rangle$ of $\hat{H}$, we see that

$$iU^\dagger_{jk}(B)d_BU_{kj}(B)=i\langle j;B|d_B|j;B\rangle \tag{2.28}$$

(no sum on j) is the Berry connection. *Therefore the Hannay angle for the field-theory Hamiltonian (2.2) equals the Berry phase for the first-quantized Hamiltonian (2.3).*

Let us now promote the parameters B to dynamical variables by adding the kinetic energy $\frac{1}{2}\dot{\mathbf{B}}^2$ to the Lagrangian (2.17) to obtain

$$L=i\tilde{\psi}^*_k\dot{\tilde{\psi}}_k-E_k\tilde{\psi}^*_k\tilde{\psi}_k+\mathcal{A}\cdot\dot{\mathbf{B}}+\tfrac{1}{2}\dot{\mathbf{B}}^2\ . \tag{2.29}$$

The momentum canonically conjugate to B^a is

$$P^a \equiv \frac{\partial L}{\partial \dot{B}^a} = \dot{B}^a + \mathcal{A}^a \tag{2.30}$$

rather than the naively expected $\dot{B}^a$. We are momentarily relaxing the adiabatic assumption so that $\mathcal{A}$ has the off-diagonal form (2.18). The parameter dependence of the transformation to normal modes, which is responsible for the classical holonomy, has modified the definition of the momentum canonically conjugate to B. The symplectic form is now

$$\omega = dP^a \wedge dB^a = d\dot{B}^a \wedge dB^a + d_B\mathcal{A} \tag{2.31}$$

with $\mathcal{A} = \mathcal{A}^a dB^a$, and the Poisson brackets are

$$\begin{aligned} &\{P^a, B^b\} = \delta^{ab} , \\ &\{P^a, P^b\} = \{B^a, B^b\} = \{P^a, \tilde{\psi}_j(0)\} \\ &\qquad = \{P^a, \tilde{\psi}_j^*(0)\} = 0 . \end{aligned} \tag{2.32}$$

Now we can ask the following: What is the correct generator for translations in parameter space? Since $\{P^a, B^b\} = \{\dot{B}^a, B^b\} = \delta^{ab}$, both P and $\dot{\mathbf{B}}$ could play this role. To choose the correct generator of translations we impose the condition that it is to be covariant under *Hannay-gauge* transformations. Before we make the adiabatic approximation the different normal modes couple and therefore the shift in angle variable $\alpha(B)$ must be the same for all modes:

$$\tilde{\psi}_j \to \tilde{\psi}'_j = \tilde{\psi}_j e^{i\alpha(B)} . \tag{2.33}$$

Let us calculate

$$\{P^a, \tilde{\psi}_j\} = \{P^a, \tilde{\psi}_j(0) e^{i\Theta_j(B)}\} = i \frac{\partial \Theta_j}{\partial B^a} \tilde{\psi}_j \tag{2.34}$$

and

$$\begin{aligned} \{P^a, \tilde{\psi}'_j\} &= e^{i\alpha(B)} i \left[\frac{\partial \Theta_j}{\partial B^a} + \frac{\partial \alpha}{\partial B^a} \right] \tilde{\psi}_j \\ &= e^{i\alpha(B)} \left[\{P^a, \tilde{\psi}_j\} + i \frac{\partial \alpha}{\partial B^a} \tilde{\psi}_j \right] . \end{aligned} \tag{2.35}$$

Therefore P^a is not *Hannay-gauge covariant.* On the other hand, since

$$\begin{aligned} &\dot{B}^a = P^a - \mathcal{A}^a , \\ &\{\mathcal{A}^a, \tilde{\psi}_j\} = \mathcal{A}^a_{lk} \{\tilde{\psi}_l^* \tilde{\psi}_k, \tilde{\psi}_j\} \\ &\qquad = -i \mathcal{A}^a_{jk} \tilde{\psi}_k \\ &\qquad = -i \left[i U^\dagger_{jl} \frac{\partial U_{lk}}{\partial B^a} - \delta_{jk} \frac{\partial \Theta_k}{\partial B^a} \right] \tilde{\psi}_k , \end{aligned} \tag{2.36}$$

we find

$$\{\dot{B}^a, \tilde{\psi}_j\} = -U^\dagger_{jl} \frac{\partial U_{lk}}{\partial B^a} \tilde{\psi}_k \tag{2.37}$$

and

$$\{\dot{B}^a, \tilde{\psi}'_j\} = e^{i\alpha} \{\dot{B}^a, \tilde{\psi}_j\} . \tag{2.38}$$

Therefore the velocity $\dot{B}^a$ is the *Hannay-covariant* generator of translations in parameter space.

Now let us calculate the Poisson brackets between two such translation generators:

$$\begin{aligned} \{\dot{B}^a, \dot{B}^b\} &= \{P^a - \mathcal{A}^a, P^b - \mathcal{A}^b\} \\ &= -\left[\frac{\partial \mathcal{A}^b}{\partial B^a} - \frac{\partial \mathcal{A}^a}{\partial B^b} \right] + \{\mathcal{A}^a, \mathcal{A}^b\} \end{aligned} \tag{2.39}$$

with

$$\begin{aligned} \{\mathcal{A}^a, \mathcal{A}^b\} &= \mathcal{A}^a_{jk} \mathcal{A}^b_{lm} \{\tilde{\psi}_j^* \tilde{\psi}_k, \tilde{\psi}_l^* \tilde{\psi}_m\} \\ &= i \tilde{\psi}_j^* \tilde{\psi}_m \left[-\left[U^\dagger \frac{\partial U}{\partial B^a} \right]_{jk} \left[U^\dagger \frac{\partial U}{\partial B^b} \right]_{km} + \left[U^\dagger \frac{\partial U}{\partial B^b} \right]_{jk} \left[U^\dagger \frac{\partial U}{\partial B^a} \right]_{km} \right] \\ &= i \tilde{\psi}_j^* \tilde{\psi}_m \left[\frac{\partial U^\dagger}{\partial B^a} \frac{\partial U}{\partial B^b} - \frac{\partial U^\dagger}{\partial B^b} \frac{\partial U}{\partial B^a} \right]_{jm} \\ &= \frac{\partial \mathcal{A}^b}{\partial B^a} - \frac{\partial \mathcal{A}^a}{\partial B^b} \end{aligned} \tag{2.40}$$

and therefore

$$\{\dot{B}^a, \dot{B}^b\} = 0 . \tag{2.41}$$

Since the translation generator differs from the canonical momentum by $\mathcal{A}$, which is like a gauge field, we would have expected the Poisson brackets between translations to yield the curvature of $\mathcal{A}$. However this curvature gets compensated for by the fermionic contribution $\{\mathcal{A}, \mathcal{A}\}$ to the Poisson brackets. Actually, this result should not be surprising. If we had added the kinetic energy term $\frac{1}{2}\dot{\mathbf{B}}^2$ to the Lagrangian (2.1), expressed in terms of the original variables ψ, we would have found that $\dot{B}$ is the canonical momentum and it obviously satisfies (2.41). The transformation

$$(\psi, \psi^\dagger, B, \dot{B}) \to (\tilde{\psi}, \tilde{\psi}^*, B, \dot{B} + \mathcal{A}) \tag{2.42}$$

is canonical and leaves the Poisson brackets invariant; it does not change the physics.

It is the adiabatic assumption that changes the physics. Let us average over the fast variables in $\dot{B} = P - \mathcal{A}$ to obtain the adiabatic translation generator

$$\bar{\dot{B}}^a = P^a - \bar{\mathcal{A}}^a , \tag{2.43}$$

where we still take P^a to be canonically conjugate to B^a. Since $\bar{\mathcal{A}}^a = \mathcal{A}^a_j I_j$ and $\{I_j, I_k\} = 0$, there is no longer a fermionic contribution to the Poisson brackets of adiabatic translations and we obtain

$$\{\bar{\dot{B}}^a, \bar{\dot{B}}^b\} = -\left[\frac{\partial \bar{\mathcal{A}}^b}{\partial B^a} - \frac{\partial \bar{\mathcal{A}}^a}{\partial B^b} \right] = -\tilde{\omega}_{ab} . \tag{2.44}$$

We call

$$\tilde{\omega} = \tfrac{1}{2} \tilde{\omega}_{ab} dB^a \wedge dB^b = d\bar{\mathcal{A}} \tag{2.45}$$

the "anomalous" symplectic form. It is an additional term in the symplectic form on the slow variable phase space[13] induced by the adiabatic holonomy, and it is related to the Hannay curvature by

$$W_j = \frac{\partial \tilde{\omega}}{\partial I_j} \ . \tag{2.46}$$

We have learned the following lessons from our toy model.

(a) The Hannay angle of *classical* field theory equals the Berry phase of the first-quantized Hamiltonian.

(b) Even before we make the adiabatic approximation, the transformation to normal modes changes the definition of the canonical momentum conjugate to the parameters, so that it differs from the Hannay-covariant translation generator. However, the translation generators still Poisson commute.

(c) The adiabatic average over fast variables "deforms" the Poisson brackets among translation generators and introduces a modification of the symplectic structure of the slow variable phase space.[13] (Kuratsuji and Iida[16] have also derived a deformation of the symplectic structure by treating the fast variables *quantum mechanically* and integrating them out of the path integral. In contrast, our treatment is purely *classical.*)

III. ROTATION SYMMETRY OF THE TOY MODEL

Let us now investigate how the modification of the symplectic structure of the slow variable phase space, found in the previous section, can influence the realization of symmetries. We consider rotational symmetry in the model (2.1) specialized to the case where there are just two complex Grassmann variables (ψ_1, ψ_2) and the first-quantized Hamiltonian

$$\hat{H}(\mathbf{B}) = \mathbf{B}\cdot\boldsymbol{\sigma} = \sum_{a=1}^{3} B_a \sigma_a \tag{3.1}$$

is a generator of SU(2). Jackiw[17] and Stone and Goff[4] have made similar analyses of this model at the quantum level. In this case we have an explicit expression[14] for the transformation matrix to normal modes:

$$U = \begin{pmatrix} \cos\left(\frac{\Theta}{2}\right) & -e^{-i\phi}\sin\left(\frac{\Theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\Theta}{2}\right) & \cos\left(\frac{\Theta}{2}\right) \end{pmatrix} , \tag{3.2}$$

valid everywhere except at $\Theta = \pi$. We have introduced here spherical coordinates on parameter space related to the Cartesian coordinates by

$$B_1 = B \sin\Theta \cos\phi \ ,$$

$$B_2 = B \sin\Theta \sin\phi \ ,$$

$$B_3 = B \cos\Theta \ .$$

Promoting the parameters $\mathbf{B}$ to dynamical variables we derive from the Lagrangian (2.29) the Hamiltonian

$$H = E_k \tilde{\psi}_k^* \tilde{\psi}_k + \tfrac{1}{2}(\mathbf{P} - \mathcal{A})^2 \ ,$$

where $E_1 = B = |\mathbf{B}|$, $E_2 = -|\mathbf{B}|$. After the adiabatic average this Hamiltonian becomes

$$\overline{H} = BI_1 - BI_2 + \tfrac{1}{2}(\mathbf{P} - \overline{\mathcal{A}})^2 \ , \tag{3.3}$$

where

$$\begin{aligned} \overline{\mathcal{A}} &= iI_j U^\dagger_{jk} d_B U_{kj} \\ &= -I_1 \sin^2(\Theta/2) d\phi + I_2 \sin^2(\Theta/2) d\phi \end{aligned} \tag{3.4}$$

or, in terms of Cartesian coordinates,

$$\overline{\mathcal{A}} = -\left(\frac{I_1 - I_2}{2}\right)\left(1 - \frac{B_3}{B}\right)\frac{B_1 dB_2 - B_2 dB_1}{B_1^2 + B_2^2} \ . \tag{3.5}$$

The anomalous symplectic form is given by

$$\tilde{\omega} = \tfrac{1}{2}\tilde{\omega}_{ab} dB_a \wedge dB_b = d\overline{\mathcal{A}} = -\frac{I}{2}\varepsilon^{abc}\frac{B_a dB_b \wedge dB_c}{B^3} \tag{3.6}$$

with $I = \frac{1}{2}(I_1 - I_2)$. We recognize (3.6) as the field of a monopole of strength I located at $B = 0$, the point in parameter space where the two normal modes $\tilde{\psi}_1$ and $\tilde{\psi}_2$ become degenerate.

The adiabatic translation generators are $\overline{\mathcal{B}}^a = P^a - \overline{\mathcal{A}}^a$; they satisfy

$$\{\overline{\mathcal{B}}^a, \overline{\mathcal{B}}^b\} = -\tilde{\omega}_{ab} = I\varepsilon^{abc}\frac{B_c}{B^3} \ . \tag{3.7}$$

The corresponding "rotation" generators are $\overline{L}^a = \varepsilon^{abc} B_b \overline{\mathcal{B}}_c$. As a consequence of (3.7) the "angular momentum" $\overline{\mathbf{L}}$ is not conserved,

$$\{H, \overline{L}^a\} = I\left(\frac{\overline{\mathcal{B}}^a}{B} - \frac{B^a \mathbf{B}\cdot\overline{\boldsymbol{\mathcal{B}}}}{B^3}\right) , \tag{3.8}$$

and satisfies, instead of the usual Poisson brackets,

$$\{\overline{L}^a, \overline{L}^b\} = \varepsilon^{abc}\overline{L}^c + \varepsilon^{abc}\frac{B^c}{B} I \ . \tag{3.9}$$

Therefore it looks as if the rotational invariance has been destroyed by the adiabatic average. However,[4] it is well known that the additional terms in (3.8) and (3.9) can be remedied by adding to $\overline{L}^a$ the term IB^a/B, which is finite and nowhere singular for $\mathbf{B} \neq 0$. The new rotation generators

$$J^a = \varepsilon^{abc} B^b \overline{\mathcal{B}}^c + I\frac{B^a}{B} \tag{3.10}$$

satisfy

$$\{H, J^a\} = 0, \quad \{J^a, J^b\} = -\varepsilon^{abc} J^c \ .$$

Thus in this model, which has a finite number of degrees of freedom (two), it is possible to preserve the symmetry by adding a finite and everywhere well-defined term to the generators. This term is in fact the relic of the spin density $S^a = \frac{1}{2}\psi^\dagger \sigma^a \psi$, left over after the adiabatic average.

IV. CLASICAL ADIABATIC HOLONOMY IN CHIRAL GAUGE THEORY

Let us now consider the classical theory of chiral fermions interacting with gauge fields. The action for the fermions (in the gauge $A^0=0$) is

$$\mathcal{S}=\int dx\, i\psi^\dagger(x)[\partial_t-(\nabla+\mathbf{A})\cdot\sigma]\psi(x)\ , \tag{4.1}$$

where $dx=d^Dx$, σ are the Pauli matrices in $(D-1)$-dimensional space, $A_j=A_j^a\lambda^a$, and λ^a are anti-Hermitian generators of the gauge group satisfying

$$[\lambda^a,\lambda^b]=f^{abc}\lambda^c,\quad \mathrm{tr}(\lambda^a\lambda^b)=-\tfrac{1}{2}\delta^{ab}\ . \tag{4.2}$$

When the gauge fields are held fixed this system is integrable in the fermions. We can expand in normal modes

$$\psi(x)=\sum_n \tilde{\psi}_n(t)u_n(\mathbf{x};\mathbf{A})\ , \tag{4.3}$$

where

$$\begin{aligned}\hat{H}(\mathbf{A})u_n(\mathbf{x},\mathbf{A})&=i(\nabla+\mathbf{A})\cdot\sigma u_n(\mathbf{x};\mathbf{A})\\&=E_n(\mathbf{A})u_n(\mathbf{x};\mathbf{A})\end{aligned} \tag{4.4}$$

and $\tilde{\psi}_n(t)$ are complex Grassmann variables. The orthonormal set of eigenfunctions $u_n(\mathbf{x};\mathbf{A})$ plays the same role as the unitary matrix U of Sec. II, transforming between the field $\psi(x,t)$ and the normal modes $\tilde{\psi}_n(t)$.

Inserting the expansion (4.3) into (4.1) we find

$$\mathcal{S}=\int dt(i\tilde{\psi}_n^*\dot{\tilde{\psi}}_n-E_n\tilde{\psi}^*\tilde{\psi}_n) \tag{4.5}$$

for fixed $\mathbf{A}(x)$. The Lagrangian is an infinite-dimensional version of (2.8). Therefore, as in Sec. II, we find the actions

$$I_n=\tilde{\psi}_n^*\tilde{\psi}_n \tag{4.6}$$

and the fixed parameter Hamiltonian

$$H=\sum_n E_nI_n\ . \tag{4.7}$$

The angles are then given by

$$\Theta_n(t)=E_nt \tag{4.8}$$

in terms of which the normal modes are given by

$$\tilde{\psi}_n(\Theta)=\tilde{\psi}_n(0)e^{-i\Theta_n},\quad \tilde{\psi}_n^*(\Theta)=\tilde{\psi}_n^*(0)e^{i\Theta_n}\ . \tag{4.9}$$

When the gauge field is held fixed in time, we have a free theory of independent fermionic oscillators.

When the gauge fields are allowed to evolve in time the normal modes become coupled and we find, upon insertion of (4.3) into (4.1), that

$$\mathcal{S}=\int dt\left[i\tilde{\psi}_n^*\dot{\tilde{\psi}}_n-E_n\tilde{\psi}_n^*\tilde{\psi}_n+\int dx\ \dot{A}_j^a(x)\mathcal{A}^{aj}(x)\right]\ . \tag{4.10}$$

Again we have found the holonomy term

$$\begin{aligned}\mathcal{A}^{aj}&=i\tilde{\psi}_n^*\tilde{\psi}_m\int d\mathbf{y}\,u_n^\dagger(\mathbf{y},\mathbf{A})\frac{\delta u_m(\mathbf{y};\mathbf{A})}{\delta A_j^a}\\&=i\tilde{\psi}_n^*\tilde{\psi}_m\langle n|\delta_{A_j^a(x)}|m\rangle\\&=\tilde{\psi}_n^*\tilde{\psi}_m\mathcal{A}_{nm}^{aj}(x)\end{aligned} \tag{4.11}$$

coming from the parameter dependence of the transformation to normal modes. We have omitted the term $\tilde{\psi}_n^*\tilde{\psi}_n\delta_A\Theta_n$ from $\mathcal{A}$ since it does not contribute to the dynamics; we know that, when we include it, $\mathcal{A}$ has the correct Hannay-gauge transformation properties discussed in Sec. II.

Now let us promote the gauge fields to dynamical variables by adding to $\mathcal{S}$ the Yang-Mills action (in the gauge $A^0=0$)

$$\begin{aligned}S_{\mathrm{YM}}&=\frac{1}{2g^2}\int dx\ \mathrm{Tr}(F_{uv}F^{uv})\\&=\int dx\left[\frac{1}{g^2}\dot{A}_j^a(\mathbf{x},t)F^{aj0}(x)-\mathcal{H}_{\mathrm{YM}}(x)\right]\end{aligned} \tag{4.12}$$

with the Hamiltonian density

$$\mathcal{H}_{\mathrm{YM}}=\frac{1}{2g^2}\left[-F_{0j}^aF^{a0j}+\sum_{i<j}F_{ij}^a(x)F^{aij}(x)\right]\ . \tag{4.13}$$

The total action of fermions interacting with gauge fields is then

$$\tilde{\mathcal{S}}=\int dt\left[\tilde{\psi}_n^*\dot{\tilde{\psi}}_n-E_n\tilde{\psi}_n^*\tilde{\psi}_n+\int d\mathbf{x}\ \dot{A}_j^a(\mathbf{x},t)\left[\frac{1}{g^2}F^{aj0}(\mathbf{x})+\mathcal{A}_{aj}(\mathbf{x})\right]-\mathcal{H}_{\mathrm{YM}}\right] \tag{4.14}$$

with $d\mathbf{x}=d^{D-1}x$. We find that in these coordinates the momentum canonically conjugate to the gauge field is

$$\tilde{P}^{aj}(\mathbf{x})=\frac{1}{g^2}F^{aj0}(\mathbf{x})+\mathcal{A}^{aj}(\mathbf{x}) \tag{4.15}$$

rather than the usual $(1/g^2)F^{aj0}$. In writing the action in terms of normal modes we have performed a canonical transformation

$$\left[\psi(\mathbf{x}),\psi^\dagger(\mathbf{x}),A_j^0(\mathbf{x}),\frac{1}{g^2}F^{aj0}(\mathbf{x})\right]\to(\tilde{\psi}_n,\tilde{\psi}_n^*,A_j^a(\mathbf{x}),\tilde{P}^{aj}(\mathbf{x}))\ . \tag{4.16}$$

In the new coordinates the Poisson brackets are

$$\begin{aligned}&\{\tilde{P}^{aj}(\mathbf{x}),\tilde{P}^{bk}(\mathbf{y})\}=\{A_j^a(\mathbf{x}),A_k^b(\mathbf{y})\}=0\ ,\\&\{\tilde{P}^{aj},\tilde{\psi}_n(0)\}=\{\tilde{P}^{aj},\tilde{\psi}_n^*(0)\}\\&\qquad=\{A,\tilde{\psi}_n(0)\}=\{A,\tilde{\psi}_n^*(0)\}=0\ ,\\&\{\tilde{P}^{aj}(\mathbf{x}),A_k^b(\mathbf{y})\}=\delta^{ab}\delta_{jk}\delta(\mathbf{x}-\mathbf{y})\ ,\end{aligned} \tag{4.17}$$

while the fermionic Poisson brackets are an infinite-dimensional generalization of (2.4), expressed in terms of normal modes. In the transformation (4.3) we have put all the gauge field dependence in $u_n(\mathbf{x}, \mathbf{A})$ so that $\tilde{\psi}_n$ and $\tilde{\psi}_n^*$ are independent of A (ignoring the Hannay-gauge freedom). This was motivated by Berry's approach to the Hannay angle as explained in Sec. II. What we have done here is unconventional because now the fermion field ψ depends on A. If we had treated $\psi(\mathbf{x})$ and $A(\mathbf{x})$ as independent dynamical variables, we would, of course, have obtained no holonomy term in the action. We therefore find it necessary to verify that the equations of motion of the new variables are equivalent to the usual ones of Yang-Mills theory. This is done in the Appendix.

As in Sec. II the Hannay covariant generator of translations in $\mathcal{M}^{D-1}$ is

$$\mathcal{E}^{aj}(\mathbf{x})=\tilde{P}^{aj}(\mathbf{x})-\mathcal{A}^{aj}(\mathbf{x}) \tag{4.18}$$

and it satisfies

$$\{\mathcal{E}^{aj}(\mathbf{x}),\mathcal{E}^{bk}(\mathbf{y})\}=0 \tag{4.19}$$

before we invoke the adiabatic assumption. The corresponding Hannay-covariant generator of gauge translations, i.e., of translations along orbits of $\mathcal{G}$ in $\mathcal{M}^{D-1}$, is $D_j^{ab}\mathcal{E}^{bj}(\mathbf{x})$, satisfying ($D_j^{ab}$ is the usual covariant derivative)

$$\{D_j^{ac}\mathcal{E}^{cj}(\mathbf{x}),D_k^{bd}\mathcal{E}^{dk}(\mathbf{y})\}=-f^{abc}\delta(\mathbf{x}-\mathbf{y})D_k^{cd}\mathcal{E}^{dk}(\mathbf{y}) \ . \tag{4.20}$$

Now we are interested in the evolution of the fermionic system as it is carried around a loop in $\mathcal{M}^{D-1}$ generated by gauge transformations. Transport around such a loop can be effected by a sequence of infinitesimal gauge transformations $\Omega(\mathbf{x},t)$ under which A^0 transforms as

$$A^0=0\rightarrow A'^0=\Omega^\dagger\partial_t\Omega(\mathbf{x},t) \ . \tag{4.21}$$

Since we want to preserve that gauge $A^0=0$, we are led to the adiabatic limit $\partial_t\Omega(\mathbf{x},t)\rightarrow 0$. Thus transport around a circuit generated by gauge transformations must be adiabatic.

Since the evolution in $\mathcal{M}^{D-1}$ is adiabatic we can average over the fast variables to obtain the Hannay-covariant generators of adiabatic translations

$$\overline{\mathcal{E}}^{aj}=P^{aj}-\overline{\mathcal{A}}^{aj}(\mathbf{x}) \tag{4.22}$$

with

$$\overline{\mathcal{A}}^{aj}(\mathbf{x})=\sum_n I_n\langle n|i\delta_{A_j^a(\mathbf{x})}|n\rangle \ . \tag{4.23}$$

They satisfy

$$\{\overline{\mathcal{E}}^{aj}(\mathbf{x}),\overline{\mathcal{E}}^{bk}(\mathbf{y})\}=-\tilde{\omega}_{ab}^{jk}(\mathbf{x},\mathbf{y}) \ , \tag{4.24}$$

where

$$\begin{aligned}\tilde{\omega}_{ab}^{jk}(\mathbf{x},\mathbf{y})&=\delta_{A_j^a(\mathbf{x})}\overline{\mathcal{A}}^{bk}(\mathbf{y})-\delta_{A_k^b(\mathbf{y})}\overline{\mathcal{A}}^{aj}(\mathbf{x})\\&=\sum_n I_n i(\langle\delta_{A_j^a(\mathbf{x})}n|\delta_{A_k^b(\mathbf{y})}n\rangle\\&\qquad-\langle\delta_{A_k^b(\mathbf{y})}n|\delta_{A_j^a(\mathbf{x})}n\rangle)\end{aligned} \tag{4.25}$$

are the coefficients of the anomalous symplectic form

$$\tilde{\omega}=\tfrac{1}{2}\int d\mathbf{x}\,d\mathbf{y}\,\delta A_j^a(\mathbf{x})\wedge\delta A_k^b(\mathbf{y})\tilde{\omega}_{ab}^{jk}(\mathbf{x},\mathbf{y})=\delta\mathcal{A} \ . \tag{4.26}$$

For a general circuit in $\mathcal{M}^{D-1}$, the Hannay angle for a normal mode $\tilde{\psi}_n$ is related to the anomalous symplectic form by

$$\Delta\Theta_n=\int\int_{\mathcal{S}}\frac{\partial\tilde{\omega}}{\partial I_n} \ . \tag{4.27}$$

For transport along gauge orbits in $\mathcal{M}^{D-1}$, the adiabatic Hannay-covariant generators of gauge translations are then $D_k^{ab}\overline{\mathcal{E}}^{bk}(\mathbf{x})$ and, due to the nonvanishing Poisson brackets (4.24), they satisfy

$$\begin{aligned}&\{D_j^{ac}\overline{\mathcal{E}}^{cj}(\mathbf{x}),D_k^{bd}\overline{\mathcal{E}}^{dk}(\mathbf{y})\}\\&\quad=-f^{abc}\delta(\mathbf{x}-\mathbf{y})D_k^{cd}\overline{\mathcal{E}}^{dk}(\mathbf{y})-D_j^{ac}(\mathbf{x})D_k^{bd}(\mathbf{y})\tilde{\omega}_{cd}^{jk}(\mathbf{x},\mathbf{y}) \ .\end{aligned} \tag{4.28}$$

The Poisson brackets between gauge transformation generators have now acquired the additional term $DD\tilde{\omega}$, which is reminiscent of the Schwinger term at the quantum level. (A similar result was also obtained by Kuratsuji and Iida[16] but *treating the fermions quantum mechanically.*)

It is easy to see that a nonvanishing symplectic form on a two-dimensional surface in $\mathcal{M}^{D-1}$ generated by gauge transformations, implies the presence of the additional term in (4.28). Consider a surface generated by a two-parameter set of gauge transformations $g(\mathbf{x};s)=g(\mathbf{x};s_1,s_2)$. On this surface

$$\delta A=Dv\equiv Dv^{(\rho)}ds_{(\rho)} \ , \tag{4.29}$$

where

$$v^{(\rho)}=g^{-1}(\mathbf{x};s)\partial_{s_{(\rho)}}g(\mathbf{x};s)=v_a^{(\rho)}\lambda^a \ ,$$

$\rho=1,2$, a is the gauge index, and D is the covariant derivative with gauge field A. Inserting (4.29) into (4.26) and partially integrating twice, we obtain

$$\tilde{\omega}=\tfrac{1}{2}\int d\mathbf{x}\,d\mathbf{y}\,v_a(\mathbf{x})\wedge v_b(\mathbf{y})D_j^{ac}(\mathbf{x})D_k^{bd}(y)\tilde{\omega}_{c}^{jk}(\mathbf{x},\mathbf{y}) \ . \tag{4.30}$$

This is as far as we can go in looking for *relics* of the anomaly at the purely classical level. We are now confronted with the problem that the action variables $I_n=\tilde{\psi}_n^*\tilde{\psi}_n$, which appear in the anomalous symplectic form $\tilde{\omega}$, have no obvious interpretation. They are not real numbers but rather even elements of the Grassmann algebra. In order to make physical sense of the actions we need to find a natural way to assign real numbers to them. The physical meaning of the Grassmann variables $\tilde{\psi}_n$ is that they carry the anticommuting nature of the fermions and they give an amplitude and phase to a mode of oscillations, i.e., $\tilde{\psi}_n=a_n\hat{\psi}_n$, where a_n is a complex number and $\hat{\psi}_n$ is some Grassmann unit. The actions are then $I_n=|a_n|^2\hat{\psi}_n^*\hat{\psi}_n$, where $\hat{\psi}_n^*\hat{\psi}_n$ should be mapped into a real number $\|\hat{\psi}_n^*\hat{\psi}_n\|$ of absolute magnitude 1. To decide whether $\|\hat{\psi}_n^*\hat{\psi}_n\|$ is 1 or -1 we can bring in the physical principle that *the energy must be bounded from below.*

Since the amplitudes $|a_n|^2$ in

$$E=\sum_n E_n|a_n|^2\|\hat{\psi}_n^*\hat{\psi}_n\|$$

are arbitrary, the energy will only be bounded from below if

$$\|\hat{\psi}_n^*\hat{\psi}_n\|=\text{sgn}(E_n)\ . \tag{4.31}$$

That means that the actions have to be positive for positive-energy modes and negative for negative-energy modes. So we have

$$I_n=|a_n|^2\text{sgn}(E_n)\quad(\text{classical})\ . \tag{4.32}$$

This prescription (4.32) is reminiscent of the filling of the Dirac sea at the quantum level. It can also be interpreted as the choice of a *complex structure*[18] necessary for the geometric quantization of the classical theory. In a forthcoming work we shall investigate geometric quantization more thoroughly and the relationship between classical and quantum holonomy in field theory. We mention here that quantizing semiclassically[19] gives each normal mode a zero-point action

$$I_n=\tfrac{1}{2}\hbar\,\text{sgn}(E_n) \tag{4.33}$$

so that the coefficients of the anomalous symplectic form become

$$\tilde{\omega}_{ab}^{jk}=i\frac{\hbar}{2}\sum_{n,m}[\text{sgn}(E_n)-\text{sgn}(E_m)]\langle\delta_{A_j^a(\mathbf{x})}n|m\rangle \times\langle m|\delta_{A_k^b(\mathbf{y})}n\rangle\ . \tag{4.34}$$

This is a formal expression which needs to be regularized; it agrees precisely with the Berry curvature of the Fock vacuum found by Niemi and Semenoff.[3]

V. AN EXPLICIT EXAMPLE IN 1+1 DIMENSIONS

In this section we calculate the anomalous symplectic form on a two-dimensional gauge orbit for a specific model in 1+1 dimensions. The model consists of a chiral fermion $\psi(x)$, on a compactified space with coordinate $0\le x\le 2\pi$, interacting with a three-parameter set of gauge fields $A(x;r,\theta,\phi)$. Various versions of this model have been considered by Forte,[20] Chang and Liang,[21] and by Hosono.[5] The gauge fields are given by

$$\begin{aligned}A^0&=0\ ,\\ A^1&=A(x;r,\theta,\phi)\\ &=rg(x;\theta,\phi)\partial_x g^{-1}(x;\theta,\phi)\\ &=\frac{r}{2}\begin{bmatrix} i(1-\cos\theta) & -\sin\theta e^{-i(x-\phi)}\\ \sin\theta e^{i(x-\phi)} & -i(1-\cos\theta)\end{bmatrix}\ ,\end{aligned} \tag{5.1}$$

where $0\le r\le 1$, $0\le\theta\le\pi$, $0\le\phi\le 2\pi$, and

$$g(x;\theta,\phi)=e^{-i\sigma_3 x/2}e^{\mathbf{n}\cdot\sigma x} \tag{5.2}$$

with $\mathbf{n}=(\sin\theta\sin\theta,\ \sin\theta\cos\phi,\ \cos\theta)$.

At $r=1$ the set of gauge fields $A(x;1,\theta,\phi)\equiv A(x;\theta,\phi)$ span a two-dimensional gauge orbit. We want to calculate the anomalous symplectic form on this orbit. The Lagrangian for the fermion is

$$L=\psi^\dagger(x,t)(i\partial_t-\hat{H})\psi(x,t) \tag{5.3}$$

with the first-quantized Hamiltonian[21]

$$\hat{H}=-i\partial_x+\tfrac{1}{2}+r\sin\frac{\theta}{2}\begin{bmatrix}\sin\dfrac{\theta}{2} & i\cos\dfrac{\theta}{2}e^{-i(x-\phi)}\\ -i\cos\dfrac{\theta}{2}e^{i(x-\phi)} & -\sin\dfrac{\theta}{2}\end{bmatrix}\ .$$

The eigenvalues of $\hat{H}$ are given by

$$E_{n,\pm}=n\pm\left[\frac{1}{4}+r(r-1)\sin^2\frac{\theta}{2}\right]^{1/2} \tag{5.4}$$

and the eigenstates become degenerate in pairs at $r=\frac{1}{2}$, $\theta=\pi$. This degeneracy is the source of the anomalous symplectic form on the surface $r=1$. (Henceforth we shall restrict our considerations to this surface.) At $r=1$ the eigenstates (these eigenstates are not well defined at $\theta=\pi$, but we would circumvent this problem by defining the eigenstates on patches[21]) are

$$\begin{aligned}u_{n,+}&=\begin{bmatrix} i\sin\dfrac{\theta}{2}e^{i(\phi-x)}\\ \cos\dfrac{\theta}{2}\end{bmatrix}\frac{e^{inx}}{\sqrt{2\pi}}\ ,\\ u_{n,-}&=\begin{bmatrix}-i\cos\dfrac{\theta}{2}e^{-ix}\\ \sin\dfrac{\theta}{2}e^{-i\phi}\end{bmatrix}\frac{e^{inx}}{\sqrt{2\pi}}\ ,\end{aligned} \tag{5.5}$$

where $n\in\mathbb{Z}$. To find the anomalous symplectic form $\tilde{\omega}$ we expand the fermion

$$\psi(x,t;\theta,\phi)=\sum_{n=-\infty,\ \alpha=\pm}^{\infty}\tilde{\psi}_{n,\alpha}(t)u_{n,\alpha}(x;\theta,\phi) \tag{5.6}$$

in normal modes

$$\tilde{\psi}_{n,\alpha}=\tilde{\psi}_{n,\alpha}(0)e^{i\Theta_{n,\alpha}(t)}$$

with frequencies

$$\dot{\Theta}_{n,\pm}(t)=E_{n,\pm}=n\pm\tfrac{1}{2}$$

and actions

$$I_{n,\alpha}=\tilde{\psi}_{n,\alpha}^*\tilde{\psi}_{n,\alpha}\ .$$

Inserting (5.6) into the Lagrangian (5.3) (with $r=1$ fixed) we get

$$L=\sum_{n,\alpha}(i\tilde{\psi}_{n,\alpha}^*\dot{\tilde{\psi}}_{n,\alpha}-E_{n,\alpha}\tilde{\psi}_{n,\alpha}^*\tilde{\psi}_{n,\alpha})+\dot{\theta}\mathcal{A}_\theta+\dot{\phi}\mathcal{A}_\phi \tag{5.7}$$

with

$$\begin{aligned}\mathcal{A}_\theta&=i\int dx\ \psi^\dagger(x;\theta,\phi)\partial_\theta\psi(x;\theta,\phi)\ ,\\ \mathcal{A}_\phi&=i\int dx\ \psi^\dagger(x;\theta,\phi)\partial_\phi\psi(x;\theta,\phi)\ .\end{aligned} \tag{5.8}$$

Taking then the exterior derivative $\delta \equiv d\theta\,\partial_\theta + d\phi\,\partial_\phi$ of the one-form

$$\mathcal{A} = \mathcal{A}_\theta d\theta + \mathcal{A}_\phi d\phi = i\int dx\, \psi^\dagger(x;\theta,\phi)\delta\psi(x;\theta,\phi) \tag{5.9}$$

and adiabatically averaging, yields the anomalous symplectic form.

There is a subtlety here: note that the normal modes $\tilde{\psi}_{n,+}$ and $\tilde{\psi}_{n+1,-}$ have the same frequency. This means that in the adiabatic average (which is really a time average) it happens that

$$\int dt\, \exp\{i[\theta_{n,+}(t) - \theta_{n+1,-}(t)]\} \neq 0$$

and there should be mixing between the degenerate modes. However, it turns out that for the choice of eigenstates (5.6), the matrix elements

$$\langle n+1,-;\theta,\phi|\delta|n,+;\theta,\phi\rangle$$

and

$$\langle n,+;\theta,\phi|\delta|n+1,-;\theta,\phi\rangle$$

vanish for all n due to an uncanceled exponential factor $e^{i(\pm x)}$ which gives zero when integrated. This would not be true for other choices of eigenstates, but the point is that it is possible to choose normal modes which do not mix under adiabatic transport on the sphere $r=1$. Using (5.5) and (5.6) we therefore obtain

$$\tilde{\omega} = i\int dx\, \delta\psi^\dagger(x;\theta,\phi)\wedge\delta\psi(x;\theta,\psi) = \tfrac{1}{2}\sin\theta\, d\theta\wedge d\phi \sum_{n=-\infty}^{\infty}(-I_{n,+} + I_{n,-}) . \tag{5.10}$$

This is the field of a "magnetic monopole" in parameter space (r,θ,ϕ). Each mode $\tilde{\psi}_{n,+}$ contributes a monopole charge of $-\frac{1}{2}I_{n,+}$ while each mode $\tilde{\psi}_{n,-}$ contributes a charge $\frac{1}{2}I_{n,-}$. If we use the semiclassical quantization condition (4.33) to make the transition to the quantum theory, all of the terms in (5.10) will cancel in pairs except for the $n=0$ terms which combine to give

$$\tilde{\omega} = -\frac{\hbar}{2}\sin\theta\, d\theta\wedge d\phi \quad (\text{quantum}) . \tag{5.11}$$

Thus the two levels which cross at zero energy when $r=\frac{1}{2}$, $\theta=\pi$ are the sole contributions to the anomalous symplectic form after the semiclassical quantization condition has been invoked. Note that these "zero-modes," at the quantum level, are the origin of the anomaly according to the analysis contained in Refs. 1, 20, and 21.

VI. CONCLUSIONS

We have seen in this paper how the influence of the *fast* fermionic variables, when they are averaged, remains in the effective theory for the *slow* bosonic variables. The effective theory is not what it would be starting from bosonic variables alone. In the bosonic theory without fermions the gauge symmetry is realized by a globally Hamiltonian action on the phase space which admits a comoment.[22] In the effective theory, on the other hand, the Hamiltonian action of the gauge symmetry does not admit a comoment, as can be seen from Eq. (4.28). The Poisson brackets between the generators of the two Lie-algebra elements are not equal to the generator corresponding to the commutator of those elements. If $v^{(\rho)}$, $\rho=1,2$ are two Lie-algebra elements, as in Eq. (4.29), and $G(v^{(\rho)}) = \int v^{(\rho)} D\bar{\mathcal{E}}$ are the corresponding generators of the Hamiltonian action, then $\{G(v^{(1)}), G(v^{(2)})\} - G([v^{(1)}, v^{(2)}])$ is a two-cocycle given by (4.30). If the cohomology group $H^2(\mathcal{G})$ is nontrivial, the obstruction (4.30) to a group action, which admits a comoment, cannot be removed. Then the generators give a representation of an extension of $\mathcal{G}$. At the quantum level this means that the group of gauge symmetries is realized projectively and we have an anomaly.[9] We have somehow an advance warning at the classical level that something is likely to go wrong when we quantize the theory. Normally in classical mechanics we are not concerned with global effects that are crucial ingredients for anomalies: Hannay's angle provides such a *global* detector of the classical phase space of the slow variables. In quantum mechanics, instead, we have naturally a *global* detector and that is the wave function.

The above conclusion depends on an *essential* assumption that we have made about putting a low bound on the energy. This assumption may seem artificial, yet without it the classical theory has *runaway* solutions. We have chosen a polarization so as to bound the energy below. Nevertheless it is clear that this ambiguity arises only in the formulation of the theory in terms of Grassmann variables. It would be interesting to see what would happen in a classical version of the theory where the fermions are represented by ordinary variables. For a $(1+1)$-dimensional field theory, such a version arises in a path integral constructed from coherent states of the relevant Kac-Moody algebra. The resulting Lagrangian always contains a Wess-Zumino term for its kinetic part. It is nothing else than the Berry phase of the Fock-space states.[23] It is thus clear that the same symplectic form is obtained as the one in this paper. The energies are positive by construction (since one starts from the states of the Fock space). Thus the polarization we have chosen really appears as the correct classical version for the standard quantum theory and its negative-energy sea.

ACKNOWLEDGMENTS

E.G. thanks S. Fubini and M. Reuter for many discussions. D.R. acknowledges conversations with K. Johnson, R. Jackiw, Y. Liang, Y. Aharonov, and K. Isler. W.D.T. thanks the members of the physics department of the University of Kaiserslautern, where part of this work was done, for their kind hospitality and acknowledges discussions with M. Berry, J. Kupsch, M. Ruijsenaars, E. Seiler, and K. Tripathy.

APPENDIX

The Hamiltonian corresponding to (4.14) is

$$\mathcal{H}=\sum_n E_n\tilde{\psi}_n^*\tilde{\psi}_n+\int d\mathbf{x}\left[-\frac{g^2}{2}(P_j^a-\mathcal{A}_j^a)(P^{aj}-\mathcal{A}^{aj})+\frac{1}{2g^2}F_{jk}^aF^{ajk}\right] \quad (\text{A1})$$

from which we derive the equations of motion

$$\begin{aligned}\dot{\tilde{\psi}}_n&=\{\mathcal{H},\tilde{\psi}_n\}\\&=-iE_n\tilde{\psi}_n-ig^2\int d\mathbf{x}[P_j^a-\mathcal{A}_j^a(\mathbf{x})]\mathcal{A}_{nm}^{aj}\tilde{\psi}_m\ ,\end{aligned} \quad (\text{A2})$$

$$\begin{aligned}\dot{\tilde{\psi}}_n^*&=\{\mathcal{H},\tilde{\psi}_n^*\}\\&=iE_n\tilde{\psi}_n+ig^2\tilde{\psi}_m^*\int d\mathbf{x}\mathcal{A}_{mn}^{aj}(\mathbf{x})[P_j^a(\mathbf{x})-\mathcal{A}_j^a(\mathbf{x})]\ ,\end{aligned} \quad (\text{A3})$$

$$\dot{A}_j^a(\mathbf{x})=\{\mathcal{H},A_j^a(\mathbf{x})\}=-g^2[P_j^a(\mathbf{x})-\mathcal{A}_j^a(\mathbf{x})]\ , \quad (\text{A4})$$

$$\begin{aligned}\dot{P}^{aj}(\mathbf{x})&=\{\mathcal{H},P^{aj}(\mathbf{x})\}\\&=-\frac{\delta E_n(A)}{A_j^a(\mathbf{x})}\tilde{\psi}_n^*\tilde{\psi}_n+\frac{1}{g^2}D_i^{ab}F^{bij}(\mathbf{x})\\&\quad-g^2\int d\mathbf{y}[P^{bk}(\mathbf{y})-\mathcal{A}^{bk}(\mathbf{y})]\frac{\delta\mathcal{A}_k^b(\mathbf{y})}{A_j^a(\mathbf{x})}\ ,\end{aligned} \quad (\text{A5})$$

where $D_j^{ab}(\mathbf{x})=\partial_j+f^{abc}A_j^c(\mathbf{x})$ is the covariant derivative. Equation (A4) is just the definition

$$F_{j0}^a(\mathbf{x})=g^2[P_j^a(\mathbf{x})-\mathcal{A}_j^a(\mathbf{x})]=-\dot{A}_j^a(\mathbf{x}) \quad (\text{A6})$$

since $A_0=0$. From Eqs. (4.3) and (4.11) it is easily seen that (A2) is equivalent to the Dirac equation

Equation (A5) is equivalent to

$$\begin{aligned}\partial_t\left[\frac{1}{g^2}F^{aj0}(\mathbf{x})+\mathcal{A}^{aj}(\mathbf{x})\right]&=-\frac{\delta E_n}{\delta A_j^a(\mathbf{x})}\tilde{\psi}_n^*\tilde{\psi}_n+\frac{1}{g^2}D_i^{ab}F^{bij}(\mathbf{x})\\&\quad+\int d\mathbf{y}\,\dot{A}^{bk}(\mathbf{y})\frac{\delta\mathcal{A}_k^b(\mathbf{y})}{\delta A_j^a(\mathbf{x})}\ .\end{aligned} \quad (\text{A8})$$

From Eqs. (A2) and (A3) we find, after some algebra, that

$$\partial_t\mathcal{A}^{aj}(\mathbf{x})=i(E_n-E_m)\mathcal{A}_{nm}^{aj}(\mathbf{x})\tilde{\psi}_n^*\tilde{\psi}_m+\int d\mathbf{y}\,\dot{A}^{bk}(\mathbf{y})\frac{\delta\mathcal{A}_k^b(\mathbf{y})}{\delta A_j^a(\mathbf{x})}\ . \quad (\text{A9})$$

We also have the identities

$$(E_n-E_m)\langle n|\delta_A|m\rangle=\langle n|\delta_A\hat{H}|m\rangle \quad (\text{A10})$$

from which it follows that

$$i(E_n-E_m)\mathcal{A}_{nm}^{aj}(\mathbf{x})=\langle n|\delta_{A_j^a(\mathbf{x})}\hat{H}|m\rangle \quad (\text{A11})$$

and

$$\delta_AE_n(A)=\langle n|\delta_A\hat{H}|n\rangle\ , \quad (\text{A12})$$

where $\hat{H}=i(\nabla+\mathbf{A})\cdot\sigma$ is the first quantized Hamiltonian. Thus we finally obtain

$$\begin{aligned}\frac{1}{g^2}D_\mu F^{a\mu j}(\mathbf{x})&=\tilde{\psi}_n^*\tilde{\psi}_m\langle n|\delta_{A_j^a(\mathbf{x})}\hat{H}|m\rangle\\&=\tilde{\psi}_n^*\tilde{\psi}_m u_n^\dagger(\mathbf{x})i\lambda^a\sigma^ju_m(\mathbf{x})\\&=J^{aj}(\mathbf{x})\ .\end{aligned} \quad (\text{A13})$$

Therefore Eqs. (A2), (A3), (A4), and (A5), are equivalent to the standard equations of Yang-Mills theory with matter fields.

[1]P. Nelson and L. Alvarez-Gaumé, Commun. Math. Phys. **99**, 103 (1985).

[2]H. Sonoda, Phys. Lett. **156B**, 220 (1985); Nucl. Phys. **B266**, 410 (1986).

[3]A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. **55**, 927 (1985); **56**, 1019 (1986).

[4]M. Stone and W. E. Goff, Nucl. Phys. **B295**, 243 (1988).

[5]S. Hosono, Nucl. Phys. **B300**, 238 (1988).

[6]A. J. Niemi, G. W. Semenoff, and Y. S. Wu, Nucl. Phys. **B276**, 173 (1986).

[7]M. N. Sanielevici and G. W. Semenoff, Phys. Rev. D **37**, 2934 (1988).

[8]M. V. Berry, Proc. R. Soc. London **A392**, 45 (1984).

[9]L. D. Faddeev, Phys. Lett. **145B**, 81 (1984).

[10]J. H. Hannay, J. Phys. A **18**, 221 (1985). Classical analogs of the Berry phase were also mentioned by F. Wilczek and A. Zee, Phys. Rev. Lett. **52**, 2111 (1984).

[11]M. V. Berry, J. Phys. A **18**, 15 (1985).

[12]G. Giavarini *et al.*, Phys. Rev. D **39**, 3007 (1989).

[13]E. Gozzi and W. D. Thacker, Phys. Rev. D **35**, 2398 (1987). A similar result is contained in R. G. Littlejohn, in *Contemporary Mathematics,* edited by J. E. Marsden (American Mathematical Society, Providence, Rhode Island, 1984), Vol. 28, p. 151.

[14]M. Stone, Phys. Rev. D **33**, 1191 (1986).

[15]R. Casalbuoni, Nuovo Cimento **A33**, 115 (1976); **A33**, 389 (1976); F. A. Berezin and M. S. Marinov, Ann. Phys. (N.Y.) **104**, 336 (1977); E. Gozzi and W. D. Thacker, Phys. Rev. D **35**, 2388 (1987).

[16]H. Kuratsuji and S. Iida, Phys. Rev. D **37**, 441 (1988).

[17]R. Jackiw, Int. J. Mod. Phys. A **3**, 285 (1988).

[18]M. J. Bowick and S. G. Rajeev, Nucl. Phys. **B296**, 1007 (1988); P. Schaller and G. Schwarz, Karlsruhe Report No. KA-THEP-3-1989 (unpublished).

[19]Similar considerations about the semiclassical quantization of Grassmannian systems have been made by S. Abe, Phys. Rev. D **39**, 2327 (1989); S. Abe and S. Naka, Prog. Theor. Phys. **72**, 881 (1984).

[20] S. Forte, Phys. Lett. B **174**, 309 (1986); Nucl. Phys. **B288**, 252 (1987).

[21] L. N. Chang and Y. Liang, Commun. Math. Phys. **108**, 139 (1987); K. Isler and M. B. Paranjape, Phys. Rev. D **41**, 561 (1990).

[22] A Hamiltonian action of $\mathcal{G}$ admits a comoment when the Lie algebra of $\mathcal{G}$ is realized by Poisson brackets among the generators of this action. See N. Woodhouse, *Geometric Quantization* (Clarendon Press, Oxford, 1980), Chap. 3.

[23] M. Stone, Nucl. Phys. **B327**, 399 (1989).