Solvable two-dimensional supersymmetric models and the supersymmetric Virasoro algebra

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(Received 14 May 1990)

The simplest extension of a variety of solvable two-dimensional models to supersymmetric models is obtained and classical solutions are discussed. It is found that these supersymmetric models are closely related.

I. INTRODUCTION

We have shown in a previous work that various solvable two-dimensional (2D) models are closely related among themselves. They can be obtained from the Virasoro algebra, which specifies the Poisson brackets (PB). The algebra constrains the dynamical variable (amplitude) of the systems, and it serves to specify its dynamics through their nontrivial commutators with the Hamiltonian.[1] Here we consider the simplest extension of these classical 2D models to supersymmetric models[2–6] and obtain the PB of the superfields and their component fields, and again verify that the corresponding supersymmetric 2D models are closely related. The PB are expected to be useful in quantizing these models.

The guiding procedure is to extend an amplitude u that satisfies the Korteweg–de Vries (KdV) equation, for example, to a corresponding Grassmann-odd superfield $W=\xi+\theta u$, where ξ is a one-component Majorana spinor and fermionic partner of u. Introduce the superderivative $D=\partial_\theta+\theta\partial_x$ ($D^2=\partial_x$) and the superdelta function $\Delta(z-z')=(\theta-\theta')\delta(x-y)$, where $z=(x,\theta)$ and $z'=(y,\theta')$. The Virasoro algebra for the classical bosonic systems is expressed in terms of the Poisson brackets

$$i\{L_n,L_m\}=(n-m)L_{n+m}+\tfrac{1}{12}c(n^3-n)\delta_{n+m,0}\ , \tag{1}$$

where L_n are the Virasoro generators and c is the central charge that is put equal to $3/\hbar$. The Fourier-transform field

$$u(x)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}L_ne^{-inx}-\frac{1}{4} \tag{2}$$

therefore satisfies

$$\{u(x),u(y)\}=\tfrac{1}{2}[-\delta_{xxx}(x-y)+4u(x)\delta_x(x-y)+2u_x(x)\delta(x-y)]\ . \tag{3}$$

The periodic boundary condition $u(t,x+2\pi)=u(t,x)$ is imposed, and it is assumed that the amplitudes have continuous derivatives of any order.

In Sec. II, the superfields U for the supersymmetric modified KdV (SMKdV) equation, Φ and S for the supersymmetric sine-Gordon (SSG) equation,[7] L for supersymmetric Liouville (SL) equation, and V for the supersymmetric nonlinear Schrödinger (SNLS) equations are introduced, and their equations of motion are obtained. In Sec. III, classical solutions are discussed, and comments are given in Sec. IV.

II. SUPERSYMMETRIC EQUATION OF MOTION

We recall that the KdV equation for the amplitude $u(x)$ follows from the evolution equation[8]

$$u_t=\{u(x),H_1\}=u_{xxx}-6uu_x\ , \tag{4}$$

provided

$$H_1=-\int dy\,u^2(y)\ .$$

The modus operandi is to write the PB of $W=\xi+\theta u$ in terms of supersymmetric quantities and adjust the coefficients so that the PB of u Eq. (3) is obtained together with the PB of $\{\xi(x),\xi(y)\}$ and $\{\xi(x),u(y)\}$. The result is the supersymmetric Virasoro[2] PB

$$\begin{aligned}\{W(z),W(z')\}&=\tfrac{1}{2}\{[-2D^2(z)+2D^2(z')-D(z)D(z')][W(z)\Delta(z-z')]-D^5(z)\Delta(z-z')\}\\&=\{\xi(x),\xi(y)\}+\theta\{u(x),\xi(y)\}-\theta'\{\xi(x),u(y)\}+\theta\theta'\{u(x),u(y)\}\ ,\end{aligned} \tag{5}$$

$$\{\xi(x),\xi(y)\}=\tfrac{1}{2}(\delta_{xx}-u\delta)\ , \tag{6}$$

$$\{u(x),\xi(y)\}=\tfrac{1}{2}(3\xi\delta_x+\xi_x\delta)\ . \tag{7}$$

The supersymmetric Hamiltonian of the SKDV model can be written as

$$\begin{aligned}H&=-\int(DW)W(z')dz'\\&=-\int(u^2-\xi\xi_y)dy\ ,\end{aligned} \tag{8}$$

where $\int dz'=\int_0^{2\pi}dy\int d\theta'$ and $\int\theta'\,d\theta'=1$. We obtain, with the aid of (3), (6), (7), and (8),

$$u_t=\{u(x),H\}=u_{xxx}-6uu_x+3\xi\xi_{xx}\ , \tag{9}$$

$$\xi_t=\{\xi(x),H\}=\xi_{xxx}-3u\xi_x-3u_x\xi\ , \tag{10}$$

where the time is a bosonic parameter. The amplitudes $u(x)$ and $\xi(x)$ are related to the superconformal generators[2,9] L_n and G_m, u is given in (2), and $\xi(x)=(4\pi)^{-1}\sum_m G_m e^{-imx}$. The set of Eqs. (9) and (10) are invariant under the supersymmetric transformation $\delta u=\xi_x$, $\delta\xi=u$, as they transform into each other.

In order to derive the SMKdV equation, let us introduce a Grassmann-odd superfield $u=\beta+\theta p$ and define a super-Miura transformation, in analogy to the bosonic case

$$W=(DU)U+U_x \ , \tag{11}$$

or

$$\xi=p\beta+\beta_x \ , \tag{12}$$

$$u=p^2+p_x-\beta\beta_x \ . \tag{13}$$

To determine the Poisson brackets of p and β, substitute (12) and (13) in (6) and (7) and find

$$\{p(x),p(y)\}=\tfrac{1}{2}\delta_x(x-y) \ , \tag{14}$$

$$\{\beta(x),\beta(y)\}=-\tfrac{1}{2}\delta(x-y) \ , \tag{15}$$

$$\{p(x),\beta(y)\}=0 \ , \tag{16}$$

or

$$\{U(z),U(z')=-\tfrac{1}{2}D\Delta(z-z') \ .$$

Substitute (12) and (13) in (8) and get the Hamiltonian for the SMKdV model:

$$\begin{aligned}H=-\int[&p^4+p_y^2-(3p^2+p_y)\beta\beta_y\\&-p\beta\beta_{yy}-\beta_y\beta_{yy}]dy \ .\end{aligned} \tag{17}$$

The equations of motion of the SMKdV model are obtained from the evolution equation $u_t=\{u,H\}$ with the aid of (14)–(17):

$$\begin{aligned}p_t&=\{p(x),H\}\\&=p_{xxx}-6p^2p_x+3p_x\beta\beta_x+3p\beta\beta_{xx} \ ,\end{aligned} \tag{18}$$

$$\beta_t=\{\beta(x),H\}=\beta_{xxx}-3ppx\beta-3p^2\beta_x \ . \tag{19}$$

Equations (18) and (19) are invariant under the supersymmetric transformation $\delta p=\beta x$, $\delta\beta=p$. Equations (18) and (19) are expressible as[2]

$$U_t=\partial_{xxx}U-3(\partial_x U)(DU)^2-3(DU)\{\partial_x(DU)\}U \ . \tag{20}$$

We next consider the sine-Gordon (SG) equation in the light-cone frame and identify the field ϕ of the SG equation as

$$p(x)=\frac{1}{2\sqrt{\pi}}\partial_x\phi \ ; \tag{21}$$

then, from (14),

$$\{p(x),\phi(y)\}=-\sqrt{\pi}\delta(x-y) \ , \tag{22}$$

or generally

$$\{p(x),\phi^n(y)\}=-\sqrt{\pi}n\phi^{n-1}\delta(x-y) \ . \tag{23}$$

We define the even superfield of the supersymmetric SG (SSG) equation

$$\begin{aligned}\Phi&=(\phi/2\sqrt{\pi})+\theta q\equiv\phi'+\theta q \ ,\\ \Psi&\equiv D\Phi=q+\theta\phi'_x\equiv q+\theta p \ ,\end{aligned} \tag{24}$$

where q is an independent field. The Poisson brackets of U and Φ are

$$\{U(z),\Phi(z')\}=-\tfrac{1}{2}\Delta(z-z') \ , \tag{25}$$

with the aid of (16), (22), and the choices

$$\begin{aligned}&\{\beta(x),q(y)\}=\tfrac{1}{2}\delta(x-y) \ ,\\&\{p(x),q(y)\}=0 \ .\end{aligned} \tag{26}$$

For the SSG equation, we express H in terms of the superfluid $S=\psi+\theta\sin\phi'(y)$:

$$\begin{aligned}H_{\text{SG}}&=-\int(DS)S(z')dz'\\&=-\int[\tfrac{1}{2}(1-\cos2\phi')-qq_y]dy \ .\end{aligned} \tag{27}$$

The SSG equation is obtained from the evolution equation $\Psi_t=\{\Psi,H_{\text{SG}}\}$ with the aid of (23) and (26):

$$\phi_{xt}=-\sin\phi \ , \tag{28}$$

$$q_t=q_x \ . \tag{29}$$

For the supersymmetric Liouville (SL) equation with

$$L=q+\theta e^{\phi'} \ , \tag{30}$$

the Hamiltonian is

$$\begin{aligned}H_L&=-\int(DL)L(z')dz'\\&=-\int(e^{2\phi'}-qq_y)dy \ .\end{aligned} \tag{31}$$

Similarly, the evolution equation $\Psi_t=\{\Psi,H_L\}$ with the aid of (23) and (26) leads to

$$\phi_{xy}=e^{\phi} \ , \tag{32}$$

$$q_t=q_x \ . \tag{33}$$

The equations for ϕ and q of the SSG and the SL models are decoupled by dint of (26). Equations (29) and (33) mean that q is a function of the light-cone variable $x+t$.

In order to introduce the supersymmetric nonlinear Schrödinger (SNLS) equation, we normalize the Poisson brackets of $p(x)$ in (14) with a factor of i,

$$\{p(x),p(y)\}=\frac{i}{2}\delta_x(x-y) \ , \tag{34}$$

and substitute $p(x)=\psi_x(x)+\psi^\dagger(x)$, where $p(x)$ is not Hermitian in (34) and find the following PB consistent with (34):

$$\begin{aligned}&\{\psi(x),\psi^\dagger(x)\}=\frac{i}{4}(x-y) \ ,\\&\{\psi(x),\psi(y)\}=\{\psi^\dagger(x),\psi^\dagger(y)\}\\&\qquad=0, \quad 0\le x\le 2\pi \ .\end{aligned} \tag{35}$$

We introduce the Grassmann-even superfield $V=\psi+\theta\rho$

and the Hamiltonian

$$H_N=-4\,\mathrm{Re}\int(DN^\dagger)N(z')dz' , \qquad (36)$$

where

$$N=DV+\theta i\sqrt{K}\,\psi^2=\rho+\theta(\psi_x+i\sqrt{K}\,\psi^2) . \qquad (37)$$

The Re (real part) is required so that H_N conserves particle number, and K is a constant.

The Poisson brackets of the component field ρ are obtained from the requirement

$$\{V(x),V^\dagger(z')\}=\frac{i}{4}D\Delta(z-z') , \qquad (38)$$

that is,

$$\{\rho(x),\rho^\dagger(y)\}=-\frac{i}{4}\delta_x(x-y) , \qquad (39)$$

$$\{\psi(x),\rho(y)\}=0 . \qquad (40)$$

Integration of (36) over θ' yields

$$H_{\mathrm{NL}}=-4\int(\psi_y^\dagger\psi_y+K\psi^{+2}\psi^2+\rho_y^\dagger\rho)dy . \qquad (41)$$

The equations of motion of the SNLS model are obtained from the evolution equation $V_t=\{V,H_{\mathrm{NL}}\}$ with the aid of (35) and (39):

$$i\psi_t+\psi_{xx}-2K|\psi|^2\psi=0 , \qquad (42)$$

$$i\rho_t=\rho_{xx} . \qquad (43)$$

Equations (42) and (43) are decoupled because of the PB (40).

III. SOLUTIONS

We rewrite the SMKdV equations (18) and (19) and regard all amplitudes as scalar functions:

$$p_t-p_{xxx}+6p^2p_x-3p_x\beta\beta_x-3p\beta\beta_{xx}=0 , \qquad (44)$$

$$\beta_t-\beta_{xxx}+3p^2\beta_x+3pp_x\beta=0 . \qquad (45)$$

In order to solve (44) and (45), we substitute the supersymmetric pair of trial functions[10] with an arbitrary constant M (fermionic): $p=2\eta\,\mathrm{sech}(-2\eta x-8\eta^3t)$ and $\beta=M2\eta\,\mathrm{sech}(-2\eta x-8\eta^3t)$ into (44) and (45), and find that they are both satisfied with $M^2=0$. We next rewrite the SKdV equations (9) and (10):

$$u_t-u_{xxx}+6uu_x-3\xi\xi_{xx}=0 , \qquad (46)$$

$$\xi_t-\xi_{xxx}+3u\xi_x+3u_x\xi=0 . \qquad (47)$$

We substitute, in a similar manner,

$$u=2\eta\,\mathrm{sech}(2\eta x+8\eta^3t) ,$$

$$\xi=M2\eta\,\mathrm{sech}(2\eta x+8\eta^3t) ,$$

into (46) and (47), and find that they are satisfied by dint of $M^2=0$ again.

The classical solutions for the SSG, SL, and SNLS equations are simple because the superfield component boson and fermion fields decoupled and are well known. The fermionic component ρ of the superfield of the SNLS equation $V=\psi+\theta\rho$ given by $\rho=2\eta\cosh(2\eta)xe^{4i\eta^2t}$ satisfies Eq. (43).

IV. REMARKS

We presented the simplest supersymmetric solvable 2D models, which are all systematically related among themselves. This should provide a common ground for quantization of these models. The next natural extension is to consider the even superfield of the type, for example, for the SSG model,

$$\Phi(x)=\phi(x)+i\bar{\theta}q(x)+\tfrac{1}{2}i\bar{\theta}\theta F(x) ,$$

where $\phi(x)$ and $F(x)$ are scalar fields, and $q(x)$ is a two-component column Majorana spinor.[4,7] The situation becomes more complicated and quantization becomes more difficult.

The Poisson brackets for the amplitudes for SKdV, SMKdV, SSG, SL, and SNLS models may all be obtained from the supersymmetries extension of the Virasoro algebra. Furthermore, the Hamiltonians that yield the equations of motion via the evolution equation all have the generic form $\int(DY)Y(z')dz'$, where Y is an appropriate superfield. Thus all these supersymmetric models are closely related and have a common origin. It is possible that theories which differ only in their Lagrangians are in fact different states of a single theory.[11]

ACKNOWLEDGMENTS

The author thanks C. Zachos for valuable discussions and improvements, and I. Yamanaka for an interesting communication. He also is grateful to R. Blankenbecler and the theory group for their hospitality at SLAC. This work was supported in part by the U.S. Department of Energy under Contract No. EY-76-C-02-141500.

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