Fermionized bosons and Szegö's theorem

Minoru Hirayama and Yoshihiro Horikawa
Department of Physics, Toyama University, Toyama 930, Japan
(Received 2 April 1990)

It is discussed how the bosonic zero-mode operator q in two-dimensional quantum field theory should be expressed in terms of fermion operators. Two expressions for q are proposed, one of which is of a clear meaning and the other of a formal nature. The idea of the boson-fermion correspondence is applied to prove simply Szegö's theorem on an infinite Toeplitz determinant. It is also seen that a fermion theoretical discussion of a special case of the Toeplitz determinant yields a generalization of Szegö's result.

I. INTRODUCTION

Recently, two-dimensional quantum field theory has attracted much attention because of its possible applications to string theory and physics on the critical point and of its mathematical interest. One of the most important aspects of two-dimensional theories is the boson-fermion correspondence:[1–7] a fermion field can be expressed as a function of a boson field, or *vice versa*. Although the original idea of bosons associated with fermion fields can be traced back to more than fifty years ago,[8] the special role of lower-dimensional space-time has been recognized only recently. In two-dimensional space-time, the correspondence is exact: a fermion field is realized by the Mandelstam exponentiation of a boson field,[3] while bosonic creation and annihilation operators of oscillation modes are constructed through Tomonaga's bilinear infinite sums of fermion operators.[1] As for bosonic zero-mode operators, there seems to remain a missing link.[9] The operator q, which corresponds to the center-of-mass coordinate of a bosonic string, has not found its appropriate fermionic expression yet. One of our purposes in this paper is to explore the expression of q in terms of fermion operators. We shall propose two expressions of q: one is not of a compact form but convenient for actual calculation, the other takes a compact form but is rather formal.

The boson-fermion correspondence can also be regarded as a convenient mathematical tool. Its usefulness is evident[6,7] in the construction of the τ functions of the Kadomtsev-Petviashvili hierarchy of soliton equations.[10] It has also been applied to recover a purely mathematical result in terms of the language of physics. It is well known that an identity for Jacobi's elliptic theta functions is derived if the partition functions of free, two-dimensional bosonic and fermionic systems are equated. Eguchi and Ooguri discussed that the boson-fermion correspondence applied to fermion Green's functions for a system on a Riemann surface yields Fay's addition theorem on Riemann's theta functions.[11] Saito[12] pointed out that Fay's theorem is also equivalent to Hirota's bilinear difference equation[13] of the above-mentioned τ functions. These kinds of line of thought remind us of the proofs of the Atiyah-Singer index theorem[14,15] and the Morse inequality[16] in terms of supersymmetric quantum mechanics supplemented by a path-integral technique. In this paper, we prove Szegö's theorem on the determinant of the Toeplitz matrix[17–19] which finds applications in several branches of mathematics. Although the previously given proofs, analytic as well as probability theoretic, of Szegö's theorem are rather skillful and lengthy, the proof aided by the boson-fermion correspondence is simple enough at least for physicists.

This paper is organized as follows. In Sec. II, in order to prepare formulas necessary for later discussions and to fix notations, we briefly recapitulate the scheme of the boson-fermion correspondence. In Sec. II B, we discuss how the bosonic zero-mode q should be constructed in terms of fermionic operators. In Sec. III, we introduce Szegö's theorem on the Toeplitz determinant and prove it with the help of boson-fermion correspondence. In Sec. III C, we consider a special case of Szegö's theorem and find that a fermion theoretical argument leads us to a conclusion stronger than the one obtained by classical analysis. Section IV is devoted to a summary and the Appendix is attached to explain some details of an equation in Sec. III.

II. FERMIONIZATION OF BOSONIC ZERO MODES

The boson-fermion correspondence is one of the most remarkable aspects of two-dimensional quantum field theory. It is widely accepted that various physical quantities can be analyzed either from a bosonic or a fermionic point of view. In this section, we discuss the correspondence[1–7] between the $R/2\pi\mathbb{Z}$-valued boson, i.e., a troidally compactified bosonic string, and the complex Neveu-Schwarz (NS) fermion. After briefly reviewing the above correspondence and setting up notation in Sec. II A, we argue in Sec. II B how bosonic zero modes are fermionized.

A. Boson-fermion correspondence

The $R/2\pi\mathbb{Z}$-valued boson field $\phi(z)$ is defined by

$$\phi(z)=q-ip\ln z+i\sum_{\substack{n\in Z\\ n\neq 0}}\frac{a_n}{n}z^{-n} \tag{2.1}$$

with the commutation relations

$$[q,p]=i,\quad [a_n,a_m]=n\delta_{n+m,0},\quad 0 \text{ otherwise} . \tag{2.2}$$

The boson Hilbert space B is given as

$$B=\{|M\rangle_b\}\otimes\{\text{Fock space}\} .$$

The zero-mode operators q,p and oscillation mode operators a_n $(n\neq 0)$ act on $\{|M\rangle_b\}$ and $\{\text{Fock space}\}$, respectively. The state $|M\rangle_b$ is an eigenstate of p belonging to the eigenvalue M:

$$p|M\rangle_b=M|M\rangle_b,\quad e^{iq}|M\rangle_b=|M+1\rangle_b,\quad M\in\mathbb{Z} . \tag{2.3}$$

The $\{\text{Fock space}\}$ is spanned by vectors

$$(a_{-1})^{k_1}(a_{-2})^{k_2}\cdots|\text{vac}\rangle,\quad k_1,k_2,\ldots\geq 0$$

with $|\text{vac}\rangle$ being defined by

$$a_n|\text{vac}\rangle=0,\quad n>0 . \tag{2.4}$$

We write $|M\rangle_b\otimes|\text{vac}\rangle$ as $|M,0\rangle$ and call it the M vacuum.

On the other hand, the complex NS fermion is defined by

$$\psi(z)=\sum_{\mu\in\mathbb{Z}+\frac{1}{2}}\psi_\mu z^{-\mu-1/2} ,\qquad \psi^*(z)=\sum_{\mu\in\mathbb{Z}+\frac{1}{2}}\psi^*_\mu z^{-\mu-1/2} \tag{2.5}$$

with anticommutation relations

$$\{\psi(z),\psi^*(w)\}=\frac{1}{z}\sum_{n\in\mathbb{Z}}(w/z)^n,\quad 0\text{ otherwise} . \tag{2.6}$$

The Laurent expansion of (2.6) yields

$$\{\psi_\mu,\psi^*_\nu\}=\delta_{\mu+\nu,0},\quad 0\text{ otherwise} . \tag{2.7}$$

The fermion Fock space is spanned by vectors

$$\psi_{\mu_1}\psi_{\mu_2}\cdots\psi^*_{\nu_1}\psi^*_{\nu_2}\cdots|0\rangle,\quad \mu_1,\mu_2,\ldots,\nu_1,\nu_2,\ldots<0$$

with the fermion vacuum $|0\rangle$ being defined by

$$\psi_\mu|0\rangle=\psi^*_\mu|0\rangle=0,\quad \mu>0 . \tag{2.8}$$

The fermionic current is defined by

$$J(z)=:\psi^*(z)\psi(z):=\sum_{n\in\mathbb{Z}}J_n z^{-n-1} , \tag{2.9}$$

where : : denotes fermion normal ordering. From (2.9), we have

$$J_n=\sum_{\mu\in\mathbb{Z}+\frac{1}{2}}\psi^*_\mu\psi_{n-\mu},\quad n\neq 0 , \tag{2.10a}$$

$$J_0=\sum_{\mu>0}(\psi^*_{-\mu}\psi_\mu-\psi_{-\mu}\psi^*_\mu) . \tag{2.10b}$$

These operators satisfy the commutation relations

$$[J_n,J_m]=n\delta_{n+m,0} , \tag{2.11a}$$

$$[J_n,\psi(z)]=-z^n\psi(z) , \tag{2.11b}$$

$$[J_n,\psi^*(z)]=z^n\psi^*(z) . \tag{2.11c}$$

The energy-momentum tensor is defined by

$$T(z)=\frac{1}{2}:\frac{d\psi(z)}{dz}\psi^*(z)-\psi(z)\frac{d\psi^*(z)}{dz}: . \tag{2.12a}$$

It is well known that the above energy-momentum tensor can be rewritten in the Sugawara form[20]

$$T(z)_S=\tfrac{1}{2}:J(z)^2:_c , \tag{2.12b}$$

where $:\ :_c$ denotes the current normal ordering which shifts J_n $(n\geq 0)$ to the right and J_n $(n<0)$ to the left.

According to the standard bosonization scheme,[1-7] fermionlike operators acting on B are constructed in the following way:

$$\psi(z)_b=:e^{i\phi(z)}:_b=e^{iq}e^{p\ln z}\exp\left[\sum_{n\geq 1}\frac{z^n}{n}a_{-n}\right]\exp\left[-\sum_{n\geq 1}\frac{z^{-n}}{n}a_n\right] , \tag{2.13a}$$

$$\psi^*(z)_b=:e^{-i\phi(z)}:_b=e^{-iq}e^{-p\ln z}\exp\left[-\sum_{n\geq 1}\frac{z^n}{n}a_{-n}\right]\times\exp\left[\sum_{n\geq 1}\frac{z^{-n}}{n}a_n\right] , \tag{2.13b}$$

where the subscript b indicates that these operators are acting on B and $:\ :_b$ is the usual normal ordering. It has been established that the analogue of (2.6) and (2.7) holds for $\psi(z)_b$, $\psi^*(z)_b$, $(\psi_\mu)_b$, and $(\psi^*_\mu)_b$, where $(\psi_\mu)_b$ and $(\psi^*_\mu)_b$ are defined as the coefficients of the Laurent expansions of $\psi(z)_b$ and $\psi^*(z)_b$, respectively. According to Eqs. (2.3) and (2.4), $(\psi_\mu)_b$ and $(\psi^*_\mu)_b$, $\mu>0$ annihilate the bosonic zero vacuum:

$$(\psi_\mu)_b|0,0\rangle=(\psi^*_\mu)_b|0,0\rangle=0,\quad \mu>0 . \tag{2.14}$$

Thus, the boson zero vacuum $|0,0\rangle$ corresponds to the fermion vacuum $|0\rangle$ in (2.8). The current operator acting on B is defined by

$$J(z)_b=\lim_{w\to z}\left[\psi^*(w)_b\psi(z)_b-\frac{1}{w-z}\right]=-i\frac{d\phi(z)}{dz}=-\sum_{n\in\mathbb{Z}}a_n z^{-n-1} , \tag{2.15}$$

where $a_0=p$.

Conversely bosonlike operators on the fermion Fock space $(a_n)_f$, are identified with fermionic current operators:

$$(a_n)_f=-\frac{1}{2\pi i}\oint dz\, z^n:\psi^*(z)\psi(z):=-J_n,\quad n\in\mathbb{Z} . \tag{2.16}$$

Equation (2.11a) reproduces the analogue of (2.2) for $(a_n)_f$. To our best knowledge, however, no appropriate definition of q_f, fermionization of q, has been given. In the next subsection, we discuss how it can be attained.

B. Construction of q_f

One candidate for q_f is

$$\tilde{q}_f=\frac{1}{2\pi}\oint dz:\psi^*(z)\psi(z):\ln z \tag{2.17}$$

since a heuristic and somewhat ambiguous calculation of $[\tilde{q}_f,p_f]$ with $p_f=-J_0$ seems to give a desired result $[\tilde{q}_f,p_f]=i$.[9] If, however, we directly integrate the right-hand side (RHS) of (2.17) by putting $z=\exp(i\theta)$, $0\le\theta<2\pi$, we obtain

$$\tilde{q}_f=-i\sum_{\substack{n\in Z\\ n\neq 0}}\frac{1}{n}(a_n)_f-\pi p_f \tag{2.18}$$

which is obviously incompatible with $[\tilde{q}_f,p_f]=i$ and $[\tilde{q}_f,(a_n)_f]=0$, $n\neq 0$. Thus, we turn to seek an alternative expression for q_f.

Simply multiplying Eqs. (2.13a) and (2.13b) by some inverse operators and fermionizing them, we are led to

$$e^{iq_f}=\exp\left[\sum_{n\ge 1}\frac{z^n}{n}J_{-n}\right]\psi(z)\exp\left[-\sum_{n\ge 1}\frac{z^{-n}}{n}J_n\right]\times\exp(J_0\ln z)\ , \tag{2.19a}$$

$$e^{-iq_f}=\exp\left[-\sum_{n\ge 1}\frac{z^n}{n}J_{-n}\right]\psi^*(z)\exp\left[\sum_{n\ge 1}\frac{z^{-n}}{n}J_n\right]\times\exp(-J_0\ln z)\ . \tag{2.19b}$$

These operators satisfy the desired commutation relations

$$[e^{iq_f},J_n]=e^{iq_f}\delta_{n,0}\ , \tag{2.20a}$$

$$[e^{-iq_f},J_n]=-e^{-iq_f}\delta_{n,0}\ . \tag{2.20b}$$

The undesirable feature of them is that the RHS's of (2.19a) and (2.19b) contain the variable z, while the LHS's do not. We see, however, in the following way that they are in fact z independent. The z independence of the RHS's of (2.19a) and (2.19b) (i.e., vanishing of their derivative) requires the equations

$$\frac{d\psi(z)}{dz}=-\left[\sum_{n=1}^{\infty}J_{-n}z^{n-1}\right]\psi(z)-\psi(z)\left[\sum_{n=0}^{\infty}J_nz^{-n-1}\right]\ , \tag{2.21a}$$

$$\frac{d\psi^*(z)}{dz}=\left[\sum_{n=1}^{\infty}J_{-n}z^{n-1}\right]\psi^*(z)+\psi^*(z)\left[\sum_{n=0}^{\infty}J_nz^{-n-1}\right]\ . \tag{2.21b}$$

It turns out that (2.21a) and (2.21b) are equivalent to

$$\begin{aligned}-(\mu+\tfrac{1}{2})\psi_\mu=&\sum_{\rho<\mu}\sum_{\nu\in Z+\frac{1}{2}}\psi_\rho\psi_\nu\psi^*_{\mu-\rho-\nu}\\&+\sum_{\rho>\mu}\sum_{\nu\in Z+\frac{1}{2}}\psi_\nu\psi^*_{\mu-\rho-\nu}\psi_\rho\\&+\sum_{\nu>0}\psi_\mu(\psi_{-\nu}\psi^*_\nu-\psi^*_{-\nu}\psi_\nu)\ ,\end{aligned} \tag{2.22a}$$

$$\begin{aligned}-(\mu+\tfrac{1}{2})\psi^*_\mu=&\sum_{\rho<\mu}\sum_{\nu\in Z+\frac{1}{2}}\psi^*_\rho\psi_\nu\psi^*_{\mu-\rho-\nu}\\&+\sum_{\rho>\mu}\sum_{\nu\in Z+\frac{1}{2}}\psi_\nu\psi^*_{\mu-\rho-\nu}\psi^*_\rho\\&+\sum_{\nu>0}\psi^*_\mu(\psi_{-\nu}\psi^*_\nu-\psi^*_{-\nu}\psi_\nu)\ .\end{aligned} \tag{2.22b}$$

We find that (2.22a) and (2.22b) are identities of our fermion system because they can be derived from (2.6) by some manipulations. Thus (2.21a) and (2.21b) are also identities of the fermion theory but not equations. So the RHS's of (2.19a) and (2.19b) are automatically z independent. We note that identities (2.21a) and (2.21b) are equivalent to the relation $T(z)=T(z)_S$: they are necessary and sufficient to ensure $T(z)=T(z)_S$. We see that $T(z)$ equals $T(z)_S$ by applying (2.21a) and (2.21b) to $T(z)$ given by (2.12a) and that the requirements $[\psi(w),T(w)]=[\psi(w),T(z)_S]$ and $[\psi^*(w),T(z)]=[\psi^*(w),T(z)_S]$ yield (2.21a) and (2.21b), respectively. The operator q_f itself should be interpreted as the Fourier series

$$q_f=\frac{1}{i}\sum_{M=1}^{\infty}\frac{1}{M}(-1)^{M-1}(e^{iMq_f}-e^{-iMq_f})\ . \tag{2.23}$$

According to Eqs. (2.19a) and (2.19b), e^{iMq_f} and e^{-iMq_f} $(M>0)$ are given by

$$\begin{aligned}e^{iMq_f}=&\left[\prod_{1\le k<k'\le M}\frac{1}{z_k-z_{k'}}\right]\exp\left[\sum_{n\ge 1}\frac{J_{-n}}{n}\left[\sum_{k=1}^{M}z_k^n\right]\right]\psi(z_1)\psi(z_2)\cdots\psi(z_M)\\&\times\exp\left[-\sum_{n\ge 1}\frac{J_n}{n}\left[\sum_{k=1}^{M}z_k^{-n}\right]\right]\exp\left[J_0\sum_{k=1}^{M}\ln z_k\right]\ ,\end{aligned} \tag{2.24a}$$

$$\begin{aligned}e^{-iMq_f}=&\left[\prod_{1\le k<k'\le M}\frac{1}{z_k-z_{k'}}\right]\exp\left[-\sum_{n\ge 1}\frac{J_{-n}}{n}\left[\sum_{k=1}^{M}z_k^n\right]\right]\psi^*(z_1)\psi^*(z_2)\cdots\psi^*(z_M)\\&\times\exp\left[\sum_{n=1}^{\infty}\frac{J_n}{n}\left[\sum_{k=1}^{M}z_k^{-n}\right]\right]\exp\left[-J_0\sum_{k=1}^{M}\ln z_k\right]\ .\end{aligned} \tag{2.24b}$$

Recalling the z independence of above operators, we see that these operators act on the fermion vacuum $|0\rangle$ as follows:

$$e^{iMq_f}|0\rangle=\begin{cases}\psi_{-M+1/2}\cdots\psi_{-3/2}\psi_{-1/2}|0\rangle, & M>0,\\ |0\rangle, & M=0,\\ \psi^*_{-|M|+1/2}\cdots\psi^*_{-3/2}\psi^*_{-1/2}|0\rangle, & M<0\end{cases}$$
$$\equiv|M\rangle\ . \tag{2.25}$$

The state $|M\rangle$ defined above satisfies

$$p_f|M\rangle=M|M\rangle,\quad e^{iq_f}|M\rangle=|M+1\rangle\ ,$$
$$(a_n)_f|M\rangle=0,\quad n>0\ . \tag{2.26}$$

Comparing (2.26) with (2.3) and (2.4), we see that $|M\rangle$ corresponds to the bosonic M vacuum $|M,0\rangle$. Equations (2.25) and (2.26) are crucial for our discussions in the next section.

Now we turn to seek a different expression for q_f which reveals another feature of q_f. If we put z_k in (2.24a) as

$$z_k=\exp[-\pi i+2\pi ik/(M+1)],\quad k=1,2,\ldots,M\ ,$$

and consider the case $M=2N+1$ with N a positive integer, we obtain

$$e^{iMq_f}=c_M S_- T_-\psi(z_1)\cdots\psi(z_N)\psi(1)$$
$$\times\psi(z_{N+2})\cdots\psi(z_M)T_+S_+\ , \tag{2.27}$$

$$c_M=\prod_{1\le k<k'\le 2N+1}(z_k-z_{k'})^{-1}$$
$$=(2N+2)^{-N}\exp[\pi iN(N+\tfrac{1}{2})]\ , \tag{2.28}$$

$$T_\pm=\exp\left[\mp\sum_{k=1}^{\infty}\frac{M-1}{k(M+1)}J_{\pm k(M+1)}\right]\ , \tag{2.29}$$

$$S_\pm=\exp\left[\mp\sum_{n=1}^{\infty}\frac{1}{n}J_{\pm n}\right]\ , \tag{2.30}$$

where the fact $z_{N+1}=1$ has been made use of. Recalling that $\exp(iq_f)=S_-\psi(1)S_+$, we have

$$\psi(1)=T_-\psi(1)\{c_M(-1)^N\exp[-i(M-1)q_f]$$
$$\times\psi(z_1)\cdots\psi(z_N)\psi(z_{N+2})\cdots\psi(z_M)\}T_+ \tag{2.31}$$

If we let M get very large, T_- and T_+ reduce to 1 as far as we work in a subspace of the fermion Fock space where the $\psi_{-\mu},\psi^*_{-\nu}$ modes with $\mu,\nu>M$ are not excited. In this sense, we write

$$\psi(1)=\psi(1)\lim_{M\to\infty}c_M(-1)^N\exp[-i(M-1)q_f]$$
$$\times\psi(z_1)\cdots\psi(z_N)\psi(z_{N+2})\cdots\psi(z_M)\ . \tag{2.32}$$

A similar line of argument yields similar expressions for $\psi(z),\psi^*(z)$ and products thereof. This observation indicates that $\exp(iq_f)$ should be identified with the $M\to\infty$ limit of the Mth root of the product of fermion fields at $z_1,z_2,\cdots,z_M$. According to conventional terminology, the latter quantity is called the geometric mean of fermion fields on the unit circle and is expressed as

$$\exp\left[(2\pi i)^{-1}\oint_{|z|=1}z^{-1}dz\ln\psi(z)\right]\ .$$

Since the above-mentioned infinite product of $\psi(z)$ vanishes, it is appropriate to introduce a normalization factor. We have thus seen that a possible expression of $\exp(iq_f)$ is given by

$$e^{iq_f}=\frac{\exp\left[\dfrac{1}{2\pi i}\oint_{|z|=1}z^{-1}dz\ln\psi(z)\right]}{\left\langle 0\left|\exp\left[\dfrac{1}{2\pi i}\oint_{|z|=1}z^{-1}dz\ln\psi(z)\right]\right|-1\right\rangle}\ . \tag{2.33}$$

This expression is of a formal nature since the clear definitions of the fractional power and logarithm of nilpotent fields ($\{\psi(z)\}^2=0$) are yet to be given.

III. BOSONIZATION AND SZEGÖ'S THEOREM

In this section, we present a mathematical application of the boson-fermion correspondence. We introduce Szegö's theorem[17–19] on determinants of the Toeplitz matrices in Sec. III A and prove it in Sec. III B on the basis of the boson-fermion correspondence. In Sec. III C we investigate a special Toeplitz determinant through fermion theoretical consideration and generalize the previously obtained result.[17]

A. Szegö's theorem and Kac's reinterpretation

Szegö's theorem[17–19] concerns the distribution of eigenvalues of $(M+1)\times(M+1)$ Toeplitz matrix

$$C_M=\begin{bmatrix}c_0 & c_{-1} & \cdots & c_{-M}\\ c_1 & c_0 & & c_{-M+1}\\ \vdots & \vdots & & \vdots\\ c_M & c_{M-1} & \cdots & c_0\end{bmatrix}\ . \tag{3.1}$$

Let the continuous function $f(\theta)$ defined by

$$f(\theta)=\sum_{l=-\infty}^{\infty}c_l e^{il\theta} \tag{3.2}$$

be positive for $0<\theta<2\pi$ and satisfy the condition

$$|f'(\theta_1)-f'(\theta_2)|<K|\theta_1-\theta_2|^\alpha,\quad 0<\theta_1,\theta_2<2\pi \tag{3.3}$$

with some constants $K>0$ and $0<\alpha<1$. We define $\{K_l\}$ and $G(f)$ by

$$K_l=\frac{1}{2\pi}\int_0^{2\pi}e^{-il\theta}\ln f(\theta)d\theta,\quad l=0,\pm1,\pm2,\ldots, \tag{3.4}$$

$$G(f)=e^{k_0}\ . \tag{3.5}$$

Szegö's theorem asserts that

$$\lim_{M\to\infty}\frac{\det C_M}{[G(f)]^{M+1}}=\exp\left[\sum_{l=1}^{\infty}lK_lK_{-l}\right]\ . \tag{3.6}$$

This theorem is proved, with the aid of Weierstrass's approximation theorem, through skillful and lengthy discussions.[17] If $f(\theta)$ in (3.6) is replaced by $1-\xi f(\theta)$, then $\det C_M/[G(f)]^{M+1}$ is replaced by

$$\frac{\det(1-\xi C_M)}{[G(1-\xi f)]^{M+1}}=\exp\left[2\sum_{n=1}^{\infty}\frac{1}{n}e_n(M)\xi^n\right], \tag{3.7}$$

$$e_n(M)=-\frac{1}{2}\left[\sum_{j=1}^{M+1}[\lambda_j(M)]^n-\frac{M+1}{2\pi}\int_0^{2\pi}f^n(\theta)d\theta\right], \tag{3.8}$$

where $\lambda_j(M)$, $j=1,2,\ldots,M+1$, are the eigenvalues of C_M. Similarly, $\exp(\sum_{l=1}^{\infty}lK_lK_{-l})$ is replaced by

$$\exp\left[\sum_{l=1}^{\infty}l\frac{1}{2\pi}\int_0^{2\pi}e^{-il\theta}\ln[1-\xi f(\theta)]d\theta\frac{1}{2\pi}\int_0^{2\pi}e^{il\theta}\ln[1-\xi f(\phi)]d\phi\right]=\exp\left[2\sum_{n=1}^{\infty}\frac{1}{n}d_n\xi^n\right], \tag{3.9}$$

where $\{d_n\}$ are given by

$$d_1=0, \tag{3.10}$$

$$d_n=\frac{1}{2}\sum_{l=1}^{\infty}l\sum_{m=1}^{n-1}\frac{n}{m(n-m)}\frac{1}{2\pi}\int_0^{2\pi}e^{-il\theta}[f(\theta)]^m d\theta\frac{1}{2\pi}\int_0^{2\pi}e^{il\phi}[f(\phi)]^{n-m}d\phi,\quad n=2,3,\ldots. \tag{3.11}$$

Appealing to Szegö's theorem, we obtain the relations

$$\lim_{M\to\infty}e_n(M)=d_n,\quad n=2,3,4,\ldots, \tag{3.12}$$

and

$$\lim_{M\to\infty}\left[\mathrm{Tr}C_M-\frac{M+1}{2\pi}K_0\right]=0, \tag{3.13}$$

which involve much information on the distribution of the eigenvalues of the infinite Toeplitz matrix $\lim_{M\to\infty}C_M$.

We now consider a set of independent, identically distributed, random variables $\{X_1,X_2,\ldots,X_n\}$ capable of integral values only. We assume that

$$\mathrm{Prob}\{X_j=k\}=\mathrm{Prob}\{X_j=-k\}=c_k=c_{-k}, \tag{3.14}$$

where $\mathrm{Prob}\{X_j=k\}$ denotes the probability that X_j takes the value k. For a probabilistic interpretation, c_k should be assumed to satisfy $c_k\ge 0$, $\sum_{k=-\infty}^{\infty}c_k=1$. The relations discussed below, however, hold valid without these restrictions. We define S_n, p_n, q_n, and r_n by

$$S_n=X_1+X_2+\cdots+X_n, \tag{3.15}$$

$$p_n=\mathrm{Prob}\{S_n=0\}E\{\max(0,S_1,S_2,\ldots,S_{n-1})|S_n=0\}, \tag{3.16}$$

$$q_n=\sum_{k=1}^{n-1}\frac{1}{k}\sum_{l=1}^{\infty}\mathrm{Prob}\{S_k=l,S_n=0\} \tag{3.17}$$

and

$$r_n=\sum_{l_1+l_2+\ldots+l_n=0}\mathrm{Max}(0,l_1,l_1+l_2,\ldots,l_1+\ldots+l_n)\times c_{l_1}c_{l_2}\cdots c_{l_n}, \tag{3.18}$$

where $E\{\alpha|\beta\}$ denotes the expectation value of α under the condition β. Making use of a curious combinatorial identity of Hunt and Dyson, it can be shown that[18,19]

$$p_n=q_n=r_n. \tag{3.19}$$

Recalling the relation

$$\mathrm{Prob}\{S_k=l\}=\sum_{l_1+l_2+\cdots+l_k=l}c_{l_1}c_{l_2}\cdots c_{lk}=\frac{1}{2\pi}\int_0^{2\pi}e^{-il\theta}[f(\theta)]^k d\theta, \tag{3.20}$$

it is easy to understand that d_n equals q_n. Kac gave an independent argument showing[19]

$$\lim_{M\to\infty}e_n(M)=r_n \tag{3.21}$$

and regarded the purely probabilistic relation $p_n=q_n$ as the reinterpretation of Szegö's theorem. He could also generalize Szegö's result and obtained a formula to calculate the Fredholm determinant of some class of integral equations.[19] It should be noted that Szegö's result has been utilized to calculate matrix elements of the fermionic many-body problem.[21]

B. Derivation of Szegö's theorem in terms of the boson-fermion correspondence

As noted in the previous subsection, both the analytic and the probabilistic proofs of Szegö's theorem are rather involved.[17–19] We are to give a simple proof of it on the basis of the boson-fermion correspondence. We consider a fermionic matrix element

$$I_M=\langle 0|\Lambda^{-1}\Psi_M\Lambda\Psi_M^*|0\rangle, \tag{3.22}$$

where Λ, Ψ_M, and Ψ_M^* are defined by

$$\Lambda=\Lambda_1\Lambda_2=\Lambda_2\Lambda_1, \tag{3.23}$$

$$\Lambda_1=\exp(-K_0J_0), \tag{3.24}$$

$$\Lambda_2=\exp\left[\frac{1}{2\pi}\int_0^{2\pi}\ln f(\theta)J(\theta)d\theta\right]$$
$$=\exp\left[\frac{F}{2}\right]\exp\left[\sum_{n\le 0}K_nJ_n\right]\exp\left[\sum_{n>0}K_nJ_n\right]$$
$$=\left[\exp\left[\frac{F}{2}\right]\exp\left[-\sum_{n\le 0}K_nJ_n\right]\times\exp\left[-\sum_{n>0}K_nJ_n\right]\right]^{-1}, \tag{3.25}$$

$$F=\sum_{n>0}nK_nK_{-n}, \tag{3.26}$$

$$\Psi_M=\psi_{1/2}\psi_{3/2}\cdots\psi_{M+1/2}, \tag{3.27}$$

$$\Psi_M^*=\psi^*_{-M-1/2}\psi^*_{-M+1/2}\cdots\psi^*_{-1/2}. \tag{3.28}$$

Here $J(\theta)$ and J_n are the current operators discussed in Sec. II and $\{K_l\}$ and $f(\theta)$ are those introduced in Sec. III A. Noticing $\langle 0|J_n=0$, $n\le 0$ and $(\Lambda_1)^{-1}\psi_\mu\Lambda_1=\exp(-K_0)\psi_\mu$, we obtain

$$I_M=e^{F/2}e^{-(M+1)K_0}\times\left\langle 0\left|\exp\left[-\sum_{n>0}K_nJ_n\right]\Psi_M\Lambda_2\Psi_M^*\right|0\right\rangle. \tag{3.29}$$

From (3.5),

$$(\Lambda_2)^{-1}\psi(\theta)\Lambda_2=f(\theta)\psi(\theta), \tag{3.30}$$

$$\langle 0|\exp\left[-\sum_{n>0}K_nJ_n\right]\Lambda_2=\exp\left[-\frac{F}{2}\right]\langle 0| \tag{3.31}$$

and Wick's theorem, we obtain

$$I_M=\frac{\det C_M}{[G(f)]^{M+1}}, \tag{3.32}$$

the $M\to\infty$ limit of which is equal to the LHS of (3.6). I_M can be written as

$$I_M=e^F\left\langle 0\left|\exp\left[-\sum_{n>0}K_nJ_n\right]\Psi_M\exp\left[\sum_{n<0}K_nJ_n\right]\exp\left[\sum_{n>0}K_nJ_n\right]\Psi_M^*\right|0\right\rangle. \tag{3.33}$$

Observing that (see the Appendix)

$$\lim_{M\to\infty}\langle 0|\exp\left[-\sum_{n>0}K_nJ_n\right]\Psi_M=\lim_{M\to\infty}\langle 0|\Psi_M, \tag{3.34}$$

we are led to

$$\lim_{M\to\infty}I_M=e^F\lim_{M\to\infty}\left\langle 0\left|\Psi_M\exp\left[\sum_{n<0}K_nJ_n\right]\exp\left[\sum_{n>0}K_nJ_n\right]\Psi_M^*\right|0\right\rangle. \tag{3.35}$$

The simplest way to calculate the matrix element on the RHS of (3.35) is to bosonize it:

$$\left\langle 0\left|\Psi_M\exp\left[\sum_{n<0}K_nJ_n\right]\exp\left[\sum_{n>0}K_nJ_n\right]\Psi_M^*\right|0\right\rangle=\left\langle 0,-M-1\left|\exp\left[-\sum_{n<0}K_na_n\right]\exp\left[-\sum_{n>0}K_na_n\right]\right|-M-1,0\right\rangle=1, \tag{3.36}$$

where we have made use of $\langle 0,-M-1|a_{-n}=a_n|-M-1,0\rangle=0$, $n>0$ and (2.25). We thus obtain

$$\lim_{M\to\infty}I_M=e^F. \tag{3.37}$$

The above result, together with (3.32), completes the proof of Szegö's theorem. We note that the above discussion indicates

$$e^F=\lim_{M\to\infty}\left|\left|\exp\left[\sum_{n<0}K_nJ_n\right]\Psi_M^*|0\rangle\right|\right|^2$$
$$=\lim_{M\to\infty}\left|\left|\exp\left[-\sum_{n<0}a_nK_n\right]|-M,0\rangle\right|\right|^2$$
$$=\lim_{M\to\infty}\left|\left|:\exp\left[\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{1}{2\pi}\int_0^{2\pi}d\theta'\ln\left[\frac{f(\theta')}{f(\theta)}\right]\frac{d\phi(\theta)}{d\theta}\right]:_b|-M,0\rangle\right|\right|^2, \tag{3.38}$$

where $:\,:_b$ denotes bosonic normal ordering. We note that some care must be taken to apply the above result to physics on the Dirac sea $\Psi_\infty^*|0\rangle$ since $\Psi_M^*|0\rangle$ with M large and the Dirac sea belong to different equivalence classes of Hilbert space.[2]

C. A special case

We here consider the special case that $f(\theta)$ is given by

$$f(\theta)=\frac{1}{|\gamma(z_0-z)(z_1-z)\cdots(z_p-z)|^2}, \quad z=e^{i\theta}, \tag{3.39}$$

where $\gamma, z_0, z_1, \ldots, z_p$ are complex numbers satisfying

$$|z_i| > 1, \quad i=0,1,\ldots,p \ , \tag{3.40}$$

$$z_i \neq z_j, \quad i \neq j \tag{3.41}$$

and p is an arbitrary non-negative integer. In this case, K_n's are calculated to be

$$K_n = \frac{1}{n}\sum_{i=0}^{p}(z_i)^{-n} = (K_{-n})^*, \quad n=1,2,3,\ldots,$$
$$K_0 = -2\ln|\gamma z_0 z_1 \cdots z_p| \ . \tag{3.42}$$

An interesting fact is that $\det C_M/[G(f)]^{M+1}$ for this case is independent of M if M is larger than or equal to p. This fact was proved with the help of ingenious inequalities among the Toeplitz determinants.[17] Translated to our words, it can be expressed as

$$|\gamma z_0 z_1 \cdots z_p|^{2M}\langle 0|\lambda^{-1}\Psi_M \lambda \Psi_M^*|0\rangle$$
$$= |\gamma z_0 z_1 \cdots z_p|^{2p}\langle 0|\lambda^{-1}\Psi_p \lambda \Psi_p^*|0\rangle, \quad M \geq p \ , \tag{3.43}$$

where λ is given by Λ_2 in (3.25) with $f(\theta)$ specified to (3.39):

$$\lambda = \exp\left\{ \sum_{n=1}^{\infty}\frac{1}{n}\left[\left[\sum_{i=0}^{p}(z_1)^{-n}\right]J_n + \left[\sum_{i=0}^{p}(z_i^*)^{-n}\right]J_{-n}\right] - 2J_0\ln|\gamma z_0 \cdots z_p| \right\} . \tag{3.44}$$

In the following, we shall discuss (3.43) in a purely fermion theoretical language and obtain a somewhat generalized result.

We first note

$$\lambda^{-1}\psi_\mu \lambda = \sum_{\nu \in Z+1/2}\psi_\nu \frac{1}{2\pi i}\oint_{|z|=1} dz\, z^{\mu-\nu-1} g(z) \ , \tag{3.45}$$

where $g(z)$ is defined by

$$g(z) = \frac{1}{\gamma\gamma^* \prod_{i=0}^{p}(z_i - z)(z_i^* - z^{-1})} = f(\theta), \quad z = e^{i\theta} \ . \tag{3.46}$$

For $\nu \leq \mu + p$, the integral on the RHS of (3.45) is given by the sum of residues at $z = (z_i^*)^{-1}$, $i=0,1,\ldots,p$, while for $\nu \geq \mu + p + 1$ the residue at $z=0$ contributes. Then, we have

$$\lambda^{-1}\psi_\mu \lambda = A_\mu + B_\mu \ , \tag{3.47}$$

$$A_\mu = \sum_{i=0}^{p} a_i(\mu)\psi[(z_i^*)^{-1}] \ , \tag{3.48}$$

$$B_\mu = \sum_{\nu \geq \mu+p+1} b_\nu(\mu)\psi_\nu \ , \tag{3.49}$$

where $a_i(\mu)$ and $b_\nu(\mu)$ are given by

$$a_i(\mu) = \frac{(-1)^{p+1}}{\gamma\gamma^*}(z_i^*)^{p-\mu-1/2}\left[\prod_{j=0}^{p}(1 - z_i^* z_j)\prod_{j\neq i}(z_j^* - z_i^*)\right]^{-1} , \tag{3.50}$$

$$b_\nu(\mu) = \frac{1}{\gamma\gamma^*}\frac{1}{(k-1)!}\left[\frac{d}{dz}\right]^{k-1}\left[\prod_{i=0}^{p}(z_i - z)(z z_i^* - 1)\right]^{-1}\Bigg|_{z=0,\, k=\nu-\mu-p \geq 1} \tag{3.51}$$

Especially we have

$$b_{\mu+p+1}(\mu) = (-1)^{p+1}(\gamma\gamma^* z_0 z_1 \cdots z_p)^{-1}, \quad \mu = \tfrac{1}{2}, \tfrac{3}{2}, \ldots, \tag{3.52}$$

and

$$(z_i^*)^\mu a_i(\mu) = (z_i^*)^\nu a_i(\nu) \ . \tag{3.53}$$

For $\mu \geq M - p + \frac{1}{2}$, B_μ consists of ψ_ν with $\nu \geq M + \frac{3}{2}$. Then we have

$$\lambda^{-1}\psi_{M-p+1/2}\psi_{M-p+3/2}\cdots\psi_{M+1/2}\lambda\Psi_M^*|0\rangle = A_{M-p+1/2}A_{M-p+3/2}\cdots A_{M+1/2}\Psi_M^*|0\rangle \ . \tag{3.54}$$

Recalling that $\{\psi(z)\}^2 = 0$, we find

$$\lambda^{-1}\Psi_M \lambda \Psi_M^*|0\rangle = \lambda^{-1}\psi_{1/2}\psi_{3/2}\cdots\psi_{M-p-1/2}\lambda\lambda^{-1}\psi_{M-p+1/2}\cdots\psi_{M+1/2}\lambda\Psi_M^*|0\rangle$$
$$= B_{1/2}B_{3/2}\cdots B_{M-p-1/2}A_{M-p+1/2}\cdots A_{M+1/2}\Psi_M^*|0\rangle$$
$$= \left[\prod_{\frac{1}{2}\leq\mu\leq M-p-\frac{1}{2}} b_{\mu+p+1}(\mu)\right](-1)^{(M-p)(p+1)}A_{M-p+1/2}\cdots A_{M+1/2}$$
$$\times(\psi_{p+3/2}\psi_{p+5/2}\cdots\psi_{M+1/2})(\psi^*_{-M-1/2}\cdots\psi^*_{-p-3/2})\Psi_p^*|0\rangle \ . \tag{3.55}$$

From Eqs. (3.52), (3.53), and (3.55), we obtain

$$\lambda^{-1}\Psi_M\lambda\Psi_M^*|0\rangle=|\gamma z_0 z_1\cdots z_p|^{2p-2M}A_{1/2}A_{3/2}\cdots A_{p+1/2}\Psi_p^*|0\rangle$$
$$=|\gamma z_0 z_1\cdots z_p|^{2p-2M}\lambda^{-1}\Psi_p\lambda\Psi_p^*|0\rangle\ . \tag{3.56}$$

Now, instead of (3.43), we have

$$|\gamma z_0\cdots z_p|^{2M}\langle\Phi|\lambda^{-1}\Psi_M\lambda\Psi_M^*|0\rangle=|\gamma z_0\cdots z_p|^{2p}\langle\Phi|\lambda^{-1}\Psi_p\lambda\Psi_p^*|0\rangle,\quad M\geq p\ , \tag{3.57}$$

where $\langle\Phi|$ is an arbitrary state vector. We realize that the fermionic algorithm leads us to (3.56) or (3.57) which is a generalization of (3.43).

IV. SUMMARY

In Sec. II in order to complete the correspondence between bosons and fermions in two-dimensional space-time, we sought the fermionized expression q_f of the bosonic zero-mode operator q. It turned out that no simple form of q_f bilinear in fermion fields exists. The expression (2.19) with J_n given by (2.10) is seemingly complicated but does not result in any inconvenience for actual calculation. Its apparent z dependence disappears thanks to the identity (2.21) which is inherent in the theory and equivalent to the property that the energy-momentum tensor (2.12a) can be written in the Sugawara form (2.12b). By the alternative expression (2.32) of q_f, q_f can be formally related to the geometric mean of the fermion field. We note, however, that a clear meaning of $\ln\psi(z)$ with $\{\psi(z)\}^2=0$ is yet to be sought.

In Sec. III, we applied the boson-fermion correspondence to prove a mathematical theorem of Szegö. Our proof was much simpler, at least for physicists, than those given previously. We could also derive a somewhat generalized version of a relation among special Toeplitz matrices. We see that bosonization technique is helpful because it automatically takes cumbersome Schwinger terms in the fermion theory into account.

ACKNOWLEDGMENTS

The authors are grateful to Professor K. Matumoto, Professor S. Hamamoto, and Professor T. Tajima for useful discussions.

APPENDIX

In this appendix, we prove (3.36). We begin with observing

$$\langle 0|J_n\Psi_M=\langle 0|\sum_{\mu\in\mathbb{Z}+\frac{1}{2}}\psi_\mu^*\psi_{n-\mu}\Psi_M$$
$$=\langle 0|\sum_{0<\nu<n}\psi_{n-\nu}^*\psi_\nu\Psi_M=0 \tag{A1}$$

for $0<n\leq M+1$ and that

$$\langle 0|J_nJ_m\Psi_M=\langle 0|\left[\sum_{0<\nu<n}\psi_{n-\nu}^*\psi_\nu\right]\left[\sum_{\mu\in\mathbb{Z}+\frac{1}{2}}\psi_\mu^*\psi_{m-\mu}\right]\Psi_M$$
$$=\langle 0|\left[\sum_{0<\nu<n}\psi_{n-\nu}^*\psi_\nu-\sum_{0<\nu<n}\sum_{\mu\in\mathbb{Z}+\frac{1}{2}}\psi_{n-\nu}^*\psi_\mu^*\psi_\nu\psi_{m-\mu}\right]\Psi_M=0 \tag{A2}$$

for $n,m>0, n+m\leq M+1$. Similarly, we have

$$\langle 0|J_{n_1}J_{n_2}\cdots J_{n_N}\Psi_M=0\ , \tag{A3}$$

for $n_1,n_2,\ldots,n_N>0, n_1+n_2+\cdots+n_N\leq M+1$. Then we obtain

$$\langle 0|\left[\exp\left[-\sum_{n>0}^{\infty}K_nJ_n\right]-1\right]\Psi_M=\langle\Phi_M|\Psi_M\ , \tag{A4}$$

where $\langle\Phi_M|$ is defined by

$$\langle\Phi_M|=\langle 0|\sum_{N=1}^{\infty}\frac{(-1)^N}{N!}\sum_{n_1+n_2+\cdots+n_N>M+1}K_{n_1}\cdots K_{n_N}J_{n_1}\cdots J_{n_N}\ . \tag{A5}$$

Noticing that $\Psi_M\Psi_M^*$ is a projection operator, we have

$$\langle\Phi_M|\Psi_M\Psi_M^*|\Phi_M\rangle\leq\langle\Phi_M|\Phi_M\rangle\ . \tag{A6}$$

On the other hand, $\langle\Phi_M|\Phi_M\rangle$ is given by

$$\langle\Phi_M|\Phi_M\rangle=\sum_{N=1}^{\infty}\sum_{L=1}^{\infty}\frac{(-1)^{N+L}}{N!L!}\sum_{n_1+\cdots+n_N>M+1}\sum_{l_1+\cdots l_L>M+1}K_{n_1}\cdots K_{n_N}K_{l_1}^*\cdots K_{l_L}^*\langle 0|J_{n_1}\cdots J_{n_N}J_{-l_1}\cdots J_{-l_L}|0\rangle. \tag{A7}$$

Bosonizing the RHS of (A7), we obtain

$$\langle \Phi_M | \Phi_M \rangle = \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{\substack{n_1>0 \\ n_1+\cdots+n_N>M+1}} \cdots \sum_{n_N>0} n_1 |K_{n_1}|^2 n_2 |K_{n_2}|^2 \cdots n_N |K_{n_N}|^2 . \tag{A8}$$

Comparing the last expression with a convergent expansion of $\exp(F)-1$, we obtain

$$\lim_{M\to\infty} \langle \Phi_M | \Phi_M \rangle = 0 . \tag{A9}$$

From Eqs. (A4), (A6), and (A9), we conclude (3.36).

[1]S. Tomonaga, Prog. Theor. Phys. **5**, 544 (1950).

[2]D. C. Mattis and E. H. Lieb, J. Math. Phys. **6**, 304 (1965).

[3]S. Mandelstam, Phys. Rev. D **11**, 3026 (1975).

[4]S. Coleman, Phys. Rev. D **11**, 2088 (1975).

[5]T. Eguchi and K. Higashijima, Prog. Theor. Phys. **86**, 192 (1986).

[6]E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, in *Non-Linear Integrable Systems—Classical Theory and Quantum Theory,* proceedings of the RIMS Symposium, Kyoto, Japan, 1981, edited by M. Jimbo and T. Miwa (World Scientific, Singapore, 1983).

[7]N. Kawamoto, Y. Namikawa, A. Tsuchiya, and Y. Yamada, Commun. Math. Phys. **116**, 247 (1988).

[8]P. Jordan, Z. Phys. **93**, 464 (1935).

[9]S. Saito, Phys. Rev. D **37**, 990 (1988).

[10]M. Sato, *Soliton Equations and Universal Grassmann Manifolds* (Sophia University Lecture Notes No. 18) (Department of Mathematics, Sophia University, Tokyo, 1984).

[11]T. Eguchi and H. Ooguri, Phys. Lett. B **187**, 127 (1987).

[12]S. Saito, Phys. Rev. Lett. **59**, 1798 (1987).

[13]R. Hirota, J. Phys. Soc. Jpn. **50**, 3785 (1981).

[14]L. Alvarez-Gaumé, Commun. Math. Phys. **90**, 161 (1983).

[15]D. Friedan and P. Windey, Nucl. Phys. **B235** [FS11], 395 (1984).

[16]E. Witten, J. Diff. Geom. **17**, 661 (1982).

[17]V. Grenander and G. Szegö, *Toeplitz Forms and Their Applications* (University of California Press, Berkeley and Los Angeles, CA, 1958).

[18]M. Kac, *Probability and Related Topics in Physical Science* (International, London, 1959).

[19]M. Kac, Duke. Math. J. **21**, 501 (1954).

[20]H. Sugawara, Phys. Rev. **170**, 1659 (1968).

[21]J. M. Luttinger, J. Math. Phys. **4**, 1154 (1963).