On Dirac's conjecture for Hamiltonian systems with first- and second-class constraints

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It is shown for a wide class of systems in the framework of the total Hamiltonian procedure that all first-class constraints generate canonical transformations connecting physically equivalent states. It occurs whenever the constraints arising in the Dirac algorithm are effective when considered in the functional form as they appear in the consistency conditions. The property of hereditary separation between first- and second-class constraints also follows from the above condition. General Poisson-brackets relations among constraints in the representation used here are also obtained. The sources of anomalies in the hereditary property reported in the literature are identified.

I. INTRODUCTION

Nowadays, the relevance of the Dirac canonical procedure for modern quantum field theory is widely recognized.[1,2] Summarizing the results of the previous years, it can be said that this method furnishes the classical basis for a powerful Becchi-Rouet-Stora-Tyutin–Batalin-Fradkin-Vilkovisky (BRST-BFV) gauge field quantization procedure. This development represents the most elegant and general way of quantizing gauge or any field theory.[3–8]

Many investigations have been dedicated to clarifying the general structure of constrained Hamiltonian systems at the classical level.[9–11] At present many points have been well understood. However, in the literature there exists some unanswered questions.[8,10,12–15] A particular question is connected with whether the so-called first-class constraints are always generators of canonical transformations mapping physical states into equivalent ones (Dirac's conjecture). This work is mainly devoted to this problem. We want to remark that in Ref. 2 Gitman and Tyutin argued a solution of this problem in an affirmative way. Their approach is based on the specific recourse of embedding the solutions of the total Hamiltonian canonical procedure in a wider problem. In this approach additional Lagrange multipliers for all the constraints appearing in the system are introduced.

In this paper we present a proof of the conjecture by working always in the context of the total Hamiltonian procedure. The meaning of the equivalence accepted by us is the same as the one used by Dirac:[1] two points in phase space (states) are considered as equivalent when they evolve from another point in a previous instant of time according to two total Hamiltonians coming from the same Lagrange system; i.e., they differ at most in the Lagrange multipliers of the primary first-class constraints. This work constitutes a generalization of the argumentation and results of Ref. 14 to include second-class as well as first-class constraints. The case in which only first-class constraints are present was also discussed in Refs. 12 and 13.

The results of this paper clarify to some extent the conditions which must obey the constraints appearing in the Dirac algorithm in order to prove the validity of the Dirac conjecture. A main requirement is to satisfy the conditions of effectiveness enunciated in Sec. II. The examples given in Ref. 15 showing the breakdown of Dirac's conjecture are related to the absence of effectiveness properties in these cases.

Section II is dedicated to showing the special structure of the set of constraints which is the central technical difficulty in showing the conjecture. After that, in Sec. III the canonical transformation generators mapping physical states into physically equivalent ones are constructed. This result opens the way to the proof of the conjecture which is the objective of Sec. VI. In Appendixes A–C some auxiliary theorems and other results are argued.

II. SEPARATION PROPERTIES BETWEEN FIRST- AND SECOND-CLASS CONSTRAINTS

This section, as was mentioned in the Introduction, is devoted to deducing the particular structure of the set of constraints which is very important in obtaining the expected results. A similar property was more directly found when there were only first-class constraints.[12–14] Here the derivation is technically more difficult. Some of the basic results implying a long algebraic development will be deferred to a few appendixes.

We start by reviewing the basic elements of Dirac's total Hamiltonian formalism as developed by Batlle, Gomis, Pons, and Roman-Roy.[11] Their presentation and results gave an optimal starting basis for the discussion in this section. Next, a modification of the set of constraints appearing in their algorithm is introduced. This will be needed for the proof of the conjecture in the following sections. General commutation relations for the modified set of constraints are obtained. The modified set of constraints retains the property that all the constraints obtained from the primary first-class ones by performing successive Poisson brackets with the so-called H' Hamiltonian are also first class. The general commutation relations allow us, in addition, to show that the determinant of the matrix formed with the Poisson brackets among all the second-class constraints is different from zero. This guarantees that no linear combination of second-class constraints turns into a first-class one.

The mechanical state of a system satisfies the following canonical equations of motion in the total Hamiltonian approach:[11]

$$\dot{q}_i=\{q_i,H_T\} \ , \tag{1}$$

$$\dot{p}_i=\{p_i,H_T\}, \quad i=1,\ldots,n \ , \tag{2}$$

$$\phi_\mu^{(0)}(q,p)=0, \quad \mu=1,\ldots,m_1 \ , \tag{3}$$

where the total Hamiltonian H_T is given as

$$H_T(q,p,\lambda)=H_c(q,p)+\lambda_\mu\phi_\mu^{(0)}(q,p) \ , \tag{4}$$

and $\phi_\mu^{(0)}$ are primary constraints. The corresponding Lagrange multipliers are denoted by λ_μ. λ_μ are functions of time and the Poisson brackets (PB) is defined as usual. Finally the Hamiltonian $H_c(q,p)$ is any function of (q,p), satisfying

$$H_c\left[q,\frac{\partial L}{\partial \dot{q}}\right]=\dot{q}_i\cdot\frac{\partial L}{\partial \dot{q}_i}-L(q,\dot{q}) \ . \tag{5}$$

Following the procedure developed in Ref. 11 it is possible to find the complete set of constraints of the system in the form

$$\begin{array}{ll} \phi_{\mu_0'}^{(0)}\phi_{\mu_1'}^{(0)}\phi_{\mu_2'}^{(0)}\cdots\phi_{\mu_f'}^{(0)}\phi_{\mu_f}^{(0)} & \text{primary constraints ,} \\ \phi_{\mu_1'}^{(1)}\phi_{\mu_2'}^{(1)}\cdots\phi_{\mu_f'}^{(1)}\phi_{\mu_f}^{(1)} & \text{secondary constraints ,} \\ \phi_{\mu_2'}^{(2)}\cdots\phi_{\mu_f'}^{(2)}\phi_{\mu_f}^{(2)} & \\ \cdots\phi_{\mu_f'}^{(f)}\phi_{\mu_f}^{(f)} & f\text{-ary constraints .} \end{array} \tag{6}$$

In our notation $\phi_{\mu_{i-1}}^{(i)}$ are all i-ary constraints. The $\phi_{\mu_i}^{(i)}$ are all i-ary constraints except $\phi_{\mu_i'}^{(i)}$. $\phi_{\mu_j}^{(i)}$ $(j\geq i)$ are all i-ary constraints except $\phi_{\mu_i'}^{(i)}\cdots\phi_{\mu_f'}^{(i)}$ $(\mu\equiv\mu_{-1})$.

The manifold in the phase space which is defined by the annulation of all the constraints (6) up to superindex k will be denoted as M_k.

The explicit form of the constraints (6) is defined in the following iterative way:[11]

$$\phi_{\mu_{k-1}}^{(k)}=\{\phi_{\mu_{k-1}}^{(k-1)},H_c^{(k)}\}, \quad k=1,2,\ldots,f \ . \tag{7}$$

The k-ary Hamiltonian $H_c^{(k)}$ is defined also iteratively through

$$H_c^{(k)}=H_c^{(k-1)}+\sigma_{\mu_{k-1}'}(q,p)\phi_{\mu_{k-1}'}^{(0)} \quad k=1,\ldots,f+1 \tag{8}$$

with $H_c^{(0)}=H_c(q,p)$ and the functions $\sigma_{\nu_k'}$ given as

$$\sigma_{\mu_k'}(q,p)=(-1)^k(C^{(k)})^{-1}_{\nu_k'\mu_k'}\{H_c^{(k)},\phi_{\nu_k'}^{(k)}\} \ . \tag{9}$$

The matrix $C^{(k)}$ is defined as

$$C_{\nu_k'\mu_k'}^{(k)}(q,p)=\{\phi_{\nu_k'}^{(k)}(q,p),\phi_{\mu_k'}^{(0)}(q,p)\} \ . \tag{10}$$

It should be remarked that $\sigma_{\mu_k'}$ are functions which coincide with the Lagrange multipliers $\lambda_{\mu_k'}$ $(k=0,\ldots,f)$ on the final manifold M_f, that is, those which are fully determined by the Dirac algorithm. The multipliers λ_{μ_f} remain completely undetermined.[11]

The total Hamiltonian may be rewritten as

$$H_T=H'+\lambda_{\mu_f}\phi_{\mu_f}^{(0)} \ , \tag{11}$$

$$H'\equiv H_c^{(f+1)} \ . \tag{12}$$

The Hamiltonian H' will be a recurrent element in the discussion below. The following Poisson-brackets relations were also obtained in Ref. 11:

$$\{\phi_{\mu_i'}^{(i)},H_c^{(i+1)}\}\underset{M_i}{=}0, \quad i=0,\ldots,f \ , \tag{13}$$

$$\{\phi_{\mu_f'}^{(f)},H_c^{(f+1)}\}\underset{M_f}{=}0 \tag{14}$$

and

$$\{\phi_{\mu_k}^{(k)},\phi_{\mu_{k-1}}^{(0)}\}\underset{M_k}{=}0, \quad \{\phi_{\mu_{k-1}}^{(k)},\phi_{\mu_k}^{(0)}\}\underset{M_k}{=}0 \ , \tag{15}$$

$$\det|\{\phi_{\mu_k'}^{(k)},\phi_{\nu_k'}^{(0)}\}|\underset{M_k}{\neq}0, \quad k=0,1,\ldots,f \ . \tag{16}$$

Let us introduce now a new collection of constraints also in an iterative way:

$$\begin{aligned} &\chi_{\mu_{-1}}^{(0)}\equiv\phi_{\mu_{-1}}^{(0)} \ , \\ &\chi_{\mu_{-1}}^{(k)}\equiv\{\chi_{\mu_{-1}}^{(k-1)},H'\}, \quad k=1,\ldots \ . \end{aligned} \tag{17}$$

Both sets of constraints $\{\phi\}$ and $\{\chi\}$ are equivalent. That is, the manifold M_f determined by the annulation of the constraints ϕ is the same as determined by the vanishing of the functions χ. The proof of this statement is given in Appendix A.

We have also the following structure for the new set of constraints:

$$\begin{array}{l} \chi_{\mu_0'}^{(0)}\chi_{\mu_1'}^{(0)}\chi_{\mu_2'}^{(0)}\cdots\chi_{\mu_f'}^{(0)}\chi_{\mu_f}^{(0)} \ , \\ \chi_{\mu_1'}^{(1)}\chi_{\mu_2'}^{(1)}\cdots\chi_{\mu_f'}^{(1)}\chi_{\mu_f}^{(1)} \ , \\ \chi_{\mu_2'}^{(2)}\cdots\chi_{\mu_f'}^{(2)}\chi_{\mu_f}^{(2)} \\ \cdots\chi_{\mu_f'}^{(f)}\chi_{\mu_f}^{(f)} \ . \end{array} \tag{18}$$

Let us define now a set of effective constraints ψ_i as one for which all the constraints have nonvanishing gradients $\partial\psi_i/\partial x_j$ $(x_j=q_j,x_{n+j}=p_j,j=1,\ldots,n)$ and all the gradients are linearly independent at any point of M_f.

It may occur, for example, that some of the constraint hypersurfaces in spite of reducing actually the dimensionality of the initial manifold have common tangent spaces. An example of this situation corresponds to the intersection of a cylinder with any of its tangential planes. In this case these constraints have nonvanishing gradients but they are not linearly independent in the final manifold. Therefore they are not effective.

An important assumption in this work will be that the set of actual constraints obtained up to any stage of the Dirac algorithm is effective in the exact functional way that they appear from the consistency conditions. Of course, we do not consider the possible identities that can appear in the Dirac algorithm as actual constraints.

It is also useful to state that in Ref. 11 and in this paper another implicit assumption is that the determinants of all the $C^{(k)}$ matrices in (9) retain their properties of being different from zero through all the stages of the Dirac algorithm. Those cases in which the determinants do not have the property mentioned above are related possibly to the discussion provided in Ref. 10.

In Appendix A it is also shown that the χ satisfy the same PB relations among themselves which obey ϕ; that is,

$$\{\chi^{(k)}_{\mu_k},\chi^{(0)}_{\mu_{k-1}}\}\underset{M_k}{=}0,\quad \{\chi^{(k)}_{\mu_{k-1}},\chi^{(0)}_{\mu_k}\}\underset{M_k}{=}0\ , \tag{19}$$

$$\det|\{\chi^{(k)}_{\mu'_k},\chi^{(0)}_{\nu'_k}\}|\underset{M_k}{\neq}0\ . \tag{20}$$

Relations (19) and (20) are not enough to prove the separation property between first- and second-class constraints. A sufficient generalization of the PB relations (19) and (20) for further discussion is given by the following theorem.

Theorem. All the constraints $\chi^{(k)}_{\mu_{k+i-1}}$ $(k=0,1,\ldots,f)$ PB commute with all the constraints $\chi^{(i)}_{\mu_{i-1}}$ except the $\chi^{(k)}_{\mu'_{k+i}}$ and $\chi^{(i)}_{\mu'_{k+i}}$ which obey

$$\det|\{\chi^{(k)}_{\mu'_{k+i}},\chi^{(i)}_{\mu'_{k+i}}\}|\underset{M_{k+i}}{\neq}0\ , \tag{21}$$

when $i\le k$, and

$$i\le f-k\quad(\text{for even } f)\ ,$$

$$i\le f-k-1\quad(\text{for odd } f)\ .$$

In addition

$$\{\chi^{(k)}_{\mu'_{k+s}},\chi^{(i)}_{\mu_{k+i-1}}\}\underset{M_{k+i}}{=}0,\quad s=0,\ldots,i-1\ . \tag{22}$$

The proof of the theorem is given in Appendix B.

The results reviewed above are sufficient to prove the hereditary property of the first-class constraints which is so decisive in demonstrating the truth of the Dirac conjecture.

First, it must be noticed that all $\chi^{(0)}_{\mu_f}$ are first-class constraints. This is a direct consequence of the theorem by putting $i=0$ and any value of k. Then

$$\{\chi^{(k)}_{\nu_{k-1}},\chi^{(0)}_{\mu_f}\}\underset{M_f}{=}0,\quad k=0,1,\ldots,f\ .$$

But, as the Hamiltonian H' is a first-class function[1] the PB of H' with any first-class constraints will be first class.[1] Then, the subfamily of constraints χ given as

$$\chi^{(0)}_{\mu_f}=\phi^{(0)}_f\ ,$$
$$\chi^{(k)}_{\mu_f}=\{\chi^{(k-1)}_{\mu_f},H'\},\quad k=1,\ldots,f \tag{23}$$

are all first-class restrictions.

Now, it is possible to demonstrate that the set (23) includes all the first-class constraints of the system. For this purpose in Appendix C we show that the determinant of the matrix formed by the PB among all the constraints not included in (23) is different from zero in M_f. This fact implies that no linear combination of the constraints which are not included in (23) can be first class. Provided that this linear combination can exist, the coefficients of such a combination will form a null vector of the above-mentioned matrix for the points $(q,p)\in M_f$. The obtainment of the hereditary structure (23) for the set of first-class constraints was the main objective of this section.

III. GENERATORS WHICH MAP SOLUTIONS INTO SOLUTIONS

In this section we introduce new nomenclature for the constraints. We denote as ψ^a_n the first-class constraints $(a=1,\ldots,m-r;n=0,\ldots,r(a))$, and ϕ_a $(\phi_a\equiv\psi^a_0)$ the primary first-class constraints. We have (as was proved in Sec. II)

$$\psi^a_{n+1}=\{\psi^a_n,H'\},\quad n=0,1,\ldots,r(a)-1\ . \tag{24}$$

Let us denote as χ^i_q all the constraints of the system $[i=1,\ldots m;q=0,\ldots,s(i)]$. We have also that

$$\chi^i_{q+1}=\{\chi^i_q,H'\},\quad q=0,\ldots,s(i)-1\ . \tag{25}$$

Let m be the number of all primary constraints and let $m-r$ be the number of all primary first-class constraints.

We may also write the relations

$$\{\psi^a_{r(a)},H'\}=\sum_{a'=1}^{m-r}\sum_{n'=0}^{r(a')}F^{aa'}_{r(a)n'}(q,p)\psi^{a'}_{n'}(q,p)+\sum_{i=1}^{m}\sum_{q=0}^{s(i)}\sum_{i'=1}^{m}\sum_{q'=0}^{s(i')}G^{a}_{r(a)}{}^{ii'}_{qq'}(q,p)\chi^i_q(q,p)\chi^{i'}_{q'}(q,p)\ , \tag{26}$$

$$\{\chi^i_{s(i)},H'\}=\sum_{i'=1}^{m}\sum_{q'=0}^{s(i')}J^{i}_{s(i)}{}^{i'}_{q'}(q,p)\chi^{i'}_{q'}(q,p) \tag{27}$$

because $\{\chi^i_{s(i)},H'\}$ must vanish in the final manifold M_f defined by all the constraints, and $\{\psi^a_{r(a)},H'\}$ is in addition a first-class function.

Let $[q(t),p(t),\lambda(t)]$ be an extremal of the action S_D:

$$S_D[q,p,\lambda]=\int_{t_1}^{t_2}[p_i\cdot\dot q_i-H_c(q,p)-\lambda_i\chi^i_0]dt\ , \tag{28}$$

where q,p,λ are independent variables.

This trajectory is a solution of the equations of motion

$$\dot q=\{q,H_T^{(\lambda)}\}\ , \tag{29}$$

$$\dot p=\{p,H_T^{(\lambda)}\}\ , \tag{30}$$

$$\chi_0^i(q,p)=0 \tag{31}$$

with the total Hamiltonian

$$H_T^{(\lambda)}(q,p,t)=H'(q,p)+\lambda_a(t)\phi_a(q,p)\ . \tag{32}$$

Note that in (32) ϕ_a are all primary first-class constraints and these λ_a their undetermined Lagrange multipliers.

Let us consider now a function $\psi^{(\lambda)}(q,p,t)$ which is supposed to be a first-class function and satisfies the relation

$$\{\psi^{(\lambda)},H'\}+\lambda_a(t)\{\psi^{(\lambda)},\phi_a\}+\frac{\partial\psi^{(\lambda)}}{\partial t}=\sum_a \omega_a^{(\lambda)}(q,p,t)\phi_a(q,p)\ , \tag{33}$$

where $w_a^{(\lambda)}$ are some functions of q, p, and t. Then the infinitesimal canonical transformation generated by $\psi^{(\lambda)}$ maps the solution $[q(t),p(t),\lambda(t)]$ into another trajectory $[q'(t),p'(t),\lambda'(t)]$:

$$q_i'(t)=q_i(t)+\epsilon\frac{\partial\psi^{(\lambda)}}{\partial p_i}(q(t),p(t),t)\ , \tag{34}$$

$$p_i'(t)=p_i(t)-\epsilon\frac{\partial\psi^{(\lambda)}}{\partial q_i}(q(t),p(t),t)\ , \tag{35}$$

$$\lambda_a'(t)=\lambda_a(t)+\epsilon w_a^{(\lambda)}(q(t),p(t),t) \tag{36}$$

which also is an extremal of the action S_D and satisfies the equations of motion with the Hamiltonian $H_T^{(\lambda')}$:

$$\begin{aligned}H_T^{(\lambda')}(q,p,t)&=H_T^{(\lambda)}(q,p,t)\\&\quad+\epsilon\left[\frac{\partial\psi^{(\lambda)}}{\partial t}(q,p,t)+\{\psi^{(\lambda)},H_T^{(\lambda)}\}\right]\\&=H'(q,p)+\lambda_a'(t)\phi_a(q,p)\ .\end{aligned} \tag{37}$$

The condition of $\psi^{(\lambda)}$ being a first-class function guarantees that $\psi^{(\lambda)}$ generates transformations of the points (q,p) of M_f into points (q',p') of the same manifold. Condition (33) furnishes that the canonical transformation of the total Hamiltonian equations of motion brings about the same total Hamiltonian equations of motion in which only the Lagrange multipliers are changed.

We will prove the existence of the function $\psi^{(\lambda)}$. For this purpose let us consider $\psi^{(\lambda)}$ expressed by a linear combination of all the first-class constraints plus another linear combination of terms quadratic in all the constraints of the system. That is,

$$\psi^{(\lambda)}(q,p,t)=\sum_{a=1}^{m-r}\sum_{n=0}^{r(a)}C^{(\lambda)a}{}_n(q,p,t)\psi_n^a(q,p)+\sum_{i=1}^{n}\sum_{q=0}^{s(i)}\sum_{i'=1}^{n}\sum_{q'=0}^{s(i')}D^{(\lambda)ii'}_{qq'}(q,p,t)\chi_q^i(q,p)\chi_{q'}^{i'}(q,p)\ . \tag{38}$$

Selecting $\psi^{(\lambda)}$ in the form (38) automatically assures that $\psi^{(\lambda)}$ is a first-class function. It remains, then, to prove that the coefficients $C^{(\lambda)}$ and $D^{(\lambda)}$ in (38) can always be selected in order to satisfy (33). Given that the PB between two first-class functions is first class and that H' is a first-class function we have that $\{\psi_n^a,H'\}$ and $\{\psi_n^a,\phi_a\}$ are first-class functions and vanish in M_f. Then they can be expressed as linear combinations of the first-class constraints plus linear combinations of terms quadratic in all the constraints. Thus, in an explicit way we have

$$\begin{aligned}\{\psi_n^a,H_T^{(\lambda)}\}&=\{\psi_n^a,H'\}+\lambda_{a'}(t)\{\psi_n^a,\phi_{a'}\}\\&=\sum_{a'=1}^{m-r}\sum_{n'=0}^{r(a')}f^{(\lambda)a a'}_{n n'}(q,p,t)\psi_{n'}^{a'}(q,p)+\sum_{i=1}^{m}\sum_{q=0}^{s(i)}\sum_{i'=1}^{m}\sum_{q'=0}^{s(i)'}g^{(\lambda)a i i'}_{n q q'}(q,p,t)\chi_q^i(q,p)\chi_{q'}^{i'}(q,p)\ .\end{aligned} \tag{39}$$

Because of the consistency conditions for all the constraints it is possible to write

$$\{\chi_q^i,H_T^{(\lambda)}\}=\sum_{i'=1}^{m}\sum_{q'=0}^{s(i')}J^{(\lambda)ii'}_{qq'}(q,p,t)\chi_{q'}^{i'}(q,p)\ . \tag{40}$$

If we substitute (38) into (33) and use relations (39) and (40) we find that the function $\psi^{(\lambda)}$ exists if the coefficients $C^{(\lambda)}$ and $D^{(\lambda)}$ satisfy the following system of equations:

$$\frac{\partial C^{(\lambda)a}{}_n}{\partial t}=-\{C^{(\lambda)a}{}_n,H_T^{(\lambda)}\}-\sum_{a'=1}^{m-r}\sum_{n'=0}^{r(a')}C^{(\lambda)a'}{}_{n'}f^{a'a}_{n'n},\quad a=1,\dots,m-r,\quad n=1,\dots,r(a)\ , \tag{41}$$

$$\frac{\partial D^{(\lambda)i i'}_{qq'}}{\partial t}=-\{D^{(\lambda)ii'}_{qq'},H_T^{(\lambda)}\}-\sum_{\bar i=1}^{m}\sum_{\bar q=0}^{s(\bar i)}J^{(\lambda)\bar i i'}_{\bar q q'}(D^{(\lambda)i\bar i}_{q\bar q}+D^{(\lambda)\bar i i}_{\bar q q})-\sum_{a=1}^{m-r}\sum_{n=0}^{r(a)}C^{(\lambda)a}{}_n g^{(\lambda)a i i'}_{n q q'}\ ,$$

$$i=1,\dots,m,\quad q=0,\dots,s(i),\quad i'=1,\dots,m,\quad q'=0,\dots,s(i)\ . \tag{42}$$

We see that (41) and (42) is a system of first-order partial differential equations which can be rewritten in a compact way as

$$\frac{\partial u}{\partial t}=F\left[q,p,t,u,\frac{\partial u}{\partial q},\frac{\partial u}{\partial p}\right] . \tag{43}$$

We consider that all the functions on the right-hand side of the equations are analytical functions of their arguments. Then, the functions F are also analytical.

Thus the Cauchy-Kowalevski theorem guarantees the existence of the solution of (43) whenever the functions $C^{(\lambda)a}_{\ o}(q,p,t)$ are arbitrary inside the class of analytical functions of their arguments. The quantities $w_a^{(\lambda)}$ may be expressed through the $C^{(\lambda)a}_{\ o}$ by the relations

$$\{C^{(\lambda)a}_{\ 0},H_T^{(\lambda)}\}+\frac{\partial C^{(\lambda)a}_{\ 0}}{\partial t}+\sum_{a'=1}^{m-r}\sum_{n'=0}^{r(a')}C^{(\lambda)a'}_{\ n'}f^{a'\ a}_{n'\ 0}=w_a^{(\lambda)} . \tag{44}$$

In this way the existence of the generator $\psi^{(\lambda)}$ has been shown for a very wide class of systems.

IV. DIRAC'S CONJECTURE

This section is devoted to present the proof of Dirac's conjecture for a general class of systems obeying our basic assumptions.

Considering a trajectory $[q(t),p(t),\lambda(t)]$ being an extremal of the equations of motion Eqs. (29)–(31), let us define the corresponding trajectory in the phase space for a particular set of $\lambda(t)$ by $\eta_\lambda(t)=[q(t),p(t)]$.

Dirac showed that all the primary first-class constraints are generators of gauge transformations of the Hamiltonian problem with H_T.[1] The conjecture states that this property is also valid for all first-class constraints.

Let us consider a point (q_0,p_0) in M_f. The infinitesimal canonical transformation

$$q'_{0i}=q_{0i}+\epsilon\frac{\partial\psi_n^a}{\partial p_i}(q_0,p_0) , \tag{45}$$

$$p'_{0i}=p_{0i}-\epsilon\frac{\partial\psi_n^a}{\partial q_i}(q_0,p_0) , \tag{46}$$

where ψ_n^a is any first-class constraint, transforms the point (q_0,p_0) into $(q'_0,p'_0)\in M_f$. The conjecture says that both points belong to the same physical state.

We must then prove that through the point (q_0,p_0) passes an extremal $\eta_\lambda(t)$ $(q_0=q(t_0),p_0=p(t_0))$ and that through the point (q'_0,p'_0) passes another extremal $\eta_{\lambda'}(t)$ $(q'_0=q'(t_0),p'_0=p'(t_0))$ in such a way that both trajectories join together at the time $t=t_1<t_0$. In other words, starting from the same initial conditions $q(t_1)=q_1,p(t_1)=p_1$ we arrive after the time interval (t_0-t_1) through two different extremals up to the points (q_0,p_0) and (q'_0,p'_0), which, therefore, describe the same physical state.

There is an infinite number of extemals of the total Hamiltonian problem which pass through the point (q_0,p_0). This is due to the arbitrariness of the Lagrange multipliers $\lambda_a(t)$. In the previous section it has been shown that for each of these trajectories (for a given set of multipliers) there is a function $\psi^{(\lambda)}(q,p,t)$ transforming it into another extremal (for another set of multipliers) via an infinitesimal canonical transformation.

Therefore, in order to prove that (q_0,p_0) and (q'_0,p'_0) related by (45) and (46) are physically equivalent it is only necessary to find the function $\psi^{(\lambda)}(q,p,t)$ for some extremal $[\lambda_a(t)$ fixed$]$ which passes through (q_0,p_0) and also satisfies the conditions

$$\frac{\partial\psi^{(\lambda)}}{\partial p}(q_0,p_0,t_0)=\frac{\partial\psi_n^a}{\partial p}(q_0,p_0) , \tag{47}$$

$$\frac{\partial\psi^{(\lambda)}}{\partial q}(q_0,p_0,t_0)=\frac{\partial\psi_n^a}{\partial q}(q_0,p_0) , \tag{48}$$

$$\frac{\partial\psi^{(\lambda)}}{\partial p}(q_1,p_1,t_1)=0 , \tag{49}$$

$$\frac{\partial\psi^{(\lambda)}}{\partial q}(q_1,p_1,t_1)=0 . \tag{50}$$

We work with the function $\psi^{(\lambda)}(q,p,t)$ as was defined in Sec. III. $\psi^{(\lambda)}$ is a first-class function and the coefficients $C^{(\lambda)}$ and $D^{(\lambda)}$ satisfy Eqs. (41) and (42).

Let us choose the extremal which passes through the point (q_0,p_0) to have a vanishing λ $(\lambda_a(t)=0,\ \forall t)$. This condition simplifies relations (39) and (40). Then we have

$$\{\psi_n^a,H_T^{(\lambda)}\}=\{\psi_n^a,H'\},\quad a=1,\ldots,m-r;\quad n=0,\ldots,r(a) , \tag{51}$$

$$\{\chi_q^i,H_T^{(\lambda)}\}=\{\chi_q^i,H'\},\quad i=1,\ldots,m;\quad q=0,\ldots,s(i) . \tag{52}$$

If we substitute (24)–(27) in Eq. (33) we find the system of equations

$$\frac{\partial C_0^a}{\partial t}+\{C_0^a,H'\}+\sum_{a'=1}^{m-r}C^{a'}_{r(a')}F^{a'\ \ a}_{r(a')o}=w_a; \tag{53}$$

$$\frac{\partial C_n^a}{\partial t}+\{C_n^a,H'\}+C^a_{n-1}+\sum_{a'=1}^{m-r}C^{a'}_{r(a')}F^{a'\ \ a}_{r(a')n}=0 ,$$
$$a=1,\ldots,m-r,\quad n=1,\ldots,r(a) ; \tag{54}$$

$$\frac{\partial D^{ii'}_{qq'}}{\partial t}+\{D^{ii'}_{qq'},H'\}+(D^{ii'}_{qq'-1}+D^{i'i}_{q'-1q})+\sum_{\bar{i}=1}^{m}J^{\bar{i}\bar{i}'}_{s(\bar{i})q'}(D^{i\bar{i}}_{qs(\bar{i})}+D^{\bar{i}\ i}_{s(\bar{i})q})+\sum_{a=1}^{m-r}C^a_{r(a)}G^{a\ \ i\ i'}_{r(a)q\,q'}=0 ,$$
$$i=1,\ldots,m,\quad q=0,\ldots,s(i) ,$$
$$i'=1,\ldots,m,\quad q'=1,\ldots,s(i') ; \tag{55}$$

$$\frac{\partial D^{ii'}_{q0}}{\partial t}+\{D^{ii'}_{q0}H'\}+\sum_{\bar{i}=1}^{m}J^{\bar{i}\ i}_{s(\bar{i})0}(D^{i\bar{i}}_{qs(\bar{i})}+D^{\bar{i}\ i}_{s(\bar{i})q})+\sum_{a=1}^{m-r}C^a_{r(a)}G^{a\ \ i\ i'}_{r(a)q0}=0 ,$$
$$i=1,\ldots,m,\quad q=0,\ldots,s(i) ,$$
$$i'=1,\ldots,m . \tag{56}$$

Thus we have to prove the existence of a solution of Eqs. (53) and (54) with the additional conditions

$$C_n^a(q_0,p_0,t_0)=\begin{cases}1, & a=\bar{a},\quad n=\bar{n}\ ,\\ 0, & a\neq\bar{a}\ \text{or}\ n\neq\bar{n}\ ;\end{cases} \tag{57}$$

$$C_n^a(q_1,p_1,t_1)=0\ ,$$
$$a=1,\ldots,m-r,\quad n=0,\ldots,r(a)\ . \tag{58}$$

The existence of $D^{(\lambda)}$ satisfying (55) and (56) is guaranteed by the Cauchy-Kowalevski theorem. It is not difficult to notice that the additional conditions (57) and (58) allow $\psi^{(\lambda)}$ to satisfy Eqs. (47)–(50).

From (53) and (54) it is clearly seen that all the $C_{r(a)}^a$ functions may be taken as arbitrary analytical functions except for their values at (q_0,p_0,t_0) and (q_1,p_1,t_1). Moreover, the set (53) and (54) may be rewritten as

$$C_{r(a)-n}^a=(-1)^n\frac{\partial^n C_{r(a)}^a}{\partial t^n}+P_n^a,\quad n=1,\ldots,r(a)\ ,$$
$$a=1,\ldots,m-r\ , \tag{59}$$

in which P_n^a is a polynomial in all the possible partial derivatives of the functions $C_{r(a)}^a$ in which the order of the derivatives runs up to the value n in the case of the q or p variables but reaches the maximum value $n-1$ for the time derivatives.

From the examination of (59), one can conclude that it is always possible to choose the values of the $C_{r(a)}^a$ quantities and of its first $r(a)$ time derivatives to bring about arbitrary selected values to the C_n^a $(a=1,\ldots,m-r;n=0,\ldots,r(a))$ quantities at the instants t_0 and t_1. In this way Dirac's conjecture is proved.

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APPENDIX A: EQUIVALENCE OF THE CONSTRAINT SETS $\{\phi\}$ AND $\{\chi\}$

Here, we want to show that the original set of constraints $\{\phi\}$ given in (6) is really equivalent to set $\{\chi\}$ defined in (17). A proof by induction will be given. In order to prove this let us suppose that up to the step k it is true that

$$\chi_{\mu_{k-1}}^{(k)}=\phi_{\mu_{k-1}}^{(k)}+L_{\mu_{k-1}}^{(k)}(\phi_{\mu_{-1}}^{(0)},\phi_{\mu_0}^{(1)},\ldots,\phi_{\mu_{k-2}}^{(k-1)})\ , \tag{A1}$$

where $L_{\mu_{k-1}}^{(k)}$ is a linear combination of all the constraints up to the step $(k-1)$. The validity of the hypothesis is evident for $k=0$ $(\chi_\mu^{(0)}\equiv\phi_\mu^{(0)})$. Then, it should be demonstrated that the constraints of the next step

$$\chi_{\mu_k}^{(k+1)}=\{\chi_{\mu_k}^{(k)},H'\} \tag{A2}$$

will satisfy

$$\chi_{\mu_k}^{(k+1)}=\phi_{\mu_k}^{(k+1)}+L_{\mu_k}^{(k+1)}(\phi_{\mu_{-1}}^{(0)},\phi_{\mu_0}^{(1)},\ldots,\phi_{\mu_{k-1}}^{(k)})\ . \tag{A3}$$

The expression (A2) may be rewritten as

$$\begin{aligned}\chi_{\mu_k}^{(k+1)}&=\{\phi_{\mu_k}^{(k)}+L_{\mu_{k-1}}^{(k)}(\phi_{\mu_{-1}}^{(0)},\phi_{\mu_0}^{(0)},\ldots,\phi_{\mu_{k-2}}^{(k-1)}),H'\}\\ &=\{\phi_{\mu_k}^{(k)},H_c^{(k+1)}+\sigma_{\bar{\nu}_k}\phi_{\bar{\nu}_k}^{(0)}\}\\ &\quad+\{\text{linear combination of } \phi_{\mu_{k-2}}^{(k-1)},H_c^{(k)}+\sigma_{\bar{\nu}_{k-1}}\phi_{\bar{\nu}_{k-1}}^{(0)}\}+\cdots\\ &\quad+\{\text{linear combination of } \phi_\mu^{(0)},H_c^{(1)}+\sigma_{\bar{\nu}_0}\phi_{\bar{\nu}_0}^{(0)}\}\ .\end{aligned} \tag{A4}$$

In (A4) a new type of subindices $\bar{\nu}_k$ is used in which an overbar over the greek letter (in this case ν) means that the set $\bar{\nu}_k$ includes all the indices of the sets ν_i' $(i=k,\ldots,f)$.

All the terms on the right-hand side of (A4) vanish except the first one. this is a result of relations (7), (13), (14), and (15).

Using again (15) the expression (A4) takes the form

$$\chi_{\mu_k}^{(k+1)}\underset{M_k}{=}\{\phi_{\mu_k}^{(k)},H_c^{(k+1)}\} \tag{A5}$$

$$\chi_{\mu_k}^{(k+1)}=\phi_{\mu_k}^{(k+1)}+L_{\mu_k}^{(k+1)}(\phi_{\mu_{-1}}^{(0)},\phi_{\mu_0}^{(1)},\ldots,\phi_{\mu_{k-1}}^{(k)})\ . \tag{A6}$$

Relation (A6) tells us that the restrictions introduced by the constraints $\chi_{\mu_k}^{(k+1)}$ in M_k are completely equivalent to the restrictions determined by the constraints $\phi_{\mu_k}^{(k+1)}$.

It is easy to prove also that

$$\{\chi_{\mu_k'}^{(k)},H'\}\underset{M_k}{=}0\ . \tag{A7}$$

This is a consequence of (13). Then the

$$\chi_{\mu_k'}^{(k+1)}\equiv\{\chi_{\mu_k'}^{(k)},H'\} \tag{A8}$$

are not actually constraints.

We see that the new set of constraints $\{\chi\}$ has the structure (18). Thus, (A6) and (A7) finish the proof of the equivalence of the sets of constraints $\{\phi\}$ and $\{\chi\}$.

Finally let us show that the constraints $\{\chi\}$ satisfy relations (19) and (20) which are similar to relations (15) and (16) for the constraints $\{\phi\}$. The proof proceeds as follows.

The fact that

$$\{\chi^{(0)}_{\mu_0},\chi^{(0)}_{\mu}\} \underset{M_0}{=} 0, \quad \det|\{\chi^{(0)}_{\mu'_0},\chi^{(0)}_{\nu'_0}\}| \underset{M_0}{\neq} 0 \tag{A9}$$

is evident because $\chi^{(0)}_{\mu}\equiv\phi^{(0)}_{\mu}$ which satisfies (15) and (16).

For a superindex $k\geq 1$ we have

$$\{\chi^{(k)}_{\mu_k},\chi^{(0)}_{\mu_{k-1}}\}=\{\phi^{(k)}_{\mu_k}+L^{(k)}_{\mu_k}(\phi^{(0)}_{\mu_{-1}},\phi^{(1)}_{\mu_0},\ldots,\phi^{(k-1)}_{\mu_{k-2}}),\phi^{(0)}_{\mu_{k-1}}\}$$
$$\underset{M_{k-1}}{=}\{\phi^{(k)}_{\mu_k},\phi^{(0)}_{\mu_{k-1}}\}\underset{M_k}{=}0 \tag{A10}$$

and

$$\{\chi^{(k)}_{\mu_{k-1}},\chi^{(0)}_{\mu_k}\}=\{\phi^{(k)}_{\mu_{k-1}}+L^{(k)}_{\mu_{k-1}}(\phi^{(0)}_{\mu_{-1}},\phi^{(1)}_{\mu_0},\ldots,\phi^{(k-1)}_{\mu_{k-2}}),\phi^{(0)}_{\mu_k}\}$$
$$\underset{M_{k-1}}{=}\{\phi^{(k)}_{\mu_{k-1}},\phi^{(0)}_{\mu_k}\}\underset{M_k}{=}0\,. \tag{A11}$$

In obtaining (A10) and (A11) we have used relations (15). The fact that the determinant (20) is not equal to zero results from the relations

$$\{\chi^{(k)}_{\mu'_k},\chi^{(0)}_{\nu'_k}\}=\{\phi^{(k)}_{\mu'_k}+L^{(k)}_{\mu'_k}(\phi^{(0)}_{\mu_{-1}},\phi^{(1)}_{\mu_0},\ldots,\phi^{(k-1)}_{\mu_{k-2}}),\phi^{(0)}_{\nu'_k}\}$$
$$\underset{M_{k-1}}{=}\{\phi^{(k)}_{\mu'_k},\phi^{(0)}_{\nu'_k}\}\underset{M_k}{=}0 \tag{A12}$$

and from (16).

APPENDIX B: THE PROOF OF THE THEOREM

The main objective of this appendix will be to prove the theorem stated in Sec. II. For this purpose two auxiliary lemmas will be demonstrated.

Lemma 1. Whenever the $\chi^{(k)}_{\mu_{k+i-1}}$ PB commute with all $\chi^{(i)}_{\mu_{i-1}}$ except $\chi^{(k)}_{\mu'_{k+i}}$ and $\chi^{(i)}_{\mu'_{k+i}}$ satisfying

$$\det|\{\chi^{(k)}_{\mu'_{k+i}},\chi^{(i)}_{\mu'_{k+i}}\}|\underset{M_{k+i}}{\neq}0 \tag{B1}$$

and in addition

$$\{\chi^{(k)}_{\mu'_{k+s}},\chi^{(i)}_{\mu_{k+i-1}}\}\underset{M_{k+i}}{=}0,\quad s=0,\ldots,i-1\,,$$
$$i=0,1,\ldots,R,\quad k=i,i+1,\ldots,R\,, \tag{B2}$$

then it follows that all $\chi^{(R+p)}_{\mu_{R+p+i-1}}$ PB commute with $\chi^{(i)}_{\mu_{i-1}}$ except $\chi^{(R+p)}_{\mu'_{R+p+i}}$ and $\chi^{(i)}_{\mu'_{R+p+i}}$ which obey

$$\det|\{\chi^{(R+p)}_{\mu'_{R+p+i}},\chi^{(i)}_{\mu'_{R+p+i}}\}|\underset{M_{R+p+i}}{\neq}0 \tag{B3}$$

and it also follows that

$$\{\chi^{(R+p)}_{\mu'_{R+p+s}},\chi^{(i)}_{\mu'_{R+p+i-1}}\}\underset{M_{R+p+i}}{=}0\,, \tag{B4}$$

where

$$s=0,1,\ldots,i-1\,,$$
$$i=0,1,\ldots,R-p$$

for all $p=0,1,\ldots,R$.

The proof will be by induction in the index p. The validity of the thesis in the first step ($p=0$) is guaranteed properly by this hypothesis. Let us suppose that the property stated in the lemma is valid for $p=0,1,\ldots,S$. Then it is necessary to demonstrate that it is also valid for $p=S+1$. In other words, all the PB $\{\chi^{(R+S+1)}_{\mu_{R+S+i}},\chi^{(i)}_{\mu_{i-1}}\}$ are equal to zero except

$$\det|\{\chi^{(R+S+1)}_{\mu'_{R+S+i+1}},\chi^{(i)}_{\mu'_{R+S+i+1}}\}|\underset{M_{R+S+i+1}}{\neq}0$$

and also

$$\{\chi^{(R+S+1)}_{\mu'_{R+S+i+s}},\chi^{(i)}_{\mu_{R+S+i}}\}\underset{M_{R+S+i+1}}{=}0\,,$$
$$[s=0,1,\ldots,i-1\,;$$
$$i=0,1,\ldots,R-(S+1)]\,.$$

Using the Jacobi identity we may write

$$\{\chi^{(R+S+1)}_{\mu_{R+S+i}},\chi^{(i)}_{\mu_{i-1}}\}=\{\{\chi^{(R+S)}_{\mu_{R+S+i}},H'\},\chi^{(i)}_{\mu_{i-1}}\}$$
$$=-\{\{H',\chi^{(i)}_{\mu_{i-1}}\},\chi^{(R+S)}_{\mu_{R+S+i}}\}$$
$$-\{\{\chi^{(i)}_{\mu_{i-1}},\chi^{(R+S)}_{\mu_{R+S+i}}\},H'\}\,. \tag{B5}$$

But, the second term in the last line is equal to zero in $M_{R+S+i+1}$ according to the hypothesis ($\{\chi^{(i)}_{\mu_{i-1}},\chi^{(R+S)}_{\mu_{R+S+i}}\}=0$).

Let us analyze relation (B5) for each set of constraints $\chi^{(i)}_{\mu'_i}$ and $\chi^{(i)}_{\mu_i}$ in which the set $\chi^{(i)}_{\mu_{i-1}}$ can be decomposed. Then we have

$$\{\chi^{(R+S+1)}_{\mu_{R+S+i}},\chi^{(i)}_{\mu'_i}\}\underset{M_{R+S+i+1}}{=}\{\{\chi^{(i)}_{\mu'_i},H'\},\chi^{(R+S)}_{\mu_{R+S+i}}\}\,. \tag{B6}$$

This is equal to zero in M_{R+S+i} because $\{\chi^{(i)}_{\mu'_i},H'\}$ vanishes in M_i and

$$\{\chi^{(i')}_{\mu_{i'-1}},\chi^{(R+S)}_{\mu_{R+S+i}}\}\underset{M_{R+S+i}}{=}0\quad(i'=0,\ldots,i)$$

by hypothesis.

It follows from (B5) that

$$\{\chi^{(R+S+1)}_{\mu_{R+S+i}},\chi^{(i)}_{\mu_i}\}\underset{M_{R+S+i+1}}{=}\{\chi^{(i+1)}_{\mu_i},\chi^{(R+S)}_{\mu_{R+S+i}}\}\,. \tag{B7}$$

By hypothesis we have that all the PB $\{\chi^{(i+1)}_{\mu_i},\chi^{(R+S)}_{\mu_{R+S+i}}\}$ are equal to zero, except $\det|\{\chi^{(R+S)}_{\mu'_{R+S+i+1}},\chi^{(i+1)}_{\mu'_{R+S+i+1}}\}|\neq 0$, where $i+1=1,\ldots,R-S$. Then we obtain that all $\chi^{(R+S+1)}_{\mu_{R+S+i}}$ PB commute with $\chi^{(i)}_{\mu_{i-1}}$ except $\det|\{\chi^{(R+S+1)}_{\mu'_{R+S+1+i}},\ \chi^{(i)}_{\mu'_{R+S+1+i}}\}|\neq 0$ $[i=0,1,\ldots,R-(S+1)]$.

We may write that

$$\{\chi^{(R+S+1)}_{\mu'_{R+S+1+s}},\chi^{(i)}_{\mu_{R+S+i}}\}=\{\{\chi^{(R+S)}_{\mu'_{R+S+1+s}},H'\},\chi^{(i)}_{\mu_{R+S+i}}\}\,. \tag{B8}$$

Using the Jacobi identity and that

$$\{\chi^{(i)}_{\mu_{R+S+i}},\chi^{(R+S)}_{\mu'_{R+S+1+s}}\}\underset{M_{R+S+i}}{=}0\,,$$
$$s=0,1,\ldots,i-1,\quad i=0,1,\ldots,R-(S+1)\,,$$

we have

$$\{\chi^{(R+S+1)}_{\mu'_{R+S+1+s}},\chi^{(i)}_{\mu_{R+S+i}}\} \underset{M_{R+S+1}}{=} \{\{\chi^{(i)}_{\mu_{R+S+i}},H'\},\chi^{(R+S)}_{\mu'_{R+S+1+s}}\}$$
$$\underset{M_{R+S+i+1}}{=} \{\chi^{(i+1)}_{\mu_{R+S+i}},\chi^{(R+S)}_{\mu'_{R+S+1+s}}\}$$
$$\underset{M_{R+S+i+1}}{=} 0 \text{ (by hypothesis)} . \quad \text{(B9)}$$

Relation (B9) completes the proof of the lemma.

In close analogy with the above discussion the following lemma may be proved.

Lemma 2. If all the constraints $\chi^{(k)}_{\mu_{k+i-1}}$ PB commute with all $\chi^{(i)}_{\mu_{i-1}}$ except $\chi^{(k)}_{\mu'_{k+i}}$ and $\chi^{(i)}_{\mu'_{k+i}}$ which satisfy

$$\det|\{\chi^{(k)}_{\mu'_{k+1}},\chi^{(i)}_{\mu'_{k+i}}\}| \underset{M_{k+i}}{\neq} 0$$

and also if

$$\{\chi^{(k)}_{\mu'_{k+s}},\chi^{(i)}_{\mu_{k+i-1}}\} \underset{M_{k+i}}{=} 0 ,$$

where $s=0,1,\ldots,i-1$; $i=0,1,\ldots,R$, and $k=i,i+1,\ldots,R,R+1$, then all the constraints $\chi^{(R+1+p)}_{\mu_{R+p+i}}$ PB commute with $\chi^{(i)}_{\mu_{i-1}}$ except $\chi^{(R+1+p)}_{\mu'_{R+1+p+i}}$ and $\chi^{(i)}_{\mu'_{R+1+p+i}}$ which obey

$$\det|\{\chi^{(R+1+p)}_{\mu'_{R+1+p+i}},\chi^{(i)}_{\mu'_{R+1+p+i}}\}| \underset{M_{R+1+p+i}}{\neq} 0$$

and also is valid that

$$\{\chi^{(R+1+p)}_{\mu'_{R+1+p+s}},\chi^{(i)}_{\mu_{R+p+i}}\} \underset{M_{R+1+p+i}}{=} 0 ,$$

where $s=0,1,\ldots,i-1$; $i=0,1,\ldots,R-p$, for all $p=0,1,\ldots,R$.

After all these previous statements we may begin the proof of the theorem which was enunciated in Sec. II.

Let us observe that the theorem predicts the PB relation properties among the constraints of two arbitrary lines (defined by the constraint superindices k and i) up to the step $S_M=f/2$ (f even) or $S_M=(f-1)/2$ (f odd). However, for the constraints in a line staying below the step S_M, the PB relation properties are predicted only with the constraints belonging to lines situated above S_M. Then it can be seen that we should demonstrate the theorem by induction up to the step S_M, and then lemma 1 guarantees the validity of the theorem completely. Below, the theorem will be demonstrated in the mentioned modified way, that is, up to the step S_M.

The modified hypothesis reads as follows.

Hypothesis. All $\chi^{(k)}_{\mu_{k+i-1}}$ PB commute with all $\chi^{(i)}_{\mu_{i-1}}$ except $\chi^{(k)}_{\mu'_{k+i}}$ and $\chi^{(i)}_{\mu'_{k+i}}$ which give

$$\det|\{\chi^{(k)}_{\mu'_{k+i}},\chi^{(i)}_{\mu'_{k+i}}\}| \underset{M_{k+i}}{\neq} 0 \quad \text{(B10)}$$

and also

$$\{\chi^{(k)}_{\mu'_{k+s}},\chi^{(i)}_{\mu_{k+i-1}}\} \underset{M_{k+i}}{=} 0 , \quad \text{(B11)}$$

where $s=0,1,\ldots,i-1$; $i=0,1,\ldots,R$, and $k=i,i+1,\ldots,R$.

The modified thesis may be enunciated in the following way.

Statement: All $\chi^{(R+1)}_{\mu_{R+i}}$ PB commute with all $\chi^{(i)}_{\mu_{i-1}}$ except $\chi^{(R+1)}_{\mu'_{R+i+1}}$ and $\chi^{(i)}_{\mu'_{R+i+1}}$ which satisfy

$$\det|\{\chi^{(R+1)}_{\mu'_{R+i+1}},\chi^{(i)}_{\mu'_{R+1+i}}\}| \underset{M_{R+1+i}}{\neq} 0 \quad \text{(B12)}$$

and also

$$\{\chi^{(R+1)}_{\mu'_{R+1+s}},\chi^{(i)}_{\mu_{R+i}}\} \underset{M_{R+1+i}}{=} 0 , \quad \text{(B13)}$$

where $s=0,1,\ldots,i-1$ and $i=0,1,\ldots,R+1$.

Firstly let us consider the cases $i=0,1,\ldots,R-1$. The validity of the thesis in these steps is a direct result of lemma 1, taking $p=1$. Then, the only cases left to be considered are $i=R$ and $i=R+1$.

1. Case $i=R$

In this case after using the iterative definition of the constraints and the Jacobi identity we have

$$\{\chi^{(R+1)}_{\mu_{2R}},\chi^{(R)}_{\mu_{R-1}}\}=\{\{\chi^{(R)}_{\mu_{2R}},H'\},\chi^{(R)}_{\mu_{R-1}}\}$$
$$=-\{\{H',\chi^{(R)}_{\mu_{R-1}}\},\chi^{(R)}_{\mu_{2R}}\}$$
$$-\{\{\chi^{(R)}_{\mu_{R-1}},\chi^{(R)}_{\mu_{2R}}\},H'\} . \quad \text{(B14)}$$

The second term in the last line vanishes in M_{2R+1} due to the fact that by hypothesis the brackets $\{\chi^{(R)}_{\mu_{R-1}},\chi^{(R)}_{\mu_{2R}}\}$ vanishes in M_{2R}. Then from (B14) it follows that

$$\{\chi^{(R+1)}_{\mu_{2R}},\chi^{(R)}_{\mu_R}\} \underset{M_{2R+1}}{=} -\{\chi^{(R)}_{\mu_{2R}},\chi^{(R+1)}_{\mu_R}\} \quad \text{(B15)}$$

and also

$$\{\chi^{(R+1)}_{\mu_{2R}},\chi^{(R)}_{\mu'_R}\} \underset{M_{2R+1}}{=} 0 , \quad \text{(B16)}$$

because

$$\{\chi^{(R)}_{\mu_{2R}},\{\chi^{(R)}_{\mu'_R},H'\}\} \underset{M_{2R}}{=} 0$$

by hypothesis. $\{\chi^{(R)}_{\mu'_R},H'\}$ being equal to zero in M_R is a linear combination of the constraints defining this manifold.

Note that relation (B15) expresses that the brackets considered in (B14) is equal to the same type of expression with the minus sign in which the superindex $R+1$ is diminished in a unit and the R superindex is increased also in a unit. The basic facts determining this property are the annulation of $\{\chi^{(R)}_{\mu_{R-1}},\chi^{(R)}_{\mu_{2R}}\}$ in M_{2R} and the vanishing of $\{H',\chi^{(R)}_{\mu'_R}\}$ in M_R.

It can be seen, for the following steps, that the validity of

$$\{\chi^{(R+p)}_{\mu_{R+p-1}},\chi^{(R-p)}_{\mu_{2R}}\} \underset{M_{2R}}{=} 0, \quad p\le R , \quad \text{(B17)}$$

$$\{\chi^{(R+p)}_{\mu'_{R+p}},H'\} \underset{M_{R+p}}{=} 0 \quad \text{(B18)}$$

implies the general relations

$$\{\chi^{(R+1)}_{\mu_{2R}},\chi^{(R)}_{\mu'_{R+p}}\}\underset{M_{2R+1}}{=}0\ , \tag{B19}$$

$$\{\chi^{(R+1)}_{\mu_{2R}},\chi^{(R)}_{\mu_{R+p}}\}\underset{M_{2R+1}}{=}(-1)^{p+1}\{\chi^{(R-p)}_{\mu_{2R}},\chi^{(R+p+1)}_{\mu_{R+p}}\} \tag{B20}$$

for $p=0,1,\ldots,R$, because of the iterative use of the Jacobi identities.

Result (B17) is a direct consequence of lemma 1, and (B18) follows automatically from the Dirac algorithm.

Putting $p=R$ we obtain

$$\{\chi^{(R+1)}_{\mu_{2R}},\chi^{(R)}_{\mu'_{2R}}\}\underset{M_{2R+1}}{=}0\ , \tag{B21}$$

$$\{\chi^{(R+1)}_{\mu_{2R}},\chi^{(R)}_{\mu_{2R}}\}\underset{M_{2R+1}}{=}(-1)^{R+1}\{\chi^{(0)}_{\mu_{2R}},\chi^{(2R+1)}_{\mu_{2R}}\}\ . \tag{B22}$$

But using the PB relations (19) and (20) among the χ constraints by substituting $k=2R+1$ we obtain

$$\begin{aligned}&\{\chi^{(2R+1)}_{\mu_{2R+1}},\chi^{(0)}_{\mu_{2R}}\}\underset{M_{2R+1}}{=}0\ ,\\ &\{\chi^{(2R+1)}_{\mu_{2R}},\chi^{(0)}_{\mu_{2R+1}}\}\underset{M_{2R+1}}{=}0\ ;\end{aligned} \tag{B23}$$

$$\det|\{\chi^{(2R+1)}_{\mu'_{2R+1}},\chi^{(0)}_{\mu'_{2R+1}}\}|\underset{M_{2R+1}}{\neq}0\ . \tag{B24}$$

Relations (B22)–(B24) and those which were obtained in previous steps give the expected result that all the constraints $\chi^{(R+1)}_{\mu_{2R}}$ PB commute with all the constraints $\chi^{(R)}_{\mu_{R-1}}$ except $\chi^{(R+1)}_{\mu'_{2R+1}}$ and $\chi^{(R)}_{\mu'_{2R+1}}$ which satisfy

$$\det|\{\chi^{(R+1)}_{\mu'_{2R+1}},\chi^{(R)}_{\mu'_{2R+1}}\}|\underset{M_{2R+1}}{\neq}0\ .$$

The rest of the relations

$$\{\chi^{(R+1)}_{\mu'_{R+1+s}},\chi^{(R)}_{\mu_{2R}}\}\underset{M_{2R+1}}{=}0,\quad s=0,\ldots,R-1$$

are a direct result from formula (B20) for $p=0$ and formula (B19) for $p=1,\ldots,R$.

Thus the statement is proved in the level $i=R$.

2. Case $i=R+1$

In this step the thesis is demonstrated in very close analogy with the case $i=R$, but the role of lemma 1 is played by lemma 2. Then, we shall not repeat the steps and only the basic results are enunciated below. The relations being analogous to (B19) and (B20) become

$$\{\chi^{(R+1)}_{\mu_{2R+1}},\chi^{(R+1)}_{\mu'_{R+p+1}}\}\underset{M_{2R+2}}{=}0\ , \tag{B25}$$

$$\{\chi^{(R+1)}_{\mu_{2R+1}},\chi^{(R+1)}_{\mu_{R+p+1}}\}\underset{M_{2R+2}}{=}(-1)^{p+1}\{\chi^{(R-p)}_{\mu_{2R+1}},\chi^{(R+p+2)}_{\mu_{R+p+1}}\}\ ,$$

$$p=0,1,\ldots,R\ . \tag{B26}$$

Relations (B25) and (B26) together with the PB relations (19) and (20) for $k=2R+2$ enable us to obtain the expected result: All the constraints $\chi^{(R+1)}_{\mu_{2R+1}}$ PB commute with all the constraints $\chi^{(R+1)}_{\mu_R}$ except $\chi^{(R+1)}_{\mu'_{2R+2}}$ which obey

$$\det|\{\chi^{(R+1)}_{\mu'_{2R+2}},\chi^{(R+1)}_{\nu'_{2R+2}}\}|\underset{M_{2R+2}}{\neq}0 \tag{B27}$$

and also

$$\{\chi^{(R+1)}_{\mu_{2R+1}},\chi^{(R+1)}_{\mu'_{R+1+s}}\}\underset{M_{2R+2}}{=}0,\quad s=0,\ldots,R\ . \tag{B28}$$

This finishes the proof of the statement for the step $i=R+1$.

APPENDIX C

Here, we will present the proof that the determinant of the matrix formed with the PB among all the second-class constraints is different from zero. By the theorem in Appendix B it follows that all second-class primary constraints PB commute with all second-class constraints except for the constraints entering in the relations

$$\det|\{\chi^{(0)}_{\mu'_s},\chi^{(s)}_{\nu'_s}\}|\underset{M_s}{\neq}0,\quad s=0,1,\ldots,f\ . \tag{C1}$$

This property means that the considered determinant is (with an indefinition only up to a sign) equal to the product of all the determinants in (C1) multiplied by the determinant of the PB matrix for all the constraints not appearing in any of the expressions in (C1).

According to the theorem, for the set of constraints not appearing in (C1) it follows that all the constraints $\chi^{(1)}_{\mu_1}$ PB commute with all the second-class constraints of the considered set except for the constraints entering in the relations

$$\det|\{\chi^{(1)}_{\mu'_{1+s}},\chi^{(1+s)}_{\nu'_{1+s}}\}|\underset{M_{s+2}}{\neq}0,\quad s=0,1,\ldots,f-2\ . \tag{C2}$$

Again, this property implies that the determinant of the matrix formed with the PB among all the constraints not appearing in (C1) reduces (up to sign indefinition) to the product of all the determinants being presented in (C2) multiplied by the determinant of the PB matrix for the constraints not appearing in (C1) and (C2).

From the first two sketched steps it may be noticed that the validity of the relations

$$\det|\{\chi^{(p)}_{\mu'_{p+s}},\chi^{(p+s)}_{\nu'_{p+s}}\}|\underset{M_{2p+s}}{\neq}0,\quad s=0,1,\ldots,f-2p\ ,$$

$$p=0,1,\ldots,S_M \tag{C3}$$

will allow us to demonstrate that the determinant of the matrix formed with the PB among all the second-class constraints is equal (up to sign) to the product of all the determinants appearing in (C3). But relations (C3) are direct consequences of the theorem. Then the considered determinant is different from zero.

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