Second-order corrections to the Gaussian effective potential of $\lambda\phi^4$ theory

I. Stancu and P. M. Stevenson
T. W. Bonner Nuclear Laboratory, Physics Department, Rice University, Houston, Texas 77251
(Received 8 September 1989)

We formulate a systematic, nonperturbative expansion for the effective potential of $\lambda\phi^4$ theory. At first order it gives the Gaussian effective potential (GEP), which itself contains the one-loop and leading-order $1/N$ results. Here, we compute the second-order terms and carry out the renormalization in the four-dimensional, "precarious" case, using dimensional regularization. (Difficulties with other regularizations are briefly discussed.) Remarkably, the final result takes the same mathematical form as the GEP, with only some numerical coefficients being changed. Indeed, in the most natural parametrization, only a single coefficient is changed, from 1 to $1-1/(N+3)^2$.

I. INTRODUCTION

The power of variational methods in quantum mechanics is well recognized. The idea of applying such methods in quantum field theory has a long history,[1,2] and has enjoyed a revival in recent years.[3] The Gaussian effective potential[2,4,5] (GEP) is a variational approximation to the effective potential which uses a Gaussian wave functional—that is, a free-field vacuum with a variable mass Ω—as the trial ground state. In addition to its intuitive appeal,[4] the GEP is known to contain the one-loop and leading-order $1/N$ results in the appropriate limiting cases.[4-6] Moreover, the GEP can be made the starting point for a systematic expansion procedure.[4,7,8] In this paper we formulate such an expansion for $\lambda\phi^4$ theory in practicable form, and we compute, and renormalize, the next-to-leading-order result; i.e., the "post-Gaussian effective potential" (PGEP).

The basic idea can be illustrated very simply. Suppose one wishes to compute the ground-state energy E_0 of the anharmonic-oscillator Hamiltonian:

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}m^2\phi^2+\lambda\phi^4 , \tag{1.1}$$

with $[\phi,p]=i$. Consider the modified Hamiltonian

$$H_\delta \equiv H_0+H_{\rm int} , \tag{1.2}$$

with

$$H_0(\Omega)=\tfrac{1}{2}p^2+\tfrac{1}{2}\Omega^2\phi^2 , \tag{1.3}$$

$$H_{\rm int}(\Omega)=\delta[\tfrac{1}{2}(m^2-\Omega^2)\phi^2+\lambda\phi^4] . \tag{1.4}$$

For $\delta=1$ this reproduces the original H, and the new parameter Ω cancels out. For $\delta=0$ the Hamiltonian is soluble; it is a simple harmonic oscillator of frequency Ω. We can therefore employ perturbation theory to obtain the ground-state energy as a power series in δ. By extrapolating this series to $\delta=1$ we obtain an approximation to E_0. To first order the result is just

$$E_0^{(1)}={}_\Omega\langle 0|H|0\rangle_\Omega , \tag{1.5}$$

which is equivalent to a variational approximation with $|0\rangle_\Omega$, the ground state of $H_0(\Omega)$, being the trial state. The approximation can therefore by optimized by minimizing with respect to Ω. The result is accurate to 2% even in the strong-coupling ($\lambda/m^3\to\infty$) limit.

The calculation can be continued to higher orders, but what should one do about Ω? If Ω is kept fixed, as in Ref. 8, then in fact the results will not converge. However, the approximations do converge steadily,[9] provided that Ω is chosen, in each order, in accordance with the "principle of minimal sensitivity."[10-12] The point is that the approximation (i.e, the extrapolation of the truncated power series from $\delta=0$ to $\delta=1$) cannot be trusted in a region where it gives a result strongly dependent on Ω. Only where the approximate result is insensitive to variations in Ω is it a credible approximation to the exact E_0, which is, of course, independent of Ω. Thus, a sensible strategy is to "optimize" the result by requiring it to be as insensitive to Ω as possible. Usually, this simply requires finding the stationary point.[13] The optimum Ω changes from one order to the next, and this is crucial for the expansion to yield convergent results.[9,12]

The same method also works for the other eigenstates of H.[9] It can also be used very successfully to calculate the eigenfunctions,[11] in which case the optimum Ω becomes a function of ϕ. The success of the approach is due to the flexibility provided by the Ω parameter, which is allowed to "adjust itself" to suit the calculation being done. The beauty of the method lies in the "benevolent paradox":[4] the results are genuinely nonperturbative—at no stage is an expansion in powers of λ invoked—but the technique of calculation is basically perturbation theory.

Evidence that the approach is indeed genuinely nonperturbative is provided by the quantum-mechanical examples of Ref. 4. The method successfully handles the quartic oscillator, and singular potentials such as the δ-function and Coulomb potentials, which are problems for which perturbation theory or the loop expansion completely fail. A field-theoretic example is provided by $\phi^6-\phi^4$ theories in three dimensions, where one can use the GEP approach to find bound states whose binding energy is exponentially small in the coupling constant.[14]

In the next section we apply the ideas described above to produce a systematic, nonperturbative expansion for the effective potential of $\lambda\phi^4$ theory in d dimensions.[15] We make use of the powerful functional methods.[16,17] In Sec. III we proceed, using dimensional regularization (DR), to renormalize the PGEP in four dimensions. [Only the so-called "precarious" ϕ^4 theory[5] is considered here. We intend to consider the "autonomous" version[18] in a future article.] Although the DR calculation works nicely, we encountered frustrating difficulties with the quadratic divergences in other types of regularization, and we briefly discuss these problems. Details of the calculation of the divergent integrals using coordinate-space methods are given in Appendix B. Section IV generalizes the results to the O(N)-symmetric ϕ^4 theory. Finally, we summarize our conclusions in Sec. V.

II. POST-GAUSSIAN EXPANSION OF THE EFFECTIVE POTENTIAL

A. General formalism

We begin by recalling the definition of the effective potential.[19,16,17] As a technical convenience we shall work throughout in the Euclidean formalism. The $\lambda\phi^4$ theory in d dimensions has the Euclidean action

$$S[\phi]=\int d^dx[\tfrac{1}{2}\phi(x)(-\partial^2+m^2)\phi(x)+\lambda\phi^4(x)] = \int d^dx\,\mathcal{L}[\phi,\partial_\mu\phi] , \tag{2.1}$$

and the generating functional for Green's functions is given by the functional integral

$$Z[j]=\int D\phi\exp\left[-S[\phi]+\int d^dx\, j(x)\phi(x)\right] . \tag{2.2}$$

Taking the logarithm produces the generating functional for the connected Green's functions:

$$W[j]=\ln Z[j] . \tag{2.3}$$

The effective action $\Gamma[\varphi]$ is then obtained by a Legendre transformation:

$$\Gamma[\varphi]=W[j]-\int d^dx\, j(x)\varphi(x) , \tag{2.4}$$

where

$$\varphi(x)=\frac{\delta W}{\delta j(x)} =Z^{-1}[j]\int D\phi\,\phi\exp\left[-S[\phi]+\int d^dx\, j(x)\phi(x)\right] \tag{2.5}$$

is the vacuum expectation value of the field $\phi(x)$ in the presence of the source $j(x)$. The effective potential $V_{\rm eff}(\varphi)$ is obtained from $\Gamma[\varphi]$ by setting $\varphi(x)$ to be a constant, φ. (Hence j will be x independent). In fact,

$$\Gamma[\varphi]\Big|_{\varphi(x)=\varphi}=-\mathcal{V}V_{\rm eff}(\varphi) , \tag{2.6}$$

where $\mathcal{V}$ is $\int d^dx$, the spacetime volume.

Thus far the discussion is quite standard. What is different is that we propose to calculate $V_{\rm eff}(\varphi)$ in a nonstandard kind of perturbation theory, in which, with

$$\hat\phi(x)=\phi(x)-\phi_0 , \tag{2.7}$$

we define

$$\mathcal{L}=(\mathcal{L}_0+\mathcal{L}_{\rm int})_{\delta=1} , \tag{2.8}$$

where $\mathcal{L}_0$ is a free-field Lagrangian, with mass Ω, for the $\hat\phi$ field,

$$\mathcal{L}_0=\tfrac{1}{2}\hat\phi(x)(-\partial^2+\Omega^2)\hat\phi(x) , \tag{2.9}$$

and the interaction Lagrangian is

$$\mathcal{L}_{\rm int}[\hat\phi]=\delta(v_0+v_1\hat\phi+v_2\hat\phi^2+v_3\hat\phi^3+v_4\hat\phi^4) , \tag{2.10}$$

in which the "coupling constants" v_i are ϕ_0 dependent:

$$\begin{aligned} v_0&=\tfrac{1}{2}m_B^2\phi_0^2+\lambda_B\phi_0^4 ,\\ v_1&=(m_B^2+4\lambda_B\phi_0^2)\phi_0 ,\\ v_2&=\tfrac{1}{2}(m_B^2-\Omega^2)+6\lambda_B\phi_0^2 ,\\ v_3&=4\lambda_B\phi_0,\quad v_4=\lambda_B . \end{aligned} \tag{2.11}$$

An artificial expansion parameter δ has been introduced in $\mathcal{L}_{\rm int}$ in order to keep track of the order of approximation: ultimately we shall set $\delta=1$. That is, our approximation consists in obtaining a (truncated) Taylor series in δ, about $\delta=0$, which is then used to extrapolate to $\delta=1$.

The shift parameter ϕ_0 is to be fixed, self-consistently, to coincide with the classical field $\varphi=\delta W/\delta j$. This choice is made purely for calculational convenience: The results are actually *independent* of the ϕ_0 used (except for the partitioning of terms between zeroth and first orders), as we explain in Appendix A. The use of ϕ_0 is, of course, very much in the spirit of the background-field method.[20] Finally, as discussed in the Introduction, the variable mass parameter Ω is to be fixed, at each order, by the principle of minimal sensitivity.

To calculate $V_{\rm eff}(\varphi)$ we can employ the usual perturbation-theoretic functional methods.[16,17] First we introduce some abbreviated notation in which we write space-time arguments as indices [e.g., ϕ_x for $\phi(x)$], and define

$$\int_x\equiv\int d^dx,\quad \delta_{xy}\equiv\delta^{(d)}(x-y) ,\qquad \int_p\equiv\int\frac{d^dp}{(2\pi)^d},\quad \delta_{pq}\equiv(2\pi)^d\delta^{(d)}(p-q) . \tag{2.12}$$

The free action can be written as

$$\int_x\mathcal{L}_{0,x}=\int_x\int_y\tfrac{1}{2}\hat\phi_x G_{xy}^{-1}\hat\phi_y , \tag{2.13}$$

in which

$$G_{xy}^{-1}=(-\partial^2+\Omega^2)\delta_{xy} , \tag{2.14}$$

whose inverse, in the "matrix" sense that

$$\int_y G_{xy}^{-1}G_{yz}=\delta_{xz} , \tag{2.15}$$

is

$$G_{xy}=\int_p\frac{1}{p^2+\Omega^2}e^{-ip\cdot(x-y)} , \tag{2.16}$$

which is the (Euclidean) x-space propagator.

Following the usual procedure we rewrite the generating functional as

$$Z[j,\phi_0]=\exp\left[\int_z j_z\phi_0\right]\exp\left[-\int_x \hat{\mathcal{L}}_{\rm int}\left[\frac{\delta}{\delta j_x}\right]\right]$$
$$\times\int D\hat{\phi}\exp\left[-\int_z \mathcal{L}_{0,z}+\int_z j_z\hat{\phi}_z\right], \qquad (2.17)$$

in which $\hat{\mathcal{L}}_{\rm int}$ is the functional differential operator obtained from $\mathcal{L}_{\rm int}$ by replacing $\hat{\phi}$ by $\delta/\delta j$. The $\hat{\phi}$ integration can be performed to yield

$$Z[j,\phi_0]=\exp(j\phi_0)({\rm Det}G^{-1})^{-1/2}\exp(-\hat{\mathcal{L}}_{\rm int})$$
$$\times\exp(\tfrac{1}{2}jGj), \qquad (2.18)$$

where, for brevity, we temporarily suppress the spacetime arguments and integrations over them. The functional determinant is well known to be

$$({\rm Det}G^{-1})^{-1/2}=\exp(-\mathcal{V}I_1), \qquad (2.19)$$

where $\mathcal{V}$ is the spacetime volume $(=\int_x=\delta_{pp})$, and I_1 is the integral

$$I_1(\Omega)=\tfrac{1}{2}\int_p \ln(p^2+\Omega^2). \qquad (2.20)$$

Taking the logarithm of Z gives

$$W[j,\phi_0]=j\phi_0-\mathcal{V}I_1$$
$$+\ln[(1-\hat{\mathcal{L}}_{\rm int}+\tfrac{1}{2}\hat{\mathcal{L}}^2_{\rm int}+\cdots)\exp(\tfrac{1}{2}jGj)]$$
$$(2.21)$$

(where $\hat{\mathcal{L}}^2_{\rm int}$ is really $\int_x\hat{\mathcal{L}}_{{\rm int},x}\int_y\hat{\mathcal{L}}_{{\rm int},y}$ etc.). Since we take $\varphi(x)$ to be a constant and we choose to set ϕ_0 equal to φ, $\Gamma[\varphi]$ is given by the above expression for W, but without the $j\phi_0$ term. The source j is to be found as a function of φ by solving the $\varphi=\delta W/\delta j$ equation.

To zeroth order in δ we have just

$$W[j,\phi_0]|_{(0)}=\int_z j_z\phi_0-\mathcal{V}I_1+\tfrac{1}{2}\int_z\int_{z'} j_zG_{z,z'}j_{z'}, \qquad (2.22)$$

so that

$$\varphi_x=\frac{\delta W}{\delta j_x}=\phi_0+(Gj)_x, \qquad (2.23)$$

where $(Gj)_x\equiv\int_z G_{xz}j_z$. Taking φ_x to be x independent, and setting $\phi_0=\varphi$, we see that j vanishes to this order. Thus, $\Gamma[\varphi]$ is just $-\mathcal{V}I_1$.

To first order in δ we have

$$W[j,\phi_0]|_{(1)}=W|_{(0)}-\delta\left[(v_0+v_2I_0+3v_4I_0^2)\mathcal{V}\right.$$
$$+(v_1+3v_3I_0)\int_z(Gj)_z$$
$$\left.+O(j^2)\right], \qquad (2.24)$$

in which I_0, arising as G_{xx}, is the integral

$$G_{xx}=I_0(\Omega)\equiv\int_p\frac{1}{p^2+\Omega^2}. \qquad (2.25)$$

From (2.24) and (2.22), removing the $j\phi_0$ term, we have

$$\Gamma[\varphi]|_{(1)}=-\mathcal{V}I_1(\Omega)+\tfrac{1}{2}jGj$$
$$-\delta[(v_0+v_2I_0+3v_4I_0^2)\mathcal{V}+O(j)]. \qquad (2.26)$$

Since j vanishes in zeroth order, and so is $O(\delta)$, it can contribute only to the $O(\delta^2)$ terms. Thus, for a first-order calculation of Γ it suffices to set $j=0$. Hence, we have immediately (substituting for v_0,v_2,v_4)

$$\Gamma[\varphi]|_{(1)}=-\mathcal{V}\{I_1(\Omega)+\delta[\tfrac{1}{2}m_B^2\varphi^2+\lambda_B\varphi^4+\tfrac{1}{2}(m_B^2-\Omega^2)I_0$$
$$+6\lambda_BI_0\varphi^2+3\lambda_BI_0^2]\}. \qquad (2.27)$$

As expected, the result inside the curly brackets, setting $\delta=1$, is precisely the GEP.

Note that the particular merit of choosing $\phi_0=\varphi$ is that then j becomes or order δ. Therefore, in an nth-order calculation, one needs only terms containing r, or fewer, factors of j in the δ^{n-r} term of $W[j,\phi_0]$. In particular, in the highest-order term one may set $j=0$. Also, when solving for j from the $\varphi=\delta W/\delta j$ equation, one may proceed iteratively, discarding terms $O(\delta^n)$.

B. Second-order Calculation

Proceedings to second order in δ we shall need j explicitly to order δ. This can be obtained by differentiating (2.24) (the Gj term was retained for just this purpose; it was not needed for the first-order calculation itself):

$$\varphi_y=\frac{\delta W}{\delta j_y}=\phi_0+(Gj)_y$$
$$-\delta(v_1+3v_3I_0)\int_z G_{yz}+O(\delta^2;\delta j). \qquad (2.28)$$

Multiplying through by G^{-1}_{xy}, with $\varphi_y=\varphi=\phi_0$, shows that j_x is indeed x independent and given by

$$j=\delta(v_1+3v_0I_0)+O(\delta^2). \qquad (2.29)$$

Next, we need

$$\tfrac{1}{2}\int_x\int_y\hat{\mathcal{L}}_{{\rm int},x}\hat{\mathcal{L}}_{{\rm int},y}\exp(\tfrac{1}{2}jGj)|_{j=0}=\tfrac{1}{2}\delta^2\int_x\int_y[A^2+v_1^2G_{xy}+6v_1v_3I_0G_{xy}+2v_2^2G_{xy}^2$$
$$+24v_2v_4I_0G_{xy}^2+v_3^2(9I_0^2G_{xy}+6G_{xy}^3)+24v_4^2(3I_0^2G_{xy}^2+G_{xy}^4)], \qquad (2.30)$$

where

$$A\equiv v_0+v_2I_0+3v_4I_0^2, \qquad (2.31)$$

which is the same combination as appears in first order. The expression also involves some new integrals:

$$\begin{aligned}
\frac{1}{1!}I^{(1)}(\Omega) &\equiv \int_y G_{xy} = \frac{1}{\Omega^2} , \\
\frac{1}{2!}I^{(2)}(\Omega) &\equiv \int_y G_{xy}^2 = \int_p \frac{1}{(p^2+\Omega^2)^2} , \\
\frac{1}{3!}I^{(3)}(\Omega) &\equiv \int_y G_{xy}^3 = \int_p \int_q \frac{1}{(p^2+\Omega^2)(q^2+\Omega^2)[(p+q)^2+\Omega^2]} , \\
\frac{1}{4!}I^{(4)}(\Omega) &\equiv \int_y G_{xy}^4 = \int_p \int_q \int_k \frac{1}{(p^2+\Omega^2)(q^2+\Omega^2)(k^2+\Omega^2)[(p+q+k)^2+\Omega^2]} .
\end{aligned} \tag{2.32}$$

Note that these are all x independent, and so the remaining x integration just produces the volume factor $\mathcal{V}$. Hence, after regrouping the terms, Eq. (2.30) becomes

$$\tfrac{1}{2}\delta^2\mathcal{V}^2A^2+\tfrac{1}{2}\delta^2\mathcal{V}\{(v_1+3v_3I_0)^2I^{(1)}+(v_2+6v_4I_0)^2I^{(2)}+v_3^2I^{(3)}+v_4^2I^{(4)}\} . \tag{2.33}$$

In combination with the earlier results we now have

$$(1-\hat{\mathcal{L}}_{\rm int}+\tfrac{1}{2}\hat{\mathcal{L}}_{\rm int}^2+\cdots)\exp(\tfrac{1}{2}jGj)=\left[1-\delta\left[\mathcal{V}A+(v_1+3v_3I_0)\int_z (Gj)_z\right]+\tfrac{1}{2}\delta^2\mathcal{V}^2A^2+\tfrac{1}{2}\delta^2\mathcal{V}\{\cdots\}\right]\exp(\tfrac{1}{2}jGj) , \tag{2.34}$$

in which $\{\cdots\}$ represents the terms inside the curly brackets in (2.33). We have discarded terms with more factors of j which would not contribute when we substitute for j, as discussed earlier. Upon taking the logarithm and reexpanding in δ we observe that the $\delta^2\mathcal{V}^2A^2$ terms cancel out. This represents the cancellation of disconnected diagrams. Upon substituting the j from (2.29) we find that the $(v_1+3v_3I_0)^2I^{(1)}$ terms cancel out. This corresponds to the removal of diagrams that are one-particle reducible. Thus, $\Gamma[\varphi]$ is given by

$$\Gamma[\varphi]|_{(2)}=\Gamma|_{(1)}+\tfrac{1}{2}\delta^2\mathcal{V}[(v_2+6v_4I_0)^2I^{(2)}+v_3^2I^{(3)}+v_4^2I^{(4)}] . \tag{2.35}$$

Removing an overall sign and the volume factor, we have the effective potential to second order. Substituting for the v_i we have, finally,

$$\begin{aligned}
V^{(2)}(\varphi,\Omega)=I_1(\Omega)&+\delta\{\tfrac{1}{2}m_B^2\varphi^2+\lambda_B\varphi^4+\tfrac{1}{2}I_0(\Omega)[m_B^2-\Omega^2+12\lambda_B\varphi^2+6\lambda_BI_0(\Omega)]\} \\
&-\delta^2\{\tfrac{1}{8}I^{(2)}(\Omega)[m_B^2-\Omega^2+12\lambda_B\varphi^2+12\lambda_BI_0(\Omega)]^2+8\lambda_B^2\varphi^2I^{(3)}(\Omega)+\tfrac{1}{2}\lambda_B^2I^{(4)}(\Omega)\} .
\end{aligned} \tag{2.36}$$

This result is in accord with the well-known fact that the effective potential is obtained from connected, one-particle-irreducible vacuum diagrams.[16,17] In our case, the relevant diagrams are those generated by the v_i interaction terms in our $\mathcal{L}_{\rm int}$. These diagrams are shown in Fig. 1. (The only problem with using the diagrams as the starting point is to ensure that each diagram is given its correct combinatorial weight. The functional-differentiation calculation provides these automatically.) Our result agrees with that of Okopinska,[7] even though our approach is slightly different.

To renormalize $V^{(2)}$ we proceed, in the usual perturbative fashion, to express the bare parameters as a series in δ:

$$m_B^2=m_{BG}^2+\delta\Delta m_B^2+O(\delta^2) , \tag{2.37}$$

$$\lambda_B=\lambda_{BG}+\delta\Delta\lambda_B+O(\delta^2) , \tag{2.38}$$

where m_{BG}^2 and λ_{BG} correspond to the bare parameters found in the Gaussian approximation, i.e., the forms needed to render the GEP finite. Below four dimensions, the coupling-constant renormalization is unnecessary; λ_B is finite. In four dimensions, though, we need the full

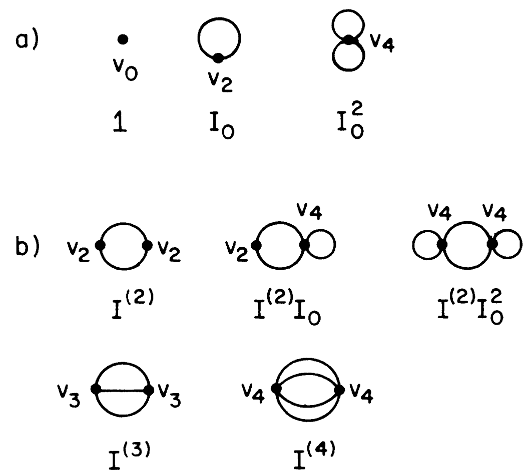

FIG. 1. Feynman diagrams contributing to the effective potential in (a) first order, and (b) second order. Note that, except in first order, v_0 produces only disconnected diagrams. Also, the one-point vertex v_1 produces only one-particle-reducible diagrams.

structure of Eqs. (2.37) and (2.38). Substituting into (2.36) produces that same expression with bare parameters replaced by their first-order forms, plus an extra second-order term:

$$V^{(2)}[m_B^2,\lambda_B]=V^{(2)}[m_{BG}^2,\lambda_{BG}]+\delta^2\{\tfrac{1}{2}\Delta m_B^2[\varphi^2+I_0(\Omega)]+\Delta\lambda_B[\varphi^4+6\varphi^2 I_0(\Omega)+3I_0^2(\Omega)]\}+O(\delta^3)\ . \tag{2.39}$$

At this point we truncate the expansion, discarding the $O(\delta^3)$ term, and set $\delta=1$. This, when optimized with respect to Ω, defines the PGEP. The task now is to find the forms of Δm_B^2 and $\Delta\lambda_B$ which will render the result finite, and to obtain the result in manifestly finite form. In order to do this one first needs to investigate and regularize the divergent integrals involved in $V^{(2)}$.

C. The divergent integrals

The integrals I_1 and I_0 are already familiar from earlier GEP studies.[2,5] In the Hamiltonian approach they appear in the form

$$I_n(\Omega)\equiv\int\frac{d^{d-1}k}{(2\pi)^{d-1}2\omega_k}(\omega_k^2)^n,\quad \omega_k^2\equiv\mathbf{k}^2+\Omega^2\ . \tag{2.40}$$

The d-dimensional integrals in Eqs. (2.20) and (2.25) are equivalent to these (up to an infinite constant in the case of I_1), as one can show by performing the extra integration over the k_0 component. These integrals can be handled conveniently by using their formal properly that

$$\frac{dI_n}{d\Omega^2}=(n-\tfrac{1}{2})I_{n-1}\ . \tag{2.41}$$

As in the GEP analysis,[5] we shall soon encounter the integral $I_{-1}(\Omega)$, defined as $-2dI_0/d\Omega^2$, which from the covariant form of I_0, (2.25), is just

$$I_{-1}(\Omega)=2\int_p\frac{1}{(p^2+\Omega^2)^2}=I^{(2)}\ . \tag{2.42}$$

It therefore coincides with the $I^{(2)}$ integral [of Eq. (2.32)]. This integral converges in less than four dimensions, while in four dimensions it is logarithmically divergent.

Useful formulas for the I_n's are given in Ref. 5. The ones we shall need are for the four-dimensional case:

$$I_0(\Omega)=I_0(0)-\tfrac{1}{2}\Omega^2\left[I_{-1}(\Omega)+\frac{1}{8\pi^2}\right]\ , \tag{2.43}$$

$$I_{-1}(\Omega)=I_{-1}(m)-\frac{1}{8\pi^2}\ln\frac{\Omega^2}{m^2}\ . \tag{2.44}$$

[Note that $I_0(0)$ is zero in dimensional regularization (DR).]

There is an important caveat to the above discussion. The equivalence of $dI_1/d\Omega^2$ with $\frac{1}{2}I_0$ and of I_{-1} with $I^{(2)}$, hinges upon a simple property of the propagator function $G(p)\equiv 1/(p^2+\Omega^2)$, namely that

$$\frac{dG(p)}{d\Omega^2}=-G(p)^2\ . \tag{2.45}$$

Provided that the regularization preserves this property, all will be well. DR, or a simple four-momentum cutoff, for instance, satisfies this criterion. However, in any regularization scheme that uses a modified propagator not obeying (2.45), one must think again. For example, in four dimensions, with such a scheme, I_{-1}, defined as $-2dI_0/d\Omega^2$, and $I^{(2)}$, defined as twice the integral over the propagator squared, would differ by a finite, regularization-scheme dependent, number. This point is closely related to the difficulties discussed in Sec. III B.

Apart from $I^{(2)}$, there are two new integrals appearing in the second-order correction terms, namely, $I^{(3)}$ and $I^{(4)}$. The calculation and regularization of these integrals are discussed in Appendix B. We record here just the results obtained in DR ($d=4-\epsilon$), which will be needed in the next section:

$$I_{-1}(\Omega)=\frac{1}{4\pi^2}\left[\frac{1}{\epsilon}-\frac{1}{2}\left[\ln\frac{\Omega^2}{\mu^2}+\gamma-\ln 4\pi\right]\right]+O(\epsilon)\ , \tag{2.46}$$

$$\frac{1}{3!}I^{(3)}(\Omega)=-\frac{3}{8}\Omega^2\left[I_{-1}^2(\Omega)+\frac{3}{8\pi^2}I_{-1}(\Omega)+O(1)\right]\ , \tag{2.47}$$

$$\frac{1}{4!}I^{(4)}(\Omega)=\frac{1}{4}\Omega^4\left[I_{-1}^3(\Omega)+\frac{23}{48\pi^2}I_{-1}^2(\Omega)+O(I_{-1})\right]\ . \tag{2.48}$$

III. RENORMALIZATION OF THE PRECARIOUS $\lambda\phi^4$ THEORY

A. Dimensional regularization

The PGEP, $\bar{V}^{(2)}(\varphi)$, is obtained from $V^{(2)}(\varphi,\Omega)$ by optimizing the Ω parameter. Differentiating the expression obtained at the end of Sec. II B gives, with the help of Eq. (2.41) and some rearrangement,

$$\frac{\partial V^{(2)}}{\partial\Omega^2}=-\tfrac{1}{8}F^2(\Omega,\varphi)\frac{dI_{-1}(\Omega)}{d\Omega^2}+\tfrac{3}{2}\lambda_{BG}I_{-1}^2(\Omega)F(\Omega,\varphi)-8\lambda_{BG}^2\varphi^2\frac{dI^{(3)}(\Omega)}{d\Omega^2}-\tfrac{1}{2}\lambda_{BG}^2\frac{dI^{(4)}(\Omega)}{d\Omega^2}-\tfrac{1}{4}I_{-1}(\Omega)\Delta m_B^2-3I_{-1}(\Omega)\Delta\lambda_B[I_0(\Omega)+\varphi^2]\ , \tag{3.1}$$

where

$$F(\Omega,\varphi)\equiv m_{BG}^2+12\lambda_{BG}[I_0(\Omega)+\varphi^2]-\Omega^2\ . \tag{3.2}$$

Requiring Eq. (3.1) to vanish determines $\bar{\Omega}$, the optimum value of Ω.

Our strategy will be as follows. First we obtain the φ^2 derivative of $\bar{V}^{(2)}$ and deal with that quantity. Once this has been rendered finite, we may integrate over φ^2 to obtain $\bar{V}^{(2)}$. To begin, we observe that

$$\frac{d\bar{V}^{(2)}}{d\varphi^2}=\frac{\partial V^{(2)}}{\partial\varphi^2}\Bigg|_{\Omega=\bar{\Omega}} , \tag{3.3}$$

since $\partial V^{(2)}/\partial\Omega^2$ vanishes at $\Omega=\bar{\Omega}$. Thus, only the partial derivative

$$\begin{aligned}\frac{\partial V^{(2)}}{\partial\varphi^2}&=\tfrac{1}{2}[m_{BG}^2+12\lambda_{BG}I_0(\Omega)+4\lambda_{BG}\varphi^2]\\&\quad-3\lambda_{BG}I_{-1}(\Omega)F(\Omega,\varphi)-8\lambda_{BG}^2I^{(3)}(\Omega)\\&\quad+\tfrac{1}{2}[\Delta m_B^2+12\Delta\lambda_BI_0(\Omega)]+2\Delta\lambda_B\varphi^2\end{aligned} \tag{3.4}$$

is needed.

The first-order forms of the bare parameters for the precarious $\lambda\phi^4$ theory are[5]

$$m_{BG}^2=m_R^2-12\lambda_{BG}I_0(m_R) , \tag{3.5}$$

$$\lambda_{BG}=\frac{-1}{6I_{-1}(\Lambda)} , \tag{3.6}$$

where m_R and Λ are finite parameters with dimensions of mass. Λ is an interaction scale, analogous to the Λ parameter of QCD. In first order m_R has a direct interpretation as the physical particle mass,[2,5] but this is no longer true at second order. Later on, we shall eliminate both m_R and Λ in favor of two other finite parameters.

In cutoff regularization schemes m_{BG}^2 would be quadratically divergent, but in DR it is actually infinitesimal[21,5]—of order ϵ, or equivalently $O(1/I_{-1})$. To see this one uses (3.6) and (2.43), with $I_0(0)=0$ in DR, which leads to

$$m_{BG}^2=\frac{m_R^2}{8\pi^2I_{-1}(\Lambda)}\left[\ln\frac{m_R^2}{\Lambda^2}-1\right]+O(1/I_{-1}^2) . \tag{3.7}$$

It seems reasonable to expect that the corrections to the bare parameters should be, at most, of the same "size" as the leading terms. We therefore postulate the forms

$$\Delta m_B^2=\frac{\mu^2}{I_{-1}(\Lambda)}+O(1/I_{-1}^2) , \tag{3.8}$$

$$\Delta\lambda_B=\frac{C_1}{I_{-1}(\Lambda)}\left[1+\frac{C_2}{I_{-1}(\Lambda)}+O(1/I_{-1}^2)\right] , \tag{3.9}$$

where μ^2, C_1, and C_2 are constants to be determined. (We shall verify later that a finite Δm_B^2 would not be possible.)

Substituting for m_{BG}^2 and λ_{BG} in the expression for F above, and using (2.43), reveals that F is infinitesimal:

$$\begin{aligned}F(\Omega,\varphi)=\frac{1}{I_{-1}(\Lambda)}&\left[\frac{m_R^2}{8\pi^2}\left[\ln\frac{m_R^2}{\Lambda^2}-1\right]\right.\\&\left.-\frac{\Omega^2}{8\pi^2}\left[\ln\frac{\Omega^2}{\Lambda^2}-1\right]-2\varphi^2\right]\\&+O(1/I_{-1}^2) .\end{aligned} \tag{3.10}$$

Similarly, the first term in $\partial V^{(2)}/\partial\varphi^2$, Eq. (3.4), reduces to $\frac{1}{2}\Omega^2$. Of the remaining terms in Eq. (3.4), the second and the last are $O(1/I_{-1})$; the third [using Eq. (2.47) for $I^{(3)}$] is

$$-8\lambda_{BG}^2I^{(3)}(\Omega)=\tfrac{1}{2}\Omega^2+O(1/I_{-1}) ; \tag{3.11}$$

and the fourth [using Eq. (2.43) for I_0, with $I_0(0)=0$] is

$$\tfrac{1}{2}[\Delta m_B^2+12\Delta\lambda_BI_0(\Omega)]=-3C_1\Omega^2+O(1/I_{-1}) . \tag{3.12}$$

Therefore, in total, we have the remarkably simple result

$$\frac{d\bar{V}}{d\varphi^2}=(1-3C_1)\bar{\Omega}^2+O(1/I_{-1}) . \tag{3.13}$$

That is, the first derivative of the PGEP, with respect to φ^2, is simply proportional to $\bar{\Omega}^2$. Of course, $\bar{\Omega}$ is a nontrivial function of φ, determined by the $\bar{\Omega}$ equation.

We therfore turn to $\partial V^{(2)}/\partial\Omega^2$, given by Eq. (3.1). The first term is only $O(1/I_{-1}^2)$ since

$$\frac{dI_{-1}}{d\Omega^2}=\frac{-1}{8\pi^2\Omega^2} . \tag{3.14}$$

The second term is finite, and given by $(-\frac{1}{4})$ times the expression inside the square brackets in (3.10). The third term reduces to $\frac{1}{2}\varphi^2$, using Eq. (2.47) for $I^{(3)}$. The fourth term, using Eq. (2.48) for $I^{(4)}$, reduces, after a short calculation, to

$$\begin{aligned}-\tfrac{1}{2}\lambda_{BG}^2\frac{dI^{(4)}}{d\Omega^2}=-\frac{\Omega^2}{6}&\left\{I_{-1}(\Lambda)\right.\\&\left.+\left[\beta_d-\frac{3}{8\pi^2}\left[\ln\frac{\Omega^2}{\Lambda^2}+\frac{1}{2}\right]\right]\right\}\\&+O(1/I_{-1}) ,\end{aligned} \tag{3.15}$$

where $\beta_d\equiv23/(48\pi^2)$ is the subleading coefficient in the DR form of $I^{(4)}$. Note that it is important to distinguish between $I_{-1}(\Omega)$ and $I_{-1}(\Lambda)$: their difference is given by Eq. (2.44). The fifth term of (3.1) reduces to $-\frac{1}{4}\mu^2$, while the last term requires a bit more calculation:

$$\begin{aligned}-3I_{-1}(\Omega)\Delta\lambda_B[I_0(\Omega)+\varphi^2]&=-3C_1\frac{I_{-1}(\Omega)}{I_{-1}(\Lambda)}\left[1+\frac{C_2}{I_{-1}(\Lambda)}+\cdots\right]\left[-\tfrac{1}{2}\Omega^2\left[I_{-1}(\Omega)+\frac{1}{8\pi^2}\right]+\varphi^2\right]\\&=\tfrac{3}{2}C_1\Omega^2\left[I_{-1}(\Lambda)+C_2-\frac{1}{4\pi^2}\left[\ln\frac{\Omega^2}{\Lambda^2}-\frac{1}{2}\right]\right]-3C_1\varphi^2+O(1/I_{-1}) .\end{aligned} \tag{3.16}$$

Therefore, in total, we have

$$\frac{\partial V^{(2)}}{\partial\Omega^2}=(-\tfrac{1}{6}+\tfrac{3}{2}C_1)\Omega^2 I_{-1}(\Lambda)+(1-3C_1)\varphi^2-\tfrac{1}{4}\mu^2-\frac{m_R^2}{32\pi^2}\left[\ln\frac{m_R^2}{\Lambda^2}-1\right]+\frac{3}{32\pi^2}(1-4C_1)\Omega^2\ln\frac{\Omega^2}{\Lambda^2}+\Omega^2\left[\frac{3C_1}{16\pi^2}+\tfrac{3}{2}C_1C_2-\tfrac{1}{6}\beta_d\right]. \tag{3.17}$$

Clearly, in order for the $\bar{\Omega}$ equation to have a nontrivial solution, the coefficient of the divergent I_{-1} term must vanish. This determines the constant C_1 in $\Delta\lambda_B$:

$$C_1=\tfrac{1}{9}. \tag{3.18}$$

[At this stage we digress briefly to explain why a finite mass correction, $\Delta m_B^2=M^2+O(1/I_{-1})$, would not work. In that case (3.17) would take the form

$$\frac{\partial V^{(2)}}{\partial\Omega^2}=[(-\tfrac{1}{6}+\tfrac{3}{2}C_1)\Omega^2-\tfrac{1}{4}M^2]I_{-1}(\Lambda)+\text{finite}, \tag{3.19}$$

which would only allow a solution $\bar{\Omega}=$const (independent of φ), producing, from (3.13), a trivial PGEP proportional to φ^2.]

With $C_1=\frac{1}{9}$, Eq. (3.17) yields the $\bar{\Omega}$ equation

$$5\bar{\Omega}^2\ln\frac{\bar{\Omega}^2}{\Lambda^2}+2[1+8\pi^2(C_2-\beta_d)]\bar{\Omega}^2+64\pi^2\varphi^2=3m_R^2\left[\ln\frac{m_R^2}{\Lambda^2}-1\right]+24\pi^2\mu^2. \tag{3.20}$$

At this point it might seem that we have too many free parameters, since, in addition to m_R^2 and Λ, we now have μ^2 and C_2 (which arise from Δm_B^2 and $\Delta\lambda_B$, respectively). Actually, the result depends only upon two independent parameters, which are combinations of these four. This is most easily seen by defining a new "renormalized mass" parameter Ω_0 as the value of $\bar{\Omega}$ at $\varphi=0$. Then, subtracting from (3.20) its $\varphi=0$ form, and rescaling the variables with respect to Ω_0, one obtains

$$x\ln x+(1-\kappa)(x-1)-\tfrac{64}{5}\pi^2\Phi^2, \tag{3.21}$$

where

$$x\equiv\bar{\Omega}^2/\Omega_0^2,\quad \Phi^2=\varphi^2/\Omega_0^2, \tag{3.22}$$

and

$$\kappa\equiv1+\tfrac{2}{5}[1+8\pi^2(C_2-\beta_d)]+\ln\frac{\Omega_0^2}{\Lambda^2}. \tag{3.23}$$

The parameter κ has a simple interpretation: it is inversely proportional to the renormalized coupling constant λ_R as canonically defined by

$$\lambda_R\equiv\frac{1}{2}\left.\frac{d^2\bar{V}^{(2)}}{(d\varphi^2)^2}\right|_{\varphi=0}. \tag{3.24}$$

This can be shown as follows. From (3.13) with $C_1=\frac{1}{9}$ one has

$$\frac{d\bar{V}^{(2)}}{d\varphi^2}=\frac{2}{3}\bar{\Omega}^2, \tag{3.25}$$

so that

$$\frac{d^2\bar{V}^{(2)}}{(d\varphi^2)^2}=\frac{2}{3}\frac{d\bar{\Omega}^2}{d\varphi^2}. \tag{3.26}$$

But the $\bar{\Omega}$ equation, (3.21), determines $\bar{\Omega}$ as a function of φ. Hence

$$\frac{d\bar{\Omega}^2}{d\varphi^2}=\frac{dx}{d\Phi^2}=\frac{-64\pi^2}{5(\kappa+\ln x)}, \tag{3.27}$$

and therefore

$$\lambda_R=-\frac{64\pi^2}{15\kappa}. \tag{3.28}$$

All that remains now is to integrate (3.25) to obtain $\bar{V}^{(2)}$ itself:

$$\begin{aligned}\bar{V}^2(\varphi)&=\tfrac{2}{3}\int d\varphi^2\bar{\Omega}^2(\varphi^2)\\&=\tfrac{2}{3}\int d\bar{\Omega}^2\bar{\Omega}^2\left[\frac{d\varphi^2}{d\bar{\Omega}^2}\right]\\&=-\frac{5}{96\pi^2}\Omega_0^4\int dx\,x(\kappa+\ln x)\\&=\frac{5}{192\pi^2}\Omega_0^4x^2(\tfrac{1}{2}-\kappa-\ln x)+\text{const}.\end{aligned} \tag{3.29}$$

Choosing the constant of integration such that $\bar{V}^{(2)}(\varphi)$ vanishes at $\varphi=0$ (where $x=1$), and using the $\bar{\Omega}$ equation (3.21) to eliminate the logarithm, produces

$$\frac{\bar{V}^2(\varphi)}{\Omega_0^4}=\tfrac{1}{3}x\Phi^2-\frac{5}{384\pi^2}(x-1)(x-1+2\kappa). \tag{3.30}$$

This expression, with x determined by (3.21), gives the PGEP in manifestly finite form, parametrized in terms of Ω_0 and κ (directly related to λ_R).

What is remarkable about this result is its extraordinary resemblance to the first-order result. The equations have exactly the same form: only the numerical coefficients are changed. In the GEP, the $64\pi^2/5$ coefficient in (3.21) used to be $16\pi^2$, while in (3.30) the coefficients $\frac{1}{3}$ and $5/(384\pi^2)$ were $\frac{1}{4}$ and $1/(128\pi^2)$, respectively. [See Ref. 5, Eqs. (5.9) and (5.10).] Also we note that in the GEP case the parameter κ was related to λ_R by $\lambda_R=-4\pi^2/\kappa$, rather than (3.28).

Perhaps the most meaningful comparison between first- and second-order renormalized results is obtained by parametrizing both $\bar{V}_G$ and $\bar{V}^{(2)}$ in terms of their first and second derivatives at the origin, i.e., in terms of

$$m^2 \equiv 2\frac{d\bar{V}}{d\varphi^2}\bigg|_{\varphi=0} = \begin{cases} \Omega_0^2 & \text{(first order)}, \\ \frac{4}{3}\Omega_0^2 & \text{(second order)}, \end{cases} \tag{3.31}$$

and λ_R defined as in (3.24). In fact, it can be shown that this m corresponds to the physical mass, as determined from the pole in the propagator.[22] When compared in this fashion, the net change from first to second order is really very small. In each case $\bar{V}$ is obtained by integrating

$$\frac{d\bar{V}}{d\varphi^2} = \frac{1}{2}m^2 x \ , \tag{3.32}$$

with x given by

$$(x-1) = \frac{\lambda_R}{4\pi^2}\left[\eta[x \ln x - (x-1)] + 16\pi^2\frac{\varphi^2}{m^2}\right] , \tag{3.33}$$

in which $\eta = 1$ at first order, and changes by a mere 6% to $\eta = \frac{15}{16}$ at second order.

Qualitatively, the shape of the effective potential is unchanged. In particular, there is a critical φ beyond which the Ω equation has no solution. We refer the reader to the extensive discussion in Ref. 5. Unfortunately, the second-order calculation does not provide any further clues as to what really happens beyond φ_{crit}.

B. Difficulties with cutoff schemes

Although DR is perfectly satisfactory and convenient, we did also try to carry out the calculation in a cutoff scheme. Our aim was twofold: (i) to explicitly verify the regularization-scheme independence of the results, and (ii) to see the cancellation of the quadratic divergences—an issue that is sidestepped in DR. Our efforts were only partly successful, as we briefly discuss.

In a cutoff scheme there are quadratic divergences in $I_1, I_0, I^{(3)}$, and $I^{(4)}$, which one would hope to absorb into m_{BG}^2 and Δm_B^2.[23] (The quartic divergences in I_1 and $I^{(4)}$, being Ω independent, contribute only to the vacuum energy constant.) In $\partial V^{(2)}/\partial\Omega^2$, Eq. (3.1), quadratic divergences arise only in the last three terms, and they will cancel if

$$2\lambda_{BG}^2\frac{dI^{(4)}(\Omega)}{d\Omega^2} + I_{-1}(\Omega)[\Delta m_B^2 + 12\Delta\lambda_B I_0(0)] = \text{n.q.d.} \ , \tag{3.34}$$

where "n.q.d." stands for "not quadratically divergent." However, the quadratic divergences must also cancel in $\partial V^{(2)}/\partial\varphi^2$, Eq. (3.4), which requires

$$16\lambda_B^2 I^{(3)}(\Omega) - [\Delta m_B^2 + 12\Delta\lambda_B I_0(0)] = \text{n.q.d.} \tag{3.35}$$

Comparing these two equations, one sees that all will be well provided that

$$\frac{dI^{(4)}(\Omega)}{d\Omega^2} + 8I_{-1}(\Omega)I^{(3)}(\Omega) = \text{n.q.d.} \tag{3.36}$$

If one takes the expressions (B14)–(B16) of Appendix B calculated with a small-x cutoff at $x=a$, and if one identifies $I^{(2)}$ with I_{-1}, then one indeed finds that this relationship is satisfied. However, this seems to be somewhat fortuitous, because the relationship is not satisfied in other similar cutoff schemes, such as the one employing a smooth cutoff function $\exp(-a/x)$. What happens is that the leading terms $O(\ln a/a^2)$, cancel properly, but the subleading terms $O(1/a^2)$, in general, do not.

We believe that this difficulty is a symptom of a faulty regularization scheme, and not a fundamental problem of the theory or of our expansion method. An x-space cutoff approach, while very convenient for the $I^{(n)}$ integrals, does not have any clear meaning for the I_n's, and it may not be valid to continue to rely on the formal result, (2.41), for $dI_n/d\Omega^2$. These problems with regularization have no counterpart in ordinary perturbation theory, in which differentiation with respect to the mass squared is not an issue.

[We have also tried various kinds of modified propagators, giving up the property (2.45), and hence (2.41). However, such schemes always had problems with I_1, the integral of the logarithm of the momentum-space propagator. Moreover, in general, Eq. (3.36) was not satisfied, and $I^{(2)}$ and I_{-1} would differ by a finite constant. One could define a modified propagator $G(p)$ which still respected (2.45), but then one could not analytically perform the Fourier transformation to obtain $G(x)$, making the calculations intractable.]

Some insight into these problems can perhaps be gained by using

$$I_{-1}(\Omega) = -2\frac{dI_0(\Omega)}{d\Omega^2} = -2\frac{dG(x)}{d\Omega^2}\bigg|_{x=0} , \tag{3.37}$$

to rewrite the combination of Eq. (3.36), formally, as

$$96\int d^dx\, G^3(x)\left[\frac{dG(x)}{d\Omega^2} - \frac{dG(x)}{d\Omega^2}\bigg|_{x=0}\right] . \tag{3.38}$$

If the expression inside the large parentheses were $O(x^2)$, all would be well. Unfortunately, it is proportional to $K_0(\Omega x) - K_0(0)$, which is wholly ill defined in the absence of regularization. We have not been able to resolve, or circumvent, this difficulty. [In three dimensions, though, this approach works very nicely to show that the combination (3.36) is finite, which is necessary for the linear divergences to cancel in the three-dimensional case. The result (B11) is thereby obtained in a regularization-scheme-independent fashion.]

If we shut our eyes to these problems and go back to the sharp x-space cutoff scheme, in which (3.36) is fortuitously satisfied, we can continue the calculation. The quadratic divergences can be consistently absorbed by a bare-mass correction term of the form

$$\Delta m_B^2 = \frac{1}{24\pi^4 a^2 I_{-1}^2(\Lambda)} - 12\Delta\lambda_B I_0(0) + \frac{\mu^2}{I_{-1}(\Lambda)} + O(1/I_{-1}^2) \ . \tag{3.39}$$

The calculation then proceeds exactly as in DR, with one difference: In DR the coefficient of the subleading I_{-1}^2

term in $I^{(4)}$ is $\beta_d \equiv 23/(48\pi^2)$ [see Eq. (2.48)], whereas in the cutoff scheme it is $\beta_c \equiv 11/(16\pi^2)$ [see Eq. (B16)]. However, since the β coefficient is ultimately absorbed (together with the arbitrary parameter C_2) into the definition of the κ parameter in (3.23), there is no difference in the end result. This gives us confidence that the final result is indeed regularization-scheme independent, even though we lack a proper demonstration of the fact.

IV. THE O(N)-SYMMETRIC $\lambda\phi^4$ THEORY

The generalization of the post-Gaussian expansion to the O(N)-symmetric model:

$$\mathcal{L}=\tfrac{1}{2}\phi_i(x)(-\partial^2+m^2)\phi_i(x)+\lambda[\phi_i(x)\phi_i(x)]^2 \quad (i=1,\ldots,N)\,, \tag{4.1}$$

is fairly straightforward. Without loss of generality, the external source, and hence the classical field, can be chosen to be in the $i=1$ direction. We therefore define

$$\hat{\phi}_1(x)=\phi_1(x)-\varphi\,, \tag{4.2}$$

$$\hat{\phi}_i(x)=\phi_i(x) \quad \text{for} \quad i=2,\ldots,N\,, \tag{4.3}$$

and take $\mathcal{L}_0$ to be a sum of N free-field Lagrangians: one with mass Ω, for the radial field, $\hat{\phi}_1$, and $N-1$ with mass ω for the transverse fields, $\hat{\phi}_2,\ldots,\hat{\phi}_N$:

$$\mathcal{L}_0=\tfrac{1}{2}\hat{\phi}_1(x)(-\partial^2+\Omega^2)\hat{\phi}_1(x)+\sum_{i=2}^{N}\tfrac{1}{2}\hat{\phi}_i(x)(-\partial^2+\omega^2)\hat{\phi}_i(x)\,. \tag{4.4}$$

The interaction Lagrangian is then

$$\mathcal{L}_{\rm int}\left[\hat{\phi}_1,\sum_{i=2}^{N}\hat{\phi}_i^2\right]=\delta\left[v_0+v_1\hat{\phi}_1+v_2\hat{\phi}_1^2+v_3\hat{\phi}_1^3+v_4\hat{\phi}_1^4+v_2'\sum_{i=2}^{N}\hat{\phi}_i^2+v_4\left(\sum_{i=2}^{N}\hat{\phi}_i^2\right)^2 +4v_4\varphi\hat{\phi}_1\sum_{i=2}^{N}\hat{\phi}_i^2+2v_4\hat{\phi}_1^2\sum_{i=2}^{N}\hat{\phi}_i^2\right]\,. \tag{4.5}$$

in which the "coupling constants" $v_0,\ldots,v_4$ are as before, while v_2' stands for

$$v_2'=\tfrac{1}{2}(m_B^2-\omega^2)+2\lambda_B\varphi^2\,. \tag{4.6}$$

Since all the transverse fields are on the same footing, the calculations can be simplified by introducing a composite field $\chi=\sum_{i=2}^{N}\hat{\phi}_i^2$ via a δ-functional constraint, and then expressing the δ-functional as a Fourier integral:

$$Z[j]=\int D[\phi_1,\ldots,\phi_N,\chi,\tilde{\chi}]\exp\left[i\tilde{\chi}\left(\chi-\sum_{i=2}^{N}\hat{\phi}_i^2\right)-\mathcal{L}_0-\mathcal{L}_{\rm int}[\hat{\phi}_1,\chi]+j(\varphi+\hat{\phi}_1)\right]\,. \tag{4.7}$$

Following standard methods, a source term K for the field χ is introduced (and set to zero in the end), which will then allow us to perform the χ and $\tilde{\chi}$ integrations, as well as all of the remaining integrations over the $\hat{\phi}_1,\ldots,\hat{\phi}_N$ fields. Hence the generating functional reduces to

$$Z[j]=\exp[j\varphi-\tfrac{1}{2}{\rm Tr}(\ln G^{-1})]\exp\left[-\hat{\mathcal{L}}_{\rm int}\left[\frac{\delta}{\delta j},\frac{\delta}{\delta K}\right]\right]\exp\left[-\frac{N-1}{2}{\rm Tr}(\ln S^{-1})+\tfrac{1}{2}jGj\right]_{K=0}\,, \tag{4.8}$$

where we have defined

$$S^{-1}=-\partial^2+\omega^2-2K\equiv G_\omega^{-1}-2K\,. \tag{4.9}$$

Expanding $\exp(-\hat{\mathcal{L}}_{\rm int})$ up to $O(\hat{\mathcal{L}}_{\rm int}^2)$, or equivalently up to $O(\delta^2)$, and performing the necessary functional differentiations, yields

$$\begin{aligned}V^{(2)}(\varphi,\Omega,\omega)=&\,I_1(\Omega)+(N-1)I_1(\omega)\\&+\delta\Big\{\tfrac{1}{2}m_B^2\varphi^2+\lambda_B\varphi^4+\tfrac{1}{2}I_0(\Omega)[m_B^2-\Omega^2+12\lambda_B\varphi^2+6\lambda_B I_0(\Omega)]\\&\quad+\frac{N-1}{2}I_0(\omega)[m_B^2-\omega^2+4\lambda_B\varphi^2+4\lambda_B I_0(\Omega)+2(N+1)\lambda_B I_0(\omega)]\Big\}\\&-\delta^2\Big[\tfrac{1}{8}I^{(2)}(\Omega)F_1^2+8\lambda_B^2\varphi^2I^{(3)}(\Omega)+\tfrac{1}{2}\lambda_B^2I^{(4)}(\Omega)\\&\quad+(N-1)\left[\tfrac{1}{8}I^{(2)}(\omega)F_2^2+\tfrac{8}{3}\lambda_B^2\varphi^2I^{(1,2)}(\Omega,\omega)+\tfrac{1}{3}\lambda_B^2I^{(2,2)}(\Omega,\omega)+\frac{N+1}{6}\lambda_B^2I^{(4)}(\omega)\right]\Big]\,,\end{aligned} \tag{4.10}$$

where F_1 and F_2 are defined by

$$F_1=F_1(\varphi,\Omega,\omega)\equiv m_B^2-\Omega^2+4\lambda_B[3I_0(\Omega)+(N-1)I_0(\omega)+3\varphi^2] , \tag{4.11}$$

$$F_2=F_2(\varphi,\Omega,\omega)\equiv m_B^2-\omega^2+4\lambda_B[I_0(\Omega)+(N+1)I_0(\omega)+\varphi^2] . \tag{4.12}$$

The two new integrals appearing in the above expression, $I^{(1,2)}(\Omega,\omega)$ and $I^{(2,2)}(\Omega,\omega)$, are just modified versions of $I^{(3)}$ and $I^{(4)}$, respectively, with two of the Ω propagators replaced by ω propagators. The DR forms of these integrals, obtained by the same x-space procedure described in Appendix B, are

$$I^{(1,2)}(\Omega,\omega)=\tfrac{1}{3}I^{(3)}(\Omega)+\tfrac{2}{3}I^{(3)}(\omega)+\text{finite}=-\tfrac{3}{4}(\Omega^2+2\omega^2)I_{-1}^2(\Omega)+O(I_{-1}) , \tag{4.13}$$

$$I^{(2,2)}(\Omega,\omega)=(\Omega^4+4\Omega^2\omega^2+\omega^4)I_{-1}^3(\Omega)+\frac{1}{16\pi^2}\left[7\Omega^4+7\omega^4+32\Omega^2\omega^2+6\omega^4\ln\frac{\Omega^2}{\omega^2}+12\Omega^2\omega^2\ln\frac{\Omega^2}{\omega^2}\right]I_{-1}^2(\Omega)+O(I_{-1}) . \tag{4.14}$$

For the renormalization we write the bare parameters as in Eqs. (2.37) and (2.38), which effectively produces an extra second-order term:

$$V^{(2)}[m_B^2,\lambda_B]=V^{(2)}[m_{BG}^2,\lambda_{BG}]+\delta^2(\tfrac{1}{2}\Delta m_B^2[\varphi^2+I_0(\Omega)+(N-1)I_0(\omega)]+\Delta\lambda_B\{\varphi^4+6\varphi^2I_0(\Omega)+3I_0^2(\Omega)+(N-1)I_0(\omega)[2\varphi^2+(N+1)I_0(\omega)+2I_0(\Omega)]\})+O(\delta^3) , \tag{4.15}$$

where[6]

$$m_{BG}^2=m_R^2-4(N+2)\lambda_BI_0(m_R) , \tag{4.16}$$

$$\lambda_{BG}=\frac{-1}{2(N+2)I_{-1}(\Lambda)} . \tag{4.17}$$

We now truncate the expansion and set $\delta=1$. This, when optimized with respect to Ω and ω, defines the PGEP. The two equations fixing $\bar{\Omega}$ and $\bar{\omega}$ are obtained by setting to zero the expressions

$$2\frac{\partial V^{(2)}}{\partial\Omega^2}=3\lambda_{BG}I_{-1}^2(\Omega)F_1+(N-1)\lambda_{BG}I_{-1}(\Omega)I_{-1}(\omega)F_2-\frac{1}{4}\frac{dI_{-1}(\Omega)}{d\Omega^2}F_1^2-16\lambda_{BG}^2\varphi^2\left[\frac{dI^{(3)}(\Omega)}{d\Omega^2}+\tfrac{1}{3}(N-1)\frac{dI^{(1,2)}(\Omega,\omega)}{d\Omega^2}\right]-\lambda_{BG}^2\left[\frac{dI^{(4)}(\Omega)}{d\Omega^2}+\tfrac{2}{3}(N-1)\frac{dI^{(2,2)}(\Omega,\omega)}{d\Omega^2}\right]-\tfrac{1}{2}\Delta m_B^2I_{-1}(\Omega)-2\Delta\lambda_BI_{-1}(\Omega)[3\varphi^2+3I_0(\Omega)+(N-1)I_0(\omega)] , \tag{4.18}$$

$$\frac{2}{N-1}\frac{\partial V^{(2)}}{\partial\omega^2}=\lambda_{BG}I_{-1}(\Omega)I_{-1}(\omega)F_1+(N+1)\lambda_{BG}I_{-1}^2(\omega)F_2-\frac{1}{4}\frac{dI_{-1}(\omega)}{d\omega^2}F_2^2-\frac{16}{3}\lambda_{BG}^2\varphi^2\frac{dI^{(1,2)}(\Omega,\omega)}{d\omega^2}-\tfrac{1}{3}\lambda_{BG}^2\left[(N+1)\frac{dI^{(4)}(\omega)}{d\omega^2}+2\frac{dI^{(2,2)}(\Omega,\omega)}{d\omega^2}\right]-\tfrac{1}{2}\Delta m_B^2I_{-1}(\omega)-2\Delta\lambda_BI_{-1}(\omega)[\varphi^2+I_0(\Omega)+(N+1)I_0(\omega)] . \tag{4.19}$$

It is noteworthy that at the origin $\bar{\Omega}_0=\bar{\omega}_0\equiv\Omega_0$, as one would expect from the unbroken O(N) symmetry. This is easily seen, once one notes that

$$2\frac{dI^{(2,2)}(\Omega,\omega)}{d\Omega^2}\bigg|_{\omega=\Omega}=\frac{dI^{(4)}(\Omega)}{d\Omega^2} . \tag{4.20}$$

As before, the corrections to the bare parameters, Δm_B^2 and $\Delta\lambda_B$ are taken to have the forms (3.8) and (3.9) (in DR). Substituting for the bare parameters in (4.18) and (4.19), one finds the divergent terms

$$2\frac{\partial V^{(2)}}{\partial\Omega^2}=\left[\frac{(2N-11)\Omega^2-5(N-1)\omega^2}{3(N+2)^2}+C_1[3\Omega^2+(N-1)\omega^2]\right]I_{-1}(\Lambda)+\text{finite}=0 , \tag{4.21}$$

$$\frac{2}{N-1}\frac{\partial V^{(2)}}{\partial\omega^2}=\left[-\left[\frac{5\Omega^2+(3N+1)\omega^2}{3(N+2)^2}\right]+C_1[\Omega^2+(N+1)\omega^2]\right]I_{-1}(\Lambda)+\text{finite}=0 . \tag{4.22}$$

Hence, in order to obtain nontrivial solutions, the two coefficients of the divergent I_{-1}'s must vanish, at least up to $O(1/I_{-1})$. This requires the ratio $y \equiv \bar{\Omega}^2/\bar{\omega}^2$ to satisfy

$$(y-1)(y-1+N)=O(1/I_{-1}) . \tag{4.23}$$

and since y is positive definite, the only solution is $y=1+O(1/I_{-1})$. Substituting back in (4.21) and (4.22) determines C_1 to be

$$C_1=\frac{1}{(N+2)^2} . \tag{4.24}$$

As in leading order,[6] although $\bar{\Omega}^2$ and $\bar{\omega}^2$ are equal up to a $1/I_{-1}$ term, one actually needs to evaluate this infinitesimal difference. Substituting

$$\bar{\omega}^2=\bar{\Omega}^2+\frac{g}{I_{-1}} \tag{4.25}$$

into (4.21) and (4.22), now including the finite terms wich we did not write out explicitly, gives two equations for Ω^2 and g. Solving for g yields

$$g=\frac{3(N+6)}{2N}\varphi^2+O(1/I_{-1}) \tag{4.26}$$

(which vanishes, at the origin, as expected). Using this one obtains the equations for Ω:

$$(N+4)\bar{\Omega}^2\ln\frac{\bar{\Omega}^2}{\Lambda^2}+(16\pi^2C_2-N-\tfrac{14}{3})\bar{\Omega}^2+\frac{N+3}{N}16\pi^2\varphi^2$$
$$=(N+2)\left[m_R^2\left[\ln\frac{m_R^2}{\Lambda^2}-1\right]+8\pi^2\mu^2\right] , \tag{4.27}$$

which generalizes Eq. (3.20) in the $N=1$ analysis. Proceeding as before, one obtains the rescaled $\bar{\Omega}$ equation

$$x-1=\frac{N\lambda_R}{4\pi^2}\frac{(N+2)(N+4)}{(N+3)^2}$$
$$\times\left[x\ln x-(x-1)+16\pi^2\frac{\Phi^2}{N}\left[\frac{N+3}{N+4}\right]\right] . \tag{4.28}$$

The effective potential can be calculated by integrating

$$\frac{d\bar{V}^{(2)}}{d\varphi^2}=\frac{(N+3)}{2(N+2)}\bar{\Omega}^2 , \tag{4.29}$$

to yield

$$\frac{1}{N}\bar{V}^{(2)}(\Phi)=\left[\frac{1}{4}\left[\frac{N+3}{N+2}\right]\frac{x\Phi^2}{N}\right.$$
$$\left.-\frac{1}{128\pi^2}\left[\frac{N+4}{N+2}\right](x-1)(x-1+2\kappa)\right] , \tag{4.30}$$

in which

$$\kappa=-\frac{4\pi^2}{N\lambda_R}\frac{(N+3)^2}{(N+2)(N+4)} . \tag{4.31}$$

For comparison, let us recall the situation in first order.[6] At the GEP level, the O(N) results in four dimensions are trivially related to those for $N=1$: one simply needs to scale $\bar{V}_G$, λ_R, and $1/\varphi^2$ by a factor of N. This means that, at the renormalized level, the GEP is identical to the leading-order $1/N$ result. (This is specific to the four-dimensional "precarious" case, and occurs only *after* renormalization. In general, the GEP reduces to the leading $1/N$ result *only* in the limit $N\to\infty$.) The equations for the GEP (Ref. 6) are the same as (4.28)–(4.31) above, except that the various factors of the form $(N+n_1)/(N+n_2)$, with $n_1,n_2=2$, 3 or 4, do not appear. The similarity is even more dramatic if we parametrize both $\bar{V}_G$ and $\bar{V}^{(2)}$ in terms of their first and second derivatives at the origin. Both results can then be expressed in the form

$$\frac{d\bar{V}}{d\phi^2}=\tfrac{1}{2}m^2x , \tag{4.32}$$

with x given by

$$(x-1)=\frac{N\lambda_R}{4\pi^2}\left[\eta[x\ln x-(x-1)]+\frac{16\pi^2\varphi^2}{Nm^2}\right] , \tag{4.33}$$

where $\eta=1$ at first order, and changes to

$$\eta=\frac{(N+2)(N+4)}{(N+3)^2}=1-\frac{1}{(N+3)^2} \tag{4.34}$$

at second order. These last three equations constitute our final result, in its simplest form.

Notice that, in this natural parametrization (in which m corresponds to the physical mass[22]), the corrections are of relative order $1/N^2$ in the large-N limit. That is, all the corrections of relative order $1/N$ have been absorbed into the renormalized parameters. It would be interesting to reexamine the next-to-leading-order $1/N$-expansion calculation of Root[24] in this light.

V. CONCLUSIONS

Our main conclusion is that a systematic post-Gaussian expansion procedure[4,7] is indeed a practical proposition. It is as well defined as the loop expansion, or the $1/N$ expansion, and is no harder to calculate. Indeed, with hindsight, the second-order calculation is really not that difficult, thanks to dimensional regularization, and the x-space technique. Moreover, it turned out that we really only needed to know the coefficient of the highest-order pole in $1/\epsilon$ in each integral.

It is remarkable that the second-order result in four dimensions has the same mathematical form as the GEP, with only the numerical coefficients changing. Qualitatively the shape of the potential remains the same, and quantitatively the changes are small. In the simplest form of our result, Eqs. (4.32)–(4.34), only a single coefficient changes, from 1 to $1-1/(N+3)^2$. The smallness of the change gives us increased confidence that the original Gaussian result is a good approximation.

An important property of the expansion is that the forms of the bare parameters change from one order to the next. That is, the bare parameters needed to make the GEP finite are not (except in two dimensions) the exact bare parameters, but only an approximation thereto.[25] In four dimensions the Gaussian approximation yields

$$\lambda_{BG}=\frac{-1}{2(N+2)I_{-1}(\Lambda)}\ , \tag{5.1}$$

which changes to

$$\begin{aligned}\lambda_B^{(2)}&=\frac{1}{I_{-1}(\Lambda)}\left[\frac{-1}{2(N+2)}+\frac{1}{(N+2)^2}\right]\\&=\frac{-N}{2(N+2)^2I_{-1}(\Lambda)}\ ,\end{aligned} \tag{5.2}$$

in second order. The change is small when N is large, but is quite substantial ($-\frac{1}{6}$ changing to $-\frac{1}{18}$) for $N=1$. Note, however, that the situation is much better than in ordinary perturbation theory, where, in oversimplified form, one has

$$\lambda_B=\lambda_R(1+a_1\lambda_R I_{-1}+a_2\lambda_R^2 I_{-1}^2+\cdots)\ , \tag{5.3}$$

in which each "correction" is much larger than the previous terms.

A resummation of perturbation theory implies the form[26]

$$\lambda_B^{(\mathrm{pert})}=\frac{-1}{2(N+8)I_{-1}(\Lambda)}+O\left[\frac{\ln I_{-1}}{I_{-1}^2}\right]\ , \tag{5.4}$$

in which the coefficient is directly related to the leading-order β function coefficient. It in encouraging that the change from (5.1) to (5.2) brings the post-Gaussian result into better agreement with (5.4). [Indeed, for $N=1$, one obtains the same coefficient, $-\frac{1}{18}$, from both (5.2) and (5.4), but this exact agreement is probably only fortuitous.]

Clearly, much work remains to be done. (i) There is the unresolved problem with quadratic divergences in non-DR schemes, discussed in Sec. III B. Presumably the problem is due to inconsistencies in regularization, and would not occur in a consistent, overall regularization of the theory. However, the calculations in, say, a lattice treatment, appear forbiddingly difficult. (ii) We have not discussed the lower-dimensional cases or the "autonomous' version of four-dimensional $\lambda\phi^4$ theory. These topics, and the generalization to scalar-fermion theories have been explored in Ref. 22, and we hope to report on them in a future paper. (iii) A calculation to third order in $\lambda\phi^4$ theory may well be quite feasible. There will be several new, nasty integrals, of course, but it is quite possible that again one would need only their leading pole terms in $1/\epsilon$. It could be that the third-result, too, has the same mathematical form as the GEP, but with further changes in the numerical coefficients. If so, one can ask if this might be true to all orders, and if there is some simple way to prove it. In any event, a third-order calculation would give further insights into the behavior of the expansion. (iv) The question of the relationship between the PGEP and the $1/N$ expansion beyond leading order deserves further attention. A careful comparison of our results with Root's[24] might be enlightening. (v) An intriguing possibility is that one surprising feature of our results might be a general property of the full theory: namely, the feature that *all corrections of relative order 1/N can be absorbed into the renormalized parameters.* If this were generally true, and true in other theories, it would go a long way towards explaining why $1/N$-expansion ideas in QCD seem to work much better than one could reasonably expect when $N=3$.

Finally, we would also like to refer the reader to various other suggestions for going beyond the Gaussian approximation,[27-30] which also deserve further study. Another interesting approach is the so-called δ expansion,[31] originally formulated in terms of a $\lambda\phi^{2(1+\delta)}$ interaction term. Recent work on that approach[32] is very close in spirit to this work.

ACKNOWLEDGMENTS

We are greatly indebted to José Latorre for collaboration in the early stages of this work: his insight and expertise provided an invaluable impetus to the project. We would also like to thank Anna Okopinska and Rolf Tarrach for useful conversations. This work was supported in part by U.S. Department of Energy Contract No. DE-AS05-76ER05096.

APPENDIX A: PROOF OF THE ϕ_0 INDEPENDENCE OF Γ

In the text we chose to identify the shift parameter ϕ_0 with the classical field φ. This choice had the great advantage that the source j is then of order δ. However, essentially the same results would be obtained (though with greater labor) for *any* ϕ_0. We sketch below a proof of this statement.

Consider Γ calculated, to some given order in δ, with some arbitrary, fixed ϕ_0. Since j is eliminated, Γ is a function only of φ and ϕ_0 (for fixed m_B, λ_B, and Ω). Therefore,

$$d\Gamma=\left.\frac{\partial\Gamma}{\partial\phi_0}\right|_{\varphi}d\phi_0+\left.\frac{\partial\Gamma}{\partial\varphi}\right|_{\phi_0}d\varphi\ . \tag{A1}$$

The first coefficient, the dependence of Γ upon ϕ_0 at fixed φ, is the focus of our interest. The second coefficient, $\partial\Gamma/\partial\varphi|_{\phi_0}$, is just $-j$ by the usual inverse Legendre transform property. The source j is some function of ϕ_0 and φ. Consider the differential of Γ at constant j:

$$\left.\frac{\partial\Gamma}{\partial\phi_0}\right|_j=\left.\frac{\partial\Gamma}{\partial\phi_0}\right|_{\varphi}-j\left.\frac{\partial\varphi}{\partial\phi_0}\right|_j\ . \tag{A2}$$

However, since $\Gamma=\ln Z-j\varphi$, we also have

$$\left.\frac{\partial\Gamma}{\partial\phi_0}\right|_j=\frac{1}{Z}\left.\frac{\partial Z}{\partial\phi_0}\right|_j-j\left.\frac{\partial\varphi}{\partial\phi_0}\right|_j\ . \tag{A3}$$

Comparing these two equations we see that

$$\left.\frac{\partial\Gamma}{\partial\phi_0}\right|_{\varphi}=\frac{1}{Z}\left.\frac{\partial Z}{\partial\phi_0}\right|_j\ . \tag{A4}$$

Now, the generating functional Z corresponds to an expansion in δ of

$$Z=\int D\phi \exp[-\tfrac{1}{2}(\phi-\phi_0)G_{\Delta}^{-1}(\phi-\phi_0)-\delta V(\phi)+j\phi] , \tag{A5}$$

where

$$(G_{\Delta}^{-1})_{xy}=[-\partial^2+(1-\delta)\Omega^2]\delta_{xy} , \tag{A6}$$

in which we have grouped together the terms, from both $\mathcal{L}_0$ and $\mathcal{L}_{\text{int}}$ which involve $\phi-\phi_0$. Differentiating Z with respect to ϕ_0 at constant j simply brings down a factor of $G_{\Delta}^{-1}(\phi-\phi_0)$ from the exponent. The right-hand side of (A4) is then just the functional integral expression for the expectation value of $G_{\Delta}^{-1}(\phi-\phi_0)$. Since $\langle \phi \rangle$ is φ [see Eq. (2.5)], we have, from (A4),

$$\left.\frac{\partial \Gamma}{\partial \phi_0}\right|_{\varphi}=G_{\Delta}^{-1}(\varphi-\phi_0) . \tag{A7}$$

For an x-independent φ this is just

$$\left.\frac{\partial \Gamma}{\partial \phi_0}\right|_{\varphi}=(1-\delta)\Omega^2(\varphi-\phi_0) . \tag{A8}$$

Therefore, ϕ_0 affects only the zeroth- and first-order terms of Γ: higher-order terms are completely independent of ϕ_0. Moreover, all ϕ_0 dependence cancels out when $\delta=1$. Thus, ϕ_0 affects only the partitioning of terms between zeroth and first order.

For example, with $\phi_0=\varphi$ we found the zeroth-order term to be just $-\mathcal{V}I_1(\Omega)$. If we had set $\phi_0=0$ the zeroth-order terms would have been $-\mathcal{V}[I_1(\Omega)+\frac{1}{2}\Omega^2\varphi^2]$, with the extra term canceling with an extra term in first order. This difference has no significance, except possibly if one were to introduce a wave-function renormalization $\varphi=Z\Phi$ in which $Z=(Z_0+Z_1\delta+Z_2\delta^2+\cdots)$.

APPENDIX B: COORDINATE-SPACE EVALUATION OF THE DIVERGENT INTEGRALS

In this appendix we derive regularized expressions for the divergent integrals $I^{(2)}$, $I^{(3)}$, and $I^{(4)}$ in four dimensions, as well as in $d=2,3$, which will be used in future work. $I^{(3)}$ and $I^{(4)}$ have degrees of divergence $2d-6$ and $3d-8$, respectively. Thus, they are finite in two dimensions; logarithmically and linearly divergent in three dimensions; and quadratically and quartically divergent in four dimensions.

In momentum space these integrals are quite nasty, but in coordinate space they are just powers of the x-space propagator function $G(x)\equiv G_{x0}$, integrated over all spacetime:

$$\frac{1}{n!}I^{(n)}(\Omega)\equiv\int_x G^n(x)=\frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty dx\, x^{d-1}G^n(x) , \tag{B1}$$

where, in the last expression, x is $|x^\mu|$, the radial distance from the origin. (For a discussion of x-space renormalization, and references to its use in other contexts, see Collins.[33]

The Euclidean x-space propagator function, $G(x)$, for a free theory of mass Ω is

$$G(x)\equiv\int_p \frac{e^{-ip\cdot x}}{p^2+\Omega^2}=\frac{1}{(2\pi)^d}\frac{2\pi^{d/2-1/2}}{\Gamma(d/2-1/2)} \times\int_0^\infty \frac{p^{d-1}dp}{p^2+\Omega^2}\int_0^\pi d\theta \sin^{d-2}\theta e^{ipx\cos\theta} . \tag{B2}$$

The θ integration can be performed [Gradshteyn-Ryzhik (GR) Eq. (3.915.5) (Ref. 34)], for $d>1$, to yield

$$G(x)=\frac{(2\pi)^{-d/2}}{x^{d/2-1}}\int_0^\infty dp\frac{p^{d/2}J_{d/2-1}(px)}{p^2+\Omega^2} , \tag{B3}$$

where $J_{d/2-1}$ is the Bessel function of the first kind. The p integration can also be done [GR Eq. (6.566.2)], for $d<5$, to produce

$$G(x)=\frac{\Omega^{d-2}}{(2\pi)^{d/2}(\Omega x)^{d/2-1}}K_{d/2-1}(\Omega x) , \tag{B4}$$

where $K_{d/2-1}(\Omega x)$ is the modified Bessel function of order $d/2-1$. The result is actually valid for $1\le d\le 5$, but note that the p integration does not converge in more than five dimensions.

Detailed properties of the $K_\nu(z)$ functions can be found in Ref. 35. For small z,

$$K_\nu(z)\sim 2^{\nu-1}\Gamma(\nu)z^{-\nu} \quad (\nu>0) , \tag{B5}$$

so that, above $d=2$, the propagator $G(x)$ has a $1/x^{d-2}$ singularity at $x=0$. This singular behavior is responsible for the ultraviolet divergences. For instance, the I_0 integral formally corresponds to $G(x=0)$.

We now examine in turn the cases $d=2$, 3, 4, and $4-\epsilon$ (i.e., dimensional regularization).

(i) $d=2$. The propagator reads

$$G(x)=\frac{1}{2\pi}K_0(\Omega x) , \tag{B6}$$

which has only a logarithmic singularity as $x\to 0$. The integrals $I^{(2)}$, $I^{(3)}$, and $I^{(4)}$ are all finite:

$$\begin{aligned}\frac{1}{2!}I^{(2)}(\Omega)&=\frac{1}{2\pi\Omega^2}\int_0^\infty du\, uK_0^2(u)=\frac{1}{4\pi\Omega^2} ,\\ \frac{1}{3!}I^{(3)}(\Omega)&=\frac{1}{4\pi^2\Omega^2}\int_0^\infty du\, uK_0^3(u)=\frac{0.586}{4\pi^2\Omega^2} ,\\ \frac{1}{4!}I^{(4)}(\Omega)&=\frac{1}{8\pi^3\Omega^2}\int_0^\infty du\, uK_0^4(u)=\frac{1.052}{8\pi^3\Omega^2} .\end{aligned} \tag{B7}$$

The first integral can be done analytically: the other two we have evaluated numerically. It is easy to see that the same mass renormalization as used in the first-order GEP will render the second-order effective potential finite.

(ii) $d=3$. The propagator is elementary,

$$G(x)=\frac{e^{-\Omega x}}{4\pi x} , \tag{B8}$$

and $I^{(2)}$ is again finite, equal to $1/(4\pi\Omega)$, but $I^{(3)}$ and $I^{(4)}$

are now divergent. It is instructive to examine these integrals with a small-x cutoff at $x=a$:

$$\frac{1}{3!}I^{(3)}(\Omega)\simeq\frac{1}{16\pi^2}\int_a^\infty dx\frac{e^{-3\Omega x}}{x}$$
$$=\frac{1}{16\pi^2}E_1(3\Omega a)$$
$$=\frac{-1}{16\pi^2}[\ln(\Omega a)+\ln 3+\gamma+O(a)]\ , \tag{B9}$$

where γ is Euler's constant. Similarly, $I^{(4)}$ becomes

$$\frac{1}{4!}I^{(4)}(\Omega)\simeq\frac{\Omega}{16\pi^3}\left[\frac{e^{-4\Omega a}}{4\Omega a}-E_1(4\Omega a)\right]$$
$$=\frac{\Omega}{16\pi^3}\left[\frac{1}{4\Omega a}+\ln(\Omega a)+\ln 4-1+\gamma+O(a)\right]. \tag{B10}$$

In other regularization schemes the finite parts will typically be different. [For example, instead of a sharp cutoff at $x=a$, one could introduce a smooth cutoff function $\exp(-a/x)$ into the integrals. Using GR (Eq. 3.471.9) one finds, in the $a\to 0$ limit, the same results as above, but with γ replaced by 2γ.] However, in any such scheme the following relationship holds:

$$\frac{dI^{(4)}}{d\Omega^2}+8I^{(3)}I^{(2)}=\frac{3}{4\pi^3\Omega}\ln\tfrac{4}{3}\ . \tag{B11}$$

The finiteness of this combination (cf. Sec. III B) is crucial in the renormalization of the post-Gaussian effective potential, allowing the new divergences to be absorbed into Δm_B^2.

(iii) $d=4$. The four-dimensional propagator is

$$G(x)=\frac{\Omega^2}{(2\pi)^2}\frac{K_1(\Omega x)}{\Omega x}\ , \tag{B12}$$

which, at small x, has the expansion

$$G(x)=\frac{1}{4\pi^2}\left[\frac{1}{x^2}+\frac{\Omega^2}{4}\left[\ln\frac{\Omega^2x^2}{\sigma^2}-1\right]+\frac{\Omega^4x^2}{32}\left[\ln\frac{\Omega^2x^2}{\sigma^2}-\frac{5}{2}\right]+O(x^4\ln x)\right]\ , \tag{B13}$$

where $\sigma\equiv 2e^{-\gamma}$. Evaluating the integrals with a small-x cutoff gives, for $I^{(2)}$,

$$I^{(2)}(\Omega)=\frac{1}{4\pi^2}\int_{\Omega a}^\infty du\ uK_1^2(u)$$
$$=\frac{u^2}{8\pi^2}[K_1^2(u)-K_0(u)K_2(u)]\Bigg|_{\Omega a}^{\infty}$$
$$=\frac{-1}{8\pi^2}\left[\ln\frac{a^2\Omega^2}{\sigma^2}+1\right]+O(a^2\ln a)\ . \tag{B14}$$

For $I^{(3)}$ and $I^{(4)}$ one may use the small-x expansion to obtain

$$\frac{1}{3!}I^{(3)}(\Omega)\simeq\frac{1}{64\pi^4a^2}-\tfrac{3}{8}\Omega^2\left[[I^{(2)}(\Omega)]^2+\frac{1}{2\pi^2}I^{(2)}(\Omega)+O(1)\right]\ , \tag{B15}$$

$$\frac{1}{4!}I^{(4)}(\Omega)\simeq\frac{1}{512\pi^6a^4}-\frac{\Omega^2}{32\pi^4a^2}\left[I^{(2)}(\Omega)+\frac{1}{8\pi^2}\right]+\tfrac{1}{4}\Omega^4\left[[I^{(2)}(\Omega)]^3+\frac{11}{16\pi^2}[I^{(2)}(\Omega)]^2+O(I^{(2)})\right]\ . \tag{B16}$$

We have discarded terms which would only produce infinitesimal contributions to $V^{(2)}(\varphi,\Omega)$. Also, we have eliminated all logarithms in favor of $I^{(2)}$, using (B14).[36] Unfortunately, the use of these x-space-cutoff type of regularizations proves to be problematic, mainly because it is not clear what one should do about I_0 and I_1. These problems are discussed in Sec. III B.

(iv) $d=4-\epsilon$. In DR the quartic and quadratic divergences, corresponding to $1/a^4$ and $1/a^2$ terms above, are simply absent. The $(\ln a)^n$ terms, however, show up as $1/\epsilon^n$ poles. $I^{(2)}$ and $I^{(3)}$ can be computed by the usual momentum-space methods,[37] but the calculation of $I^{(3)}$ is already quite lengthy, and to calculate $I^{(4)}$ in the same manner would be very hard. However, one may profitably combine the x-space approach with DR.

The small-x expansion of the propagator in $d=4-\epsilon$ dimensions[33] can be rewritten in the form

$$G(x)=\frac{A(x)}{x^2}+\sum_{k=0}^{\infty}x^{2k}B_k(x)\ , \tag{B17}$$

where

$$A(x)\equiv\frac{\Gamma(1-\epsilon/2)}{4\pi^{2-\epsilon/2}}x^\epsilon\ , \tag{B18}$$

$$B_k(x)=\frac{1}{4\pi^{2-\epsilon/2}}\left[-\frac{\Omega^2}{4}\right]^{k+1}\frac{1}{k!}\times\left[\frac{x^\epsilon}{k+1}\Gamma(-k-\epsilon/2)-\left[\frac{2}{\Omega}\right]^\epsilon\Gamma(-1-k+\epsilon/2)\right]. \tag{B19}$$

Note that as $\epsilon\to 0$ the Γ-function poles in B_k cancel, leaving behind the logarithms of Eq. (B13) above. Therefore, in integrals over $G(x)$, the $1/\epsilon^n$ poles can only arise from the terms in the integrand which are singular as $x\to 0$.

Consider the $I^{(3)}$ integral

$$\frac{1}{3!}I^{(3)}(\Omega)=\frac{2\pi^{2-\epsilon/2}}{\Gamma(2-\epsilon/2)}\int_0^\infty dx\ x^{3-\epsilon}G^3(x)\ . \tag{B20}$$

For our purposes we need the pole terms, but not the finite part.[38] We may therefore replace the upper limit by some arbitrary finite value, taken to be 1 in some arbitrary units. Furthermore, we may substitute

$$G^3(x)=\frac{A^3}{x^6}+\frac{3A^2B_0}{x^4}+\cdots\ , \tag{B21}$$

since the $+\cdots$ terms would contribute only to the finite part of $I^{(3)}$. The first term produces an integral:

$$\int_0^1 dx\, x^{2\epsilon-3}=\left.\frac{x^{-2+2\epsilon}}{-2+2\epsilon}\right|_0^1\ , \tag{B22}$$

which would ordinarily produce a quadratic divergence, but which is formally finite in DR. To see this, one may rearrange it as

$$\begin{aligned}\int_0^1 dx\, x^{2\epsilon-3}(1-x^2)+\int_0^1 dx\, x^{2\epsilon-1} &=\tfrac{1}{2}B(\epsilon-1,2)+\frac{1}{2\epsilon}\\ &=\frac{1}{2\epsilon}\left[\frac{1}{\epsilon-1}+1\right]=-\tfrac{1}{2}+O(\epsilon)\ .\end{aligned} \tag{B23}$$

[Note that the same result is obtained from (B22) if one pretends that the evaluation at $x=0$ gives zero, not infinity.] Thus, the pole terms arise solely from the second term of (B21). The evaluation of this is quite straightforward, and yields

$$\frac{1}{3!}I^{(3)}=-\frac{3\Omega^2}{128\pi^4}\left[\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\left[\ln\frac{\Omega^2}{\mu^2}+\gamma-\ln4\pi-\tfrac{3}{2}\right]\right] +\text{finite} \tag{B24}$$

(where μ is the arbitrary "unit of mass" used when expanding Ω^ϵ). The result agrees with that of the momentum-space calculation.[37]

The calculation of $I^{(4)}$ proceeds in the same manner. The poles arise solely from the term

$$(4A^3B_1+6A^2B_0^2)\frac{1}{x^4}\ , \tag{B25}$$

in the expansion of $G^4(x)$, and can be calculated without too much labor.

$I^{(2)}$ can also be obtained simply in the x-space approach, since it is equivalent to $I_{-1}\equiv-2\,dI_0/d\Omega^2$, which is $-2\,dG(x=0)/d\Omega^2$. Taking the Ω^2 derivative of (B17) and then taking $x\to0$, one finds that only the B_0 term contributes (and only the second part of B_0, at that). Therefore, one obtains

$$I_{-1}(\Omega)=\frac{\Gamma(\epsilon/2)}{8\pi^2}\left[\frac{4\pi}{\Omega^2}\right]^{\epsilon/2}\ , \tag{B26}$$

which leads to the result quoted in the text, Eq. (2.46). $I^{(3)}$ and $I^{(4)}$ can then be expressed in terms of $I^{(2)}\equiv I_{-1}$ giving us the results quoted in Eqs. (2.47) and (2.48).

---

[1]L. I. Schiff, Phys. Rev. **130**, 458 (1963); G. Rosen, *ibid.* **173**, 1632 (1968); in *Path Integrals and Their Applications in Quantum, Statistical, and Solid-State Physics*, edited by G. J. Papadopoulos and J. T. Devreese (Plenum, New York, 1978); J. Kuti (unpublished), as discussed in J. M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D **10**, 2428 (1974); S. J. Chang, *ibid.* **12**, 1071 (1975).

[2]T. Barnes and G. I. Ghandour, Phys. Rev. D **22**, 924 (1980).

[3]*Variational Calculations in Quantum Field Theory*, proceedings of the Workshop, Wangerooge, West Germany, 1987, edited by L. Polley and D. E. L. Pottinger (World Scientific, Singapore, 1988).

[4]P. M. Stevenson, Phys. Rev. D **30**, 1712 (1984).

[5]P. M. Stevenson, Phys. Rev. D **32**, 1389 (1985); Z. Phys. C **24**, 87 (1984).

[6]P. M. Stevenson, B. Alles, and R. Tarrach, Phys. Rev. D **35**, 2407 (1987).

[7]A. Okopinska, Phys. Rev. D **35**, 1835 (1987).

[8]P. Cea, Phys. Lett. B **236**, 191 (1990).

[9]W. E. Caswell, Ann. Phys. (N.Y.) **123**, 153 (1979); J. Killingbeck, J. Phys. A **14**, 1005 (1981).

[10]P. M. Stevenson, Phys. Rev. D **23**, 2916 (1981).

[11]S. K. Kaufmann and S. M. Perez, J. Phys. A **17**, 2027 (1984).

[12]P. M. Stevenson, Nucl. Phys. **B231**, 65 (1984).

[13]In some cases, when no stationary point exists, one must seek instead a "plateau" region where the slope is minimized, i.e., where the second derivative vanishes. See Refs. 10–12 for further discussion.

[14]J. W. Darewych, M. Horbatsch, and R. Koniuk, Phys. Rev. Lett. **54**, 2188 (1985); Phys. Rev. D **33**, 2317 (1986); P. M. Stevenson and I. Roditi, *ibid.* **33**, 2305 (1986).

[15]This expansion is equivalent to the "optimized expansion" of Ref. 7. Okopinska gives explicit results in 0 and 1 dimensions, but does not tackle the higher-dimensional cases which we study here.

[16]R. Jackiw, Phys. Rev. D **9**, 1686 (1974).

[17]For good pedagogical treatments, see, e.g., C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980); P. Ramond, *Field Theory, A Modern Primer* (Benjamin/Cummings, Reading, MA, 1981).

[18]P. M. Stevenson and R. Tarrach, Phys. Lett. B **176**, 436 (1986); M. Consoli and A. Ciancitto, Nucl. Phys. **B254**, 653 (1985).

[19]J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962); G. Jona-Lasinio, Nuovo Cimento **34**, 1790 (1964).

[20]B. S. DeWitt, Phys. Rev. **162**, 1195 (1967); **162**, 1239 (1967); L. F. Abbott, Nucl. Phys. **B185**, 189 (1981).

[21]C. G. Bollini and J. J. Giambiagi, Nuovo Cimento **A93**, 113 (1986).

[22]I. Stancu, Ph.D. thesis, Rice University report, 1990.

[23]Note that there is no chance of canceling the quadratic divergences with a single overall substitution for m_B^2, so that the "order-by-order" procedure with $m_B^2=m_{BG}^2+\delta\Delta m_B^2+O(\delta^2)$ is necessary, just as in ordinary perturbation theory.

[24]R. G. Root, Phys. Rev. D **10**, 3322 (1974).

[25]Note that our viewpoint is quite different from that of J. Wudka, Phys. Rev. D **36**, 1745 (1987); **37**, 1464 (1988); **39**, 3000 (1989).

[26]G. 't Hooft, Nucl. Phys. **B61**, 455 (1973).

[27]I. Yotsuyanagi, Z. Phys. C **35**, 453 (1987).

[28]U. Ritschel and L. Polley, in *Variational Calculations in Quantum Field Theory* (Ref. 3); Phys. Lett. B **221**, 44 (1989); U. Ritschel, Z. Phys. C (to be published).

[29]M. Altenbokum and U. Kaulfuss, Phys. Rev. D **33**, 3658 (1986); **35**, 609 (1987); Illinois Report No. ILL-TH-86-13 (unpublished).

[30]A. Kovner and B. Rosenstein, Phys. Rev. D **39**, 2332 (1989); **40**, 504 (1989).

[31]C. M. Bender, K. A. Milton, M. Moshe, S. S. Pinsky, and L. M. Simmons, Jr., Phys. Rev. Lett. **58**, 2615 (1987); Phys. Rev. D **37**, 1472 (1988); N. Brown, Brookhaven Report No. BNL 40971, 1988 (unpublished).

[32]A. Duncan and M. Moshe, Phys. Lett. B **215**, 352 (1988); H. F. Jones and M. Monoyios, Int. J. Mod. Phys. A **4**, 1735 (1989); H. F. Jones and M. Moshe, Phys. Lett. B **234**, 492 (1990).

[33]J. C. Collins, *Renormalization* (Cambridge University Press, Cambridge, England, 1984).

[34]Here and below, these code numbers refer to formulas in I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Corrected and Enlarged Edition) (Academic, San Diego, 1980).

[35]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).

[36]One curiosity is the following: If one employs a different cutoff scheme [e.g., a smooth cutoff function $\exp(-a/x)$], the general structure of these formulas remains the same, but almost all the numerical coefficients change. In particular, the finite part of $I^{(2)}$ is different. In fact, *only* the coefficient of the "leading logarithm," $(\ln a)^{n-1}$, in each $I^{(n)}$ remains the same, in general. However, we have observed that if the logarithms are reexpressed in terms of $I^{(2)}$ (i.e., the $I^{(2)}$ of the same regularization scheme) then the coefficient of the "subleading logarithm," $(I^{(2)})^{n-2}$, seems to be the same in all x-space cutoff schemes. This apparent invariance does not carry over to DR, though, as can be seen from the DR results (2.47) and (2.48). However, for the PGEP calculation, the value of the "subleading coefficient" does not actually matter, in the end.

[37]See, e.g., Ramond (Ref. 17), Sec. IV.4.

[38]The finite part could be obtained, if needed, by carefully isolating the divergent terms and then performing the remaining finite integration numerically.