Scattering in unbroken and broken phases of (2+1)-dimensional gravity

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Two-particle gravitational scattering is compared in the Chern-Simons and standard perturbative approaches. In the limit where one particle's energy dominates, the cross sections agree to order M^3; in general they disagree. This is attributed to the nonexistence of quantum-mechanical fluctuations in the topological theory.

I. INTRODUCTION

New interest in (2+1)-dimensional gravity has been prompted by Witten's solution as an ISO(2,1) Chern-Simons theory,[1,2] which has been interpreted as describing an unbroken phase of general coordinate invariance. This point of view differs from more traditional approaches to quantum gravity and is intimately connected with the finiteness and renormalizability of the Chern-Simons version. It is therefore interesting to contrast the physics of this phase with that of the broken phase. The theory we choose to represent the broken phase is the standard gravitational perturbation theory as developed in the 1960s by analogy with QED.[3] This is manifestly in the broken phase because we expand about the Minkowski metric.

One way of probing to the theory, with the advantage of being very physical, is to consider scattering of particles due to their gravitational interaction. Specifically, we shall be calculating the tree-level cross section for two scalar particles. This becomes particularly simple in the kinematical limit where one particle is much heavier than the other; we shall henceforth call this the massive limit. In this case we merely solve for the motion of the light particle in the frozen classical geometry due to the heavy one. The problem reduces to solving the wave equation on a cone with opening angle $\alpha_M = M(\kappa/2)^2$.[4,5] 't Hooft[6] has argued that this procedure is also correct for the relative coordinate of two general mass particles in the center-of-mass frame. The addition formula for deficit angles[7] must be used to determine α in this case. More recently, Carlip[8] has shown that a treatment based on Wilson-line information in the Chern-Simons theory gives the same intuitive picture.

We shall be comparing these exact results, expanded in powers of α, with those derived from standard perturbation theory. We shall always work in a regime in which all particle energies are much lighter than the natural gravitational scale κ^{-2}. The lowest-order tree-level diagram gives, in the center-of-mass frame,

$$\frac{d\sigma}{d\theta} = \frac{\alpha_{E_1+E_2}^2}{2\pi p}\left[\frac{\mathbf{p}^2}{\mathbf{q}^2} - \frac{1}{2}\,\frac{(E_1E_2+\mathbf{p}^2)}{(E_1+E_2)^2}\right]^2 . \tag{1}$$

Note that this perturbative expression also simplifies in the massive limit $E_2 = M \gg E_1$ in that the s-wave term vanishes.

After the discussion of broken and unbroken phases motivating this study, the reader may be perplexed as to what scattering means in an unbroken theory. In fact, Carlip's work is based on there being an asymptotic conical spacetime, and so there is no problem in interpreting scattering. However, this means that the unbroken theory looks distinctly broken. It is the purpose of this paper to show that it still retains unbroken characteristics that are revealed through scattering. Broken and unbroken should be understood in this context.

We devote Sec. II to working within the massive limit described above. By careful analysis of asymptotic states, we shall show agreement between the perturbative result and the exact result expanded to order α_M^3.

Section III considers the general kinematical situation. We discuss Carlip's approach in some detail to show that it does not reproduce the s-wave term in (1). The physical origin of this term is then proposed and reasons for the discrepancy discussed.

II. MASSIVE LIMIT

We commence with the perturbative approach and first make some technical points concerning the formalism. The variable $\tilde{g}^{\mu\nu} = \sqrt{g}\, g^{\mu\nu}$, which makes the three-graviton vertex simpler, is used, and as is almost universal, we work in the harmonic gauge. In the limit we work in, the source can be taken to be classical of mass M. The metric signature is $(+,-,-)$. The Feynman rules are then standard as given below:

$$D_{\mu\nu\alpha\beta} = \frac{i}{2(p^2+i\epsilon)}(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta}) ,$$

$$G = \frac{i}{p^2-m^2+i\epsilon} ,$$

$$J_{\mu\nu} = -i\frac{\kappa M}{2}(\eta_{\mu 0}\eta_{\nu 0}-\eta_{\mu\nu})2\pi\delta(q_0) ,$$

$$\Gamma^1_{\mu\nu} = i\kappa(p_\mu p'_\nu + m^2\eta_{\mu\nu}) ,$$

$$\Gamma^2_{\mu\nu\alpha\beta} = \frac{-i}{2}\kappa^2 m^2[\eta_{\mu\nu}\eta_{\alpha\beta} - \tfrac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha})] .$$

The three-graviton vertex is too long to include here; it is given correctly by Capper and Namazie.[9]

The relevant diagrams for scattering in first and second orders are shown in Fig. 1. The problem is the analog of Coulomb scattering with a semiclassical expansion in the source charge. The first-order diagram [Fig. 1(a)] may be simply computed and is the same as the massive limit of (1):

$$\mathcal{M}^{(a)}=2iM\left[\frac{\kappa}{2}\right]^2\left[\frac{\mathbf{p}^2}{\mathbf{q}^2}\right]2\pi\delta(q_0)\ . \tag{2}$$

In second order the diagram shown in Fig. 1(d) vanishes; the integrals for the other graphs may be evaluated straightforwardly. (I thank K. Suehiro for correcting a factor of 2 in Ref. 4.) An infrared regulator in the form of a small graviton mass μ is introduced. The diagram shown in Fig. 1(b) yields

$$\mathcal{M}^{(b)}=i\frac{\alpha^2}{\pi}\frac{\mathbf{p}^2}{\mathbf{q}^2}\left[i\pi\frac{p}{\mu}+1-\ln\left[\frac{\mathbf{q}^2}{\mu^2}\right]\right]2\pi\delta(q_0)\ . \tag{3}$$

The diagram shown in Fig. 1(c), which has a symmetry factor of $\frac{1}{2}$, consists only of a logarithmic infrared divergence. In the full second-order amplitude these logarithmic terms cancel leaving

$$\mathcal{M}^{(b+c+d)}=i\frac{\alpha^2}{\pi}\frac{\mathbf{p}^2}{\mathbf{q}^2}\left[i\pi\frac{p}{\mu}+1\right]2\pi\delta(q_0)\ . \tag{4}$$

The remaining infrared divergence corresponds to an infinite phase shift and will not appear in the cross section. Neither are there any difficulties with bremsstrahlung because there are no free gravitons.

Using the matrix elements determined above, the cross section can be evaluated to order α^3:

$$\frac{d\sigma}{d\theta}=\frac{\alpha^2}{2\pi p}\left[\frac{\mathbf{p}^2}{\mathbf{q}^2}\right]^2\left[1+\frac{\alpha}{\pi}\right]\ . \tag{5}$$

To understand the meaning of this coordinate-dependent quantity, we can work out the expectation value of the metric. The first term leads to a spatial metric $d\tau^2=[1+(\alpha/\pi)K_0(\mu s)](ds^2+s^2d\theta^2)$. Beyond the cutoff $s\gg 1/\mu$, space is unperturbed and flat. Within the cutoff $s\ll 1/\mu$, we recognize the expansion of a cone in conformal coordinates (s,θ), $d\tau^2=(\mu s)^{-\alpha/\pi}(ds^2+s^2d\theta^2)$. Incidentally, this allows a direct comparison to be made with an alternative method which starts with the Klein-Gordon equation (in conformal coordinates) and perturbs in the opening angle of the cone.[4] For example, the Born term comes from the Fourier transform of the leading $\alpha\ln\mu s$ correction to the flat metric.

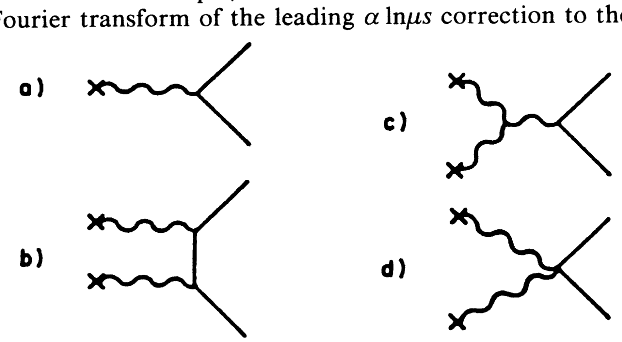

FIG. 1. Scattering in the massive limit.

The exact solution of the wave equation is most easily determined in the flat, wedge form of the metric. The coordinates are (r,ϕ), $-\pi/\beta<\phi<\pi/\beta$, where it is convenient to define $\beta=2\pi/(2\pi-\alpha)$. The general solution may be written as a sum of Bessel functions. The coefficients are fixed by matching onto the incoming part of the following asymptotic form:

$$\psi(r,\phi)\to e^{-ikr\cos\phi}+\frac{1}{\sqrt{2\pi kr}}e^{ikr}e^{i\pi/4}f(\phi)$$
$$\to\frac{e^{i\pi/4}}{\sqrt{2\pi kr}}[e^{-ikr}\delta(\phi)+e^{ikr}f(\phi)]\ . \tag{6}$$

This procedure leads to the series form of the complete wave function:

$$\psi(r,\phi)=\beta\sum_{-\infty}^{\infty}e^{-i|n|\beta\pi/2}J_{|\beta|n}(kr)e^{in\beta\phi}\ . \tag{7}$$

Using the techniques of Ref. 10, this is readily shown to be the same as the contour form in Ref. 6. The scattering amplitude $f(\phi)$ of Eq. (6) is simply determined by formally summing the outgoing asymptotic waves:

$$f(\phi)=\frac{\beta}{2}\left[\cot\frac{\beta}{2}(\phi-\pi)-\cot\frac{\beta}{2}(\phi+\pi)\right]$$
$$=\frac{-\beta\sin\beta\pi}{(\cos\beta\phi-\cos\beta\pi)}\ . \tag{8}$$

The form of incident wave $e^{-ikr\cos\phi}$ in (6) is physically correct for the problem of scattering on a cone. It is what an experimentalist would set up in a local region of flat space on the cone but away from the apex. However, we can imagine a different form of incoming wave that would be appropriate if asymptotically the cone were to become globally flat—imagine a witch's hat placed on a table. This is the situation that occurs if the infrared cutoff in the conical metric is finite and, as we shall argue, is relevant to the perturbative calculation. A plane wave in background space becomes refracted when it starts to move onto the cone. It develops a curved wave front, and a simple geometric optics argument shows that the beam will be defocused.

To analyze this effect we consider a sharp boundary between the cone and background. The boundary is asymptotically far from the apex, and we expect the approximation of a sharp cutoff to be negligible. The calculation leading to the wave function (7) is repeated, but this time matching onto an incident plane wave in the external region. Within the cutoff we work in conformal coordinates (s,θ), $d\tau^2=(\mu s)^{-\alpha/\pi}(ds^2+s^2d\theta^2)$, $-\pi<\theta<\pi$. The general series solution, with unknown coefficients a_n, has the asymptotic form

$$\psi(s,\theta)\to\sum_0^{\infty}a_n\left[\frac{2}{\pi(k\beta/\mu)(\mu s)^{1/\beta}}\right]^{1/2}$$
$$\times\cos\left[\frac{k\beta}{\mu}(\mu s)^{1/\beta}-\beta n\frac{\pi}{2}-\frac{\pi}{4}\right]\cos(n\theta)\ . \tag{9}$$

In the external region, using ordinary radial coordinates $(\tilde{s},\theta)$, the wave function consists of an incident plane wave and scattered pieces:

$$\psi(\tilde{s},\theta)=e^{ik\tilde{s}\cos\theta}+\frac{1}{\sqrt{2\pi k\tilde{s}}}e^{ik\tilde{s}}e^{i\pi/4}g(\theta)$$
$$\rightarrow\frac{e^{i\pi/4}}{\sqrt{2\pi k\tilde{s}}}[e^{-ik\tilde{s}}\delta(\theta)+e^{ik\tilde{s}}g(\theta)]\ . \tag{10}$$

The waves are matched at the boundary in the asymptotic region $\tilde{s}=s=1/\mu\rightarrow\infty$ to leading order in μ. Both the wave function and radial derivative are to be continuous. We thereby derive the following form for the scattering amplitude in the external region:

$$g(\theta)=\frac{1}{\beta}f(\theta/\beta)e^{2i(\beta-1)k/\mu}\ . \tag{11}$$

The defocusing effect gives the same angular dependence to the scattering amplitude, but reduces its strength.

The above formalism is appropriate for making a comparison with perturbation theory. In perturbation theory an incident plane wave is set up in the flat space outside the range of the perturbation, in this case limited by the cutoff μ. The cross section is determined using $g(\theta)$ and may be expanded in α for comparison with the perturbative result (5):

$$\frac{d\sigma}{d\theta}=\frac{1}{2\pi\kappa}|g(\theta)|^2$$
$$=\frac{\alpha^2}{2\pi k}\left[\frac{1}{2\sin\theta/2}\right]^4\left[1+\frac{\alpha}{\pi}\right]+O(\alpha^4)\ . \tag{12}$$

Despite apparent difficulties due to the infinite range of the perturbation, a careful treatment has led to agreement. The calculations we have presented for this massive limit are based on earlier unpublished work in Ref. 4.

III. GENERAL KINEMATICAL SITUATION

The general kinematical situation in which neither particle's energy dominates can be approached by analyzing the ISO(2,1) Chern-Simons theory in the presence of Wilson lines, each containing an infinite-dimensional particlelike representation of SO(2,1).[8] Carlip is able to give a concrete description of the quantum-mechanical Hilbert space and Hamiltonian that constitute the theory.

One is obliged to work in the center-of-mass frame. In the present context this means the frame in which space is asymptotically conical; it is only then that time translations are asymptotic symmetries and a Hamiltonian can exist. This frame can be analyzed classically using the techniques of Ref. 7, where flat pieces of spacetime are patched together by ISO(2,1) transformations. The condition that the matching of two particles yields an asymptotic cone is that

$$\Omega_{\text{total}}=L_1\Omega_1L_1^{-1}L_2\Omega_2L_2^{-1}\ , \tag{13}$$

where Ω_i are rotations by α_i, the respective masses in gravitational units; and L^{-1} are boosts that bring the particles to their respective rest frames. This geometry (shown in Fig. 2 for zero impact parameter) is a simple extension of the static case, but the matching is no longer under simple rotations.

This equation may be rewritten in terms of exponentiated SO(2,1) generators [we use the representation $(J^a)^{\mu}_{\nu}=-\epsilon^{a\mu b}\eta_{b\nu}$]. The combination $L\Omega L^{-\nu}$ is equal to $e^{p_aJ^a}$, where if the rotation is $\Omega=e^{mJ^0}$, then p_a is a boost of the timelike vector $(m,0,0)$ by L. So according to the picture of Ref. 7, where L^{-1} is the boost required to bring a particle to its rest frame, p_a is the momentum in the center of mass:

$$e^{HJ^0}=e^{p_a^{(1)}J^a}e^{p_a^{(2)}J^a}\ , \tag{14}$$

which, with the above interpretation, we recognize as Carlip's definition of the center of mass. $H=\alpha_{\text{total}}$ is the total opening angle of the asymptotic cone (we have absorbed the gravitational scale in all these formulas). In this form the equation leads simply to the following relations between masses, momenta, and Hamiltonian:

$$\frac{1}{m_1}\sin\frac{m_1}{2}p_1=\frac{1}{m_2}\sin\frac{m_2}{2}p_2=\bar{p}\ ,$$
$$\cos\frac{H}{2}=\frac{1}{1+\bar{p}^2}\left[\cos\frac{m_1}{2}\cos\frac{m_2}{2}-\sin\frac{m_1}{2}\sin\frac{m_2}{2}\frac{E_1E_2}{m_1m_2}\right]\ . \tag{15}$$

Once we have fixed the frame, the Hilbert space reduces to wave functions of a momentum variable $\mathbf{p}$ (for example, the symmetric one in Ref. 8), which obey the braid constraint $e^{(2\pi-H)J^0}\psi(\mathbf{p})=\psi(\mathbf{p})$. It is convenient to Fourier transform the wave function and work in terms of $\mathbf{r}$, which is some kind of relative vector. The braid constraint becomes

$$\psi(r,\phi+2\pi-\alpha)=\psi(r,\phi)\ . \tag{16}$$

The resulting picture is very similar to that in the massive limit. The wave function lives on a cone with opening angle $\alpha=H$ (becoming α_M in the massive limit). To calculate amplitudes we may simply take over the expressions developed in the previous section:

$$\frac{d\sigma}{d\theta}=\frac{1}{2\pi p}|f(\theta)|^2\ , \tag{17}$$

where we must use α given by (15) in the definition (8) of $f(\phi)$.

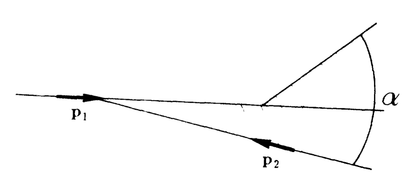

FIG. 2. Classical two-particle geometry in the center-of-mass frame.

Expanding this expression in α, as in (12), we find the leading term.

$$\frac{d\sigma}{d\theta}=\frac{\alpha^2}{2\pi p}\left[\frac{\mathbf{p}^2}{\mathbf{q}^2}\right]^2 . \tag{18}$$

This is in clear disagreement with the direct perturbative evaluation (1). The s-wave term is completely missing.

Before going on to discuss the origin of this discrepancy in detail, it is useful to have a physical interpretation for the origin of the s-wave component in the perturbative cross section. Such intuition is easy to come by. Let us start with the massive limit and consider moving away from it in some expansion in E_1/M. Familiarity with quantum mechanics leads us to expect that the massive particle would fluctuate on the scale of its Compton wavelength. This would cause the apex of the cone to become fuzzy and would affect the low-angular-momentum waves that penetrate close to it.

We can support this intuition by performing a quantum-mechanical calculation on a smoothed-off cone. Explicitly, consider a spherical cap cutoff (Fig. 3), though we do not expect precise details to matter. The dimensionless cutoff δ is the smoothing distance b, measured in incident wavelengths. We shall set it to be of order $\delta \sim \lambda_M/\lambda_{\rm inc} \sim \tilde{p}/M$.

In the exterior region we must also now include Bessel functions of the second kind, $Y_{\beta n}$, in the expansion of the wave function. These components, notably Y_0 and Y_β, will be responsible for the change in scattering amplitude.

Rather than perform the straightforward calculation, which would require peculiar limits of the interior Legendre functions in the matching condition, it is helpful to remember that we will only work to leading order in α. The sphere can be taken to have a large radius which allows the Legendre functions to be expanded about Bessel functions, and we set up the problem perturbatively in δ from the start.

The coefficients of the $Y_{\beta n}$ in the exterior scattering wave function are suppressed by powers of δ. At leading order δ^2 only $n=0,1$ contribute to the cross section:

$$\frac{d\sigma}{d\tilde{\theta}}=\frac{\alpha_M^2}{2\pi\tilde{p}}\left[\frac{1}{4\sin^2\tilde{\theta}/2}+\frac{\delta^2}{4}(1+2\cos\tilde{\theta})\right]^2 , \tag{19}$$

where we have put tildes on the variables so as not to forget that this calculation is done in the rest frame of the massive particle.

To make a comparison with the first-order perturbative result (1), we must reevaluate it in this new frame. Although in the center-of-mass frame leading corrections to the $E_1/M=0$ case are order $1/M$, in the lab frame the first terms are order $1/M^2$;

$$\frac{d\sigma}{d\tilde{\theta}}=\frac{\alpha_M^2}{2\pi\tilde{p}}\left[\frac{1}{4\sin^2\tilde{\theta}/2}-\frac{\tilde{p}^2}{4M^2}-\frac{m_1^2}{4M^2}\cos\tilde{\theta}\right]^2 . \tag{20}$$

Bearing in mind that the cutoff δ is order $\tilde{p}/M$, we find qualitative agreement that the corrections are both of order $1/M^2$. We should not be too concerned about the slight differences in angular dependence because the quantum-mechanical calculation is not treating recoil correctly. We conclude that this result supports the physical picture of quantum-mechanical fluctuations causing the s-wave terms.

The origin of the discrepancy becomes clear if we return to the basis of Carlip's approach and look at how its topological character affects the scattering calculation. The very description of the Hilbert space is in terms of topological information: the holonomies about the punctures in the spatial slice due to the Wilson lines. Usually, to obtain particle amplitudes, we must sum over all world lines. In this theory world lines in the same topological class give identical contributions because the $\int \mathcal{D}e\,\mathcal{D}\omega$ has rendered the theory general coordinate invariant. Here the sum is only over world lines with a different topology, which gives rise to the braid constraint (16). In ordinary quantum mechanics it is precisely the sum over world lines in the same topological class, differing slightly in action as they take slightly different trajectories, that gives rise to the characteristic spatial fluctuations. This is what happens in the perturbative treatment, which corresponds to the broken phase.

So we see that in the unbroken phase the fluctuations are necessarily dismissed, and hence we do not see the s-wave terms that would characterize such behavior.

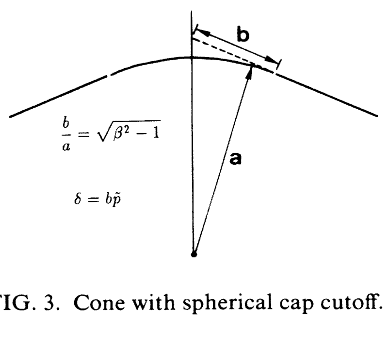

FIG. 3. Cone with spherical cap cutoff.

IV. CONCLUSION

In the massive limit, after taking account of subtleties concerning the form of asymptotic states, we have demonstrated that the direct solution of the wave equation on a cone and gravitational perturbation theory give the same cross section to order α_M^3. On the other hand, when neither particle's energy dominates, even at the lowest order in α, the cross section distinguishes broken and unbroken phases. The difference, an s-wave term which only arises in the broken phase, is physically due to quantum-mechanical fluctuations which cause uncertainty in the position of the conical apex. Such fluctuations are suppressed in the massive limit. The unbroken theory is topological in nature, and fluctuations such as this form no part of it.

The difference can also be seen in terms of the scale of the conical spacetime. In the topological theory it is fixed by the relative momentum scale, but in the broken theory there is an additional scale from the fluctuations. In the Arnowitt-Deser-Misner (ADM) canonical theory,[11] which is presumably broken, a dynamical scale appears.

Throughout this paper, broken and unbroken phases should be understood only in terms of the short "distance" behavior which is relevant to the differences in scattering that we have been discussing. As mentioned in the Introduction, at long distances, both theories are broken; they have asymptotic spacetimes and scattering makes good sense. This phenomenon of breaking at long distances has been identified as being the underlying reason for the difficulty of coupling second-quantized matter fields to Chern-Simons theory.[2] Even in the present case, the situation is not completely clear. Carlip, by working with noncompact space, implicitly assumes asymptotic spacetime. This is reasonable in the light of Witten's argument;[2] that the infrared divergences in the partition function for manifolds allowing a classical solution lead to spacetime becoming macroscopic.

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