Spectrum of wormholes

S. W. Hawking

*Institute for Theoretical Physics, University of California, Santa Barbara, California 93106 and Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, United Kingdom**

Don N. Page

*Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, Department of Physics, Pennsylvania State University, University Park, Pennsylvania 16802, and Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Alberta, Canada T6G 2J1**

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Wormholes have been studied mainly in the semiclassical approximation as solutions of the classical Euclidean field equations. However, such solutions are rather special, and exist only for certain kinds of matter. On the other hand, one can represent wormholes in a more general manner as solutions of the Wheeler-DeWitt equation with appropriate boundary conditions. Minisuperspace models with massless minimal or conformal scalar fields have a discrete spectrum of these solutions. The Giddings-Strominger instanton solution corresponds to a sum of an infinite number of these solutions. Minisuperspace models with a massive scalar field also appear to have a discrete spectrum of such solutions, whose asymptotic form is given.

I. INTRODUCTION

Wormholes are Euclidean metrics that consist of two large regions joined by a narrow tube or throat. Macroscopic wormholes may provide the mechanism for black holes to evaporate and disappear completely,[1] while microscopic wormholes seem to have an important effect on physical constants, particularly the cosmological constant.[2–9] Wormholes have been studied mainly as instantons, solutions of the classical Euclidean field equations. These are saddle points in the path integral. They can form the basis in a semiclassical treatment in which one makes the dilute wormhole approximation of neglecting the interaction between the ends of different wormholes joining on the same large region.

However, real wormholelike solutions occur only for certain special kinds of matter that allow the Ricci tensor to have negative eigenvalues. These do not include a minimally coupled scalar (unless it is pure imaginary), but include an antisymmetric tensor field whose field equations in four dimensions are equivalent to those of a scalar field.[10] There are no known electromagnetic wormhole solutions in four dimensions, but there are Yang-Mills solutions.[11] These, however, in general do not seem to be local minima of the action.[12] It is not clear, therefore, that they contribute to the semiclassical approximation. There are Yang-Mills solutions which are local minima of the action, but they exist only when the Yang-Mills field is not coupled to any fields in the fundamental representation.[12] Moreover, these solutions have a maximum throat size of a few Planck units, which makes it difficult to see how they could carry away all the particles and information that are lost when a macroscopic black hole evaporates.

Is one, therefore, to assume that wormholes are important only in the very restricted class of theories in which the matter content allows wormhole instantons? This would make it difficult to believe that wormholes are the mechanism for black-hole evaporation, because this will occur for any matter content, or even no matter content, but just pure gravity. It would also cast doubt on whether wormholes are the reason why the cosmological constant is zero. We will, therefore, advocate a different approach,[1] in which wormholes are regarded, not as solutions of the classical Euclidean field equations, but as solutions of the quantum-mechanical Wheeler-DeWitt equation. These wave functions have to obey certain boundary conditions in order that they represent wormholes. The boundary conditions seem to be that the wave function is exponentially damped for large three-geometries, and is regular in some suitable way when the three-geometry collapses to zero. We shall argue that there is a discrete spectrum of solutions of the Wheeler-DeWitt equation that obey these boundary conditions. We shall illustrate this with a discussion of minisuperspace solutions of the Wheeler-DeWitt equation with a scalar field. There is a continuous family of solutions with a massless scalar field that are eigenfunctions of the scalar charge operator. They correspond to the instanton solutions found by Giddings and Strominger.[10] The wave functions are damped at infinity, but they oscillate infinitely near zero radius. However, these solutions can be expressed as an infinite sum of a discrete family of solutions that are well behaved both at infinity and at zero radius. Such solutions can also be constructed, although only approximately, for a massive scalar field. We give explicit formulas for their asymptotic form here.

II. THE BOUNDARY CONDITIONS

In the dilute-wormhole approximation, one can treat each wormhole separately, as joining two asymptotically

Euclidean regions. We shall, therefore, consider Euclidean metrics of topology $R^1 \times S^3$ which are asymptotically Euclidean at each end of the R^1. The idea is to study the effect of the wormhole on physics in the two asymptotic regions at energies low compared to the Planck scale. For this purpose, one wants to calculate the Green's functions

$$\langle \phi(x_1)\phi(x_2)\cdots\phi(y_1)\phi(y_2)\cdots\rangle , \tag{1}$$

where $x_1 x_2, \ldots$ and $y_1, y_2, \ldots$ are points in the two asymptotic regions far from the wormhole. The points $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ can effectively be taken to be at infinity in flat space. One can then factorize the Green functions by introducing a complete set of wormhole states:

$$\begin{aligned}\langle \phi(x_1)\phi(x_2)\cdots\phi(y_1)\phi(y_2)\cdots\rangle &= \sum_k \langle 0|\phi(x_1)\phi(x_2)\cdots|\psi_k\rangle \\ &\quad \times \langle \psi_k|\phi(y_1)\phi(y_2)\cdots|0\rangle ,\end{aligned} \tag{2}$$

where $|0\rangle$ is the vacuum state in the absence of wormholes, and $|\psi_i\rangle$ are a complete orthonormal set of wormhole states.[1]

What are these wormhole states $|\psi_k\rangle$? Let S be a cross section, a three-surface that separates the two asymptotically Euclidean regions. Then the quantum states of the wormhole can be described by wave functions $\Psi_k(h_{ij},\phi_0)$ which depend on the three-metric h_{ij} and matter fields ϕ_0 on S. The wave functions will obey the Wheeler-DeWitt and momentum constraint equations

$$H\Psi_k = \left[-\tfrac{1}{2} m_P^{-2} h^{1/2}(h_{il}h_{jm} + h_{im}h_{jl} - h_{ij}h_{lm}) \frac{\delta^2}{\delta h_{ij}\delta h_{lm}} - m_P^2 h^{1/2}\,{}^{(3)}R + \tfrac{1}{2} h^{1/2} T^{mn}\left[\phi_0, -i\frac{\delta}{\delta\phi_0}\right]\right]\Psi_k(h_{ij},\phi_0) = 0 , \tag{3}$$

$$H^i\Psi_k = \left[-2im_P^2 \left[\frac{\delta}{\delta h_{ij}}\right]_{;j} + T^{ni}\left[\phi_0, -i\frac{\delta}{\delta\phi_0}\right]\right]\Psi_k(h_{ij},\phi_0) = 0 , \tag{4}$$

at all finite nonzero three-metrics h_{ij}. However, if the wave functions are to correspond to wormholes rather than other kinds of spacetime, they should also obey certain boundary conditions when the three-metric h_{ij} degenerates or become infinite.

The boundary conditions when h_{ij} degenerates should express the fact that the four-metric is nonsingular. It is not clear what these boundary conditions should be in the full superspace of all three-metrics, but in minisuperspace models, such as those considered in the next section, it seems reasonable to suppose that the wave function should be regular, or (depending on the factor ordering) maybe go as a power of the radius a as a approaches zero. It certainly should not oscillate an infinite number of times.

The boundary conditions when h_{ij} is large should express that the four-metric is asymptotically Euclidean. One can interpret this as saying that there are no gravitational excitations in the asymptotic state. If one also imposed the boundary condition that there were no matter excitations in the asymptotic region, one would get a "ground state" or vacuum wave function Ψ_0. Like the no-boundary wave function, one can obtain the vacuum wave function from a path integral

$$\Psi_0(h_{ij},\phi_0) = \int d[g_{\mu\nu}]d[\phi]e^{-I[g,\phi]} . \tag{5}$$

In the case of the no-boundary state, the path integral is over all compact metrics and matter fields with the given boundary values. But in the case of the vacuum state, the path integral is over all asymptotically Euclidean metrics, and all matter fields that are zero, or gauge equivalent to zero, at infinity.

In minisuperspace models, the no-boundary wave function increases as $e^{a^2/2}$, where a is the radius of the three-surface S. On the other hand, the vacuum-state wave function *decreases* for large a like $e^{-a^2/2}$. This difference comes about because the main term in the gravitational action is the surface term

$$-\frac{m_P^2}{8\pi}\int d^3x\sqrt{h}K , \tag{6}$$

where K is the trace of the second fundamental form of the outward-directed normal to the surface S. In the case of the no-boundary wave function, the stationary phase metric for zero matter field is flat space inside a three-sphere of radius a. The outward normals are diverging, so the action is negative. This makes the no-boundary wave function grow with the size of the three-surface. On the other hand, the stationary phase metric for the vacuum wave function is flat space *outside* a three-sphere of radius a. The outward normals will be converging, so the action will be positive, and the wave function will be damped at large radius a.

However, there are other solutions of the Wheeler-DeWitt equation for the minisuperspace models that are also regular at $a=0$, and are damped at large radius. Some of then solutions can be expressed as superpositions of solutions that have a nonzero flux of a conserved quantity across the three-surface S. Such solutions cannot close off with a compact four-geometry, for then the flux would be zero. The behavior at large radius indicates that these solutions are asymptotically Euclidean, and the regularity at $a=0$ indicates that it is nonsingular. Thus, these solutions must correspond to wormholes that connect two asymptotically Euclidean regions.

The ground state for the wormhole will be defined by a path integral over all metrics of the topology $S^3 \times R^1$ which are asymptotically Euclidean at each end of the

R^1. The matter fields in the path integral will be gauge equivalent to zero at each end of the R^1. This means that the wave function for the ground state of the wormhole will be identical to that for the vacuum state, and will be given by a path integral over all asymptotic Euclidean metrics and all asymptotically zero matter fields that have the given values on the surface S. On the other hand, the other solutions of the Wheeler-DeWitt equation that are regular at $a=0$, and damped at large radius, can be interpreted as "excited states" of wormholes. Such solutions were interpreted in Ref. 13 as excited states of a closed universe. This was because the wave function oscillates at small a, and so corresponds to a Lorentzian closed Friedmann metric. However, one can equally well interpret the wave function at large a, where it is exponential, as corresponding to a Euclidean wormhole metric. In fact, the wormhole metric is the analytic continuation of the Friedmann metric.

The wave functions of the excited-wormhole states can also be represented by path integrals. The metrics in the path integrals are asymptotically Euclidean, which means that there are no gravitational excitations asymptotically. But the matter fields have sources at infinity, which can be interpreted as saying that there are matter particles passing through the wormhole. Here, "at infinity" means at distances large compared to the characteristic scale of the wormhole. This will be true of sources introduced to calculate low-energy Green's functions, and also, in the dilute-wormhole approximation, of the effective sources provided by other wormholes. One can interpret the dilute-wormhole approximation as the statement that the wormholes are "on shell." One then has boundary conditions on the Wheeler-DeWitt equation that, at least in minisuperspace examples, allow only a discrete spectrum of solutions. However, when one goes beyond the dilute-wormhole approximation and considers wormholes that are close together, one will have to include a continuous family of "off-shell" wormhole states in the sum over states used to factorize the Green's functions.

III. MINISUPERSPACE MODELS WITH A MASSLESS SCALAR

We shall consider metrics of the form

$$ds^2=\sigma^2[N^2(t)dt^2+a^2(t)d\Omega_3^2] , \tag{7}$$

where $\sigma^2=2G/3\pi$ and $d\Omega_3^2$ is the metric of a three-sphere of unit radius. If N is imaginary, this is the Lorentzian metric of a Friedmann universe, while if N is real, it is the metric of an Euclidean wormhole. However, solutions of the Wheeler-DeWitt equation are independent of N and t. So they can be interpreted either as Friedmann universes, or as wormholes, according to whether the wave is oscillatory or exponential.

We shall consider first a zero-mass minimally coupled scalar field ϕ. The Wheeler-DeWitt equation is

$$\left[\frac{1}{a^2}\frac{\partial}{\partial a}a\frac{\partial}{\partial a}-\frac{1}{a^3}\frac{\partial^2}{\partial\phi^2}-a\right]\Psi(a,\phi)=0 , \tag{8}$$

where the factor ordering is the one that is invariant under changes of coordinates in minisuperspace.[14] One can separate the Wheeler-DeWitt equation by writing

$$\Psi(a,\phi)=c(a)e^{ik\phi} , \tag{9}$$

where

$$\frac{d^2c}{da^2}+\frac{1}{a}\frac{dc}{da}+\left[\frac{k^2}{a^2}-a^2\right]c=0 . \tag{10}$$

The two independent solutions are[15,16]

$$J_{\pm ik/2}\left[\frac{i}{2}a^2\right] . \tag{11}$$

There is a linear combination of these that goes as $e^{-a^2/2}$ at large radius a. However, near $a=0$, the solutions go like $a^{\pm ik}$, so they oscillate an infinite number of times.

These solutions are eigenstates of the quantum-mechanical operator $\pi_\phi=-i\partial/\partial\phi$ with eigenvalue k. Classically,

$$\pi_\phi=\frac{i}{N}a^3\frac{\partial\phi}{\partial t} . \tag{12}$$

Thus these solutions carry a conserved scalar flux $q=2\pi^2k$ where

$$q=i\int_S\phi_{;\mu}d\sigma^\mu . \tag{13}$$

They will oscillate in a for $0<a<k^{1/2}$. Thus in this region they can be interpreted as corresponding to classical Lorentzian-Friedmann solutions with a scalar flux q. These solutions will expand from $a=0$ to a maximum radius $a=(q/2\pi^2)^{1/2}$ and collapse again to $a=0$. The infinite number of oscillations of the wave function near $a=0$ will correspond to the initial and final singularities of the Friedmann solution.

For $a>k^{1/2}$, the wave functions will decrease exponentially like $e^{-a^2/2}$. This indicates that they will correspond to asymptotically Euclidean classical solutions. The lower bound $k^{1/2}$ on the radius a, and the existence of the nonzero scalar flux q, indicate that the solution will have the form of a wormhole connecting two asymptotically Euclidean regions. For real q, the gradient of ϕ on the Euclidean solution will be imaginary. This means the energy-momentum tensor of the scalar field will be of the opposite sign to that of a scalar field that was real on the Euclidean section. The classical Euclidean solution will be the same as that found by Giddings and Strominger.[10] This is just the analytical continuation of the classical Friedmann solution with real ϕ.

In the semiclassical approach to wormholes, one considers instantons, which are classical Euclidean solutions. If one requires that the matter fields be real, such solutions exist only in special cases, like an antisymmetric tensor field,[10] or the Yang-Mills field.[11,12] They do not exist for pure gravity. This would suggest that wormholes would not be a general solution to the cosmological-constant problem. On the other hand, in the quantum-mechanical wave function approach, one might expect that solutions of the Wheeler-DeWitt equation with appropriate boundary conditions would exist for all reasonable forms of matter.

Of course, the solutions given above do not satisfy the regularity condition at $a=0$. However, we shall show that there is another class of solutions of the Wheeler-DeWitt equation, that are regular at $a=0$, and are damped at large radius. We introduce new coordinates in minisuperspace defined by

$$x=a\sinh\phi,\quad y=a\cosh\phi\ . \tag{14}$$

The Wheeler-DeWitt equation then becomes

$$\left[\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial x^2}-y^2+x^2\right]\Psi=0\ . \tag{15}$$

This is the equation for two harmonic oscillators with opposite signs of the energy. The solutions that are regular at the origin and damped at infinity are just products of harmonic-oscillator wave functions[1,16]

$$\Psi=\psi_n(x)\psi_n(y)\ , \tag{16}$$

where

$$\psi_n(x)=(2^n n!)^{-1/2}H_n(x)e^{-x^2/2}\ . \tag{17}$$

These harmonic-oscillator solutions, which will be denoted $|n\rangle$, form a basis for solutions of the Wheeler-DeWitt equation that are regular at the origin, and damped at infinity. Thus they must transform into each other under the symmetry of the Wheeler-DeWitt equation, generated by adding a constant to ϕ. One can regard this as a Lorentz transformation in the x,y plane. This is generated by the Killing vector $\partial/\partial\phi=y\partial/\partial x+x\partial/\partial y$. One can express this in terms of annihilation and creation operators for the two harmonic oscillators, using

$$x=\frac{1}{\sqrt{2}}(a_x+a_x^\dagger)\ , \tag{18}$$

$$\frac{\partial}{\partial x}=\frac{1}{\sqrt{2}}(a_x-a_x^\dagger)\ . \tag{19}$$

The symmetry generator $K=i\pi_\phi$ is then

$$\frac{\partial}{\partial\phi}=(a_x a_y-a_x^\dagger a_y^\dagger)\ . \tag{20}$$

One can use this to express the K eigenstates in terms of harmonic-oscillator states and vice versa. Let

$$|k\rangle=\sum_{n=0}^{\infty}c_n(k)|n\rangle\ . \tag{21}$$

Operating with K, one gets

$$ikc_n=(n+1)c_{n+1}-nc_{n-1}\ . \tag{22}$$

One can solve this iteratively for $c_n(k)$ in terms of $c_0(k)$, which can be fixed by normalization. One gets[16]

$$\begin{aligned}c_n(k)&=(-1)^n\pi^{3/2}\operatorname{sech}(\tfrac{1}{2}\pi k)F(-n,\tfrac{1}{2}+\tfrac{1}{2}ik;1;2)\\&=\pi^{3/2}\operatorname{sech}(\tfrac{1}{2}\pi k)F(-n,\tfrac{1}{2}-\tfrac{1}{2}ik;1;2)\end{aligned} \tag{23}$$

in terms of the hypergeometric function F, which here is an even or odd polynomial of degree n in ik. One can, therefore, regard the singular K eigenstates as being superpositions of an infinite number of regular harmonic-oscillator solutions. Similarly, the harmonic-oscillator solutions can be regarded as superpositions of different K eigenstates,[16] just as wave-packet solutions of the wave equation can be thought of as superpositions of plane waves. Thus, the harmonic-oscillator solutions can be interpreted as coherent states of classical solutions.

There is a similar discrete spectrum of harmonic-oscillator solutions for a minisuperspace model with a conformally invariant scalar field ϕ. The Wheeler-DeWitt equation is (modulo factor ordering)

$$\left[\frac{\partial^2}{\partial a^2}-\frac{\partial^2}{\partial\xi^2}-a^2+\xi^2\right]\Psi=0\ , \tag{24}$$

where $\xi=a\phi$. This obviously has harmonic-oscillator solutions in a and ξ. However, one can always use the freedom to make a conformal transformation of the metric, to establish an equivalence between the conformal and minimal scalar fields coupled to gravity.[17,16] In the present case, one can see the equivalence by defining a new radius and scalar field

$$\tilde{a}=a\sqrt{1-\phi^2},\quad \tilde{\phi}=\operatorname{arctanh}\phi\ . \tag{25}$$

In these variables, the Wheeler-DeWitt equation becomes the same as for the minimal massless scalar field. Thus there will be K eigenstate solutions. These will be singular at $a=0$, which is the singularity at $a=0$ in the analytically continued Euclidean wormhole instanton. They will also be singular at $\phi=\pm1$. This corresponds to the fact that the Newtonian gravitational constant G becomes infinite and changes sign where $\phi=\pm1$. However, the harmonic-oscillator solutions are well behaved at both $a=0$ and $\phi=\pm1$. Thus they can provide a quantum-mechanical description of wormholes in a theory with a conformally invariant scalar field.

IV. MINISUPERSPACE MODELS WITH A MASSIVE SCALAR

Conformally invariant and massless minimally coupled scalar fields are rather special forms of matter. Are there solutions of the Wheeler-DeWitt equation, for other forms of matter, which are regular at $a=0$ and damped at large radii? This is difficult to answer with complete confidence, because one apparently cannot get explicit exact solutions in closed form, even in the simple case of a minimally coupled scalar field with a nonzero self-coupling potential $U(\phi)$, such as $U=\frac{1}{2}m^2\phi^2$ for the free massive scalar field. However, one can find explicit asymptotic solutions for the massive scalar-field case, and similar methods of construction should work for a rather arbitrary potential.

For a Friedmann-Robertson-Walker three-sphere geometry ($k=+1$) of radius a coupled to a homogeneous scalar field ϕ, the Wheeler-DeWitt equation with a suitable choice of units is

$$\begin{aligned}W\Psi&\equiv\left[a\frac{\partial}{\partial a}a\frac{\partial}{\partial a}-\frac{\partial^2}{\partial\phi^2}-a^4+2a^6U(\phi)\right]\Psi(a,\phi)\\&\equiv\ddot{\Psi}-\Psi''-a^4\Psi+2a^6U\Psi=0\ ,\end{aligned} \tag{26}$$

where an overdot denotes $a\partial/\partial a$ and a prime denotes $\partial/\partial\phi$. As a starting point, construct a zeroth-order WKB approximation e^{-I}, where $I(a,\phi)$ is the action of the classical Euclidean solution which goes from the point (a,ϕ) to $a=\infty$ and to ϕ at a minimum of $U(\phi)$ (which will be taken to be at $\phi=0$; the value there must be $U=0$ in order that the solution be asymptotically Euclidean).

The Euclidean action $I(\alpha,\phi)$ obeys the Hamilton-Jacobi equation

$$\dot{I}^2-I'^2-a^4+2a^6U=0\ . \tag{27}$$

(The signs of the quadratic terms in I are opposite what they would be for a Lorentzian action.) For large a one may write the action in the asymptotic form

$$I(a,\phi)=a^3E(\phi)+\tfrac{1}{2}a^2F(\phi)+aG(\phi)+H(\phi)+O(a^{-1})\ , \tag{28}$$

where

$$E'^2-9E^2=2U \tag{29}$$

with $E(\phi)$ having a minimum value of 0 at $\phi=0$, and where one gets differential equations for the other functions of ϕ that may be integrated to give

$$F(\phi)=\exp\left[\int_0^{\phi}\frac{6E(x)dx}{E'(x)}\right]\ , \tag{30}$$

$$G(\phi)=[F(\phi)]^{1/2}\int_0^{\phi}[E'(x)]^{-3}[F(x)]^{-1/2}\times\{U(x)[F(x)]^2-\tfrac{1}{2}[E'(x)]^2\}dx\ , \tag{31}$$

$$H(\phi)=\int_0^{\phi}[E'(x)]^{-1}[F(x)G(x)-\tfrac{1}{2}F'(x)G'(x)]dx\ . \tag{32}$$

Only $F(\phi)$ is nonzero at $\phi=0$ (where $F=1$), so the action is exactly $I(a,0)=\frac{1}{2}a^2$ along the $\phi=0$ line, which corresponds to the empty flat-space Euclidean solution. One may readily see from Eqs. (29) and (30) that $E(\phi)$ and $F(\phi)$ are both monotonically increasing functions of $|\phi|$ as one goes away from $\phi=0$, provided that $U(\phi)$ stays non-negative or at least does not drop down to $-\frac{9}{2}E(\phi)$ (where E' would become zero, giving a caustic), as we shall henceforth assume.

For $U(\phi)=\frac{1}{2}m^2\phi^2$ and for $\phi^2\ll1$, one can find power-series solutions for E, F, G, and H:

$$E(\phi)=m\left[\tfrac{1}{2}\phi^2+\frac{3^2}{2^5}\phi^4+\frac{3^3}{2^8}\phi^6+\frac{3^5}{2^{14}}\phi^8+0\phi^{10}+\frac{3^7}{2^{21}}\phi^{12}-\frac{3^9}{2^{24}7}\phi^{14}+O(\phi^{16})\right]\ , \tag{33}$$

$$F(\phi)=1+\tfrac{3}{2}\phi^2+\frac{3^25}{2^6}\phi^4+\frac{3^2}{2^8}\phi^6-\frac{3^313}{2^{16}}\phi^8+\frac{3^441}{2^{15}5}\phi^{10}-\frac{3^42603}{2^{21}5}\phi^{12}-\frac{3^5904\,417}{2^{24}7^2}\phi^{14}+O(\phi^{16})\ , \tag{34}$$

$$G(\phi)=m^{-1}\left[\frac{3}{2^4}\phi^2-\frac{3^3}{2^8}\phi^4+\frac{3\times41}{2^{12}}\phi^6+\frac{3^2115}{2^{19}}\phi^8-\frac{3^3410\,953}{2^{21}5^2}\phi^{10}-\frac{3^32\,319\,221}{2^{25}5^2}\phi^{12}+O(\phi^{14})\right]\ , \tag{35}$$

$$H(\phi)=m^{-2}\left[-\frac{3}{2^4}\phi^2-\frac{3^2}{2^8}\phi^4-\frac{43}{2^9}\phi^6+\frac{3^326\,267}{2^{22}}\phi^8-\frac{3^41\,377\,457}{2^{19}5^5}\phi^{10}+O(\phi^{12})\right]\ . \tag{36}$$

For $U(\phi)=(\lambda/2p)\phi^{2p}$, one gets

$$E=\left[\frac{\lambda}{p}\right]^{1/2}\frac{\phi^{p+1}}{p+1}\left[1+\frac{3^2\phi^2}{2(p+1)(p+3)}+\frac{3^4(3p+1)\phi^4}{2^3(p+1)^3(p+3)(p+5)}+\frac{3^6(5p^3+15p^2-5p+1)\phi^6}{2^4(p+1)^5(p+3)^2(p+5)(p+7)}+O(\phi^8)\right]\ , \tag{37}$$

$$F=1+\frac{3\phi^2}{p+1}+\frac{3^2p(p+4)\phi^4}{2(p+1)^3(p+3)}+\frac{3^2(p^4+10p^3+23p^2-34p+6)\phi^6}{2(p+1)^5(p+3)(p+5)}+O(\phi^8)\ , \tag{38}$$

$$G=\left[\frac{\lambda}{p}\right]^{-1/2}\frac{3(2p-1)\phi^{3-p}}{2(p+1)^2(3-p)}\left[1-\frac{3(4p^4-14p^3-61p^2+134p-39)}{2(p+1)^2(p+3)(2p-1)(5-p)}\phi^2+O(\phi^4)\right]\ , \tag{39}$$

$$H=-\left[\frac{\lambda}{p}\right]^{-1}\frac{3(2p-1)\phi^{4-2p}}{(p+1)^3(3-p)}[1+O(\phi^2)]\ . \tag{40}$$

If we write the asymptotic expansion (28) as

$$I(a,\phi)=\tfrac{1}{2}a^2+\sum_{l=-3}^{\infty}a^{-l}F_l(\phi)\ , \tag{41}$$

so $E=F_{-3}$, $F=1+2F_{-2}$, $G=F_{-1}$, and $H=F_0$, then we see that for small ϕ, F_l goes as $\lambda^{-1-l/2}\phi^{2-(p-1)(l+2)}$, so if $p>1$, this expansion breaks down at small ϕ for $l>(4-2p)/(p-1)$. However, for $U=\frac{1}{2}m^2\phi^2$, the $p=1$ case, the expansion is apparently valid at small ϕ for all l. Hence we shall assume that at small enough ϕ the potential goes as $\frac{1}{2}m^2\phi^2$ with $m^2>0$.

Assuming that $U(\phi)$ does not grow faster than a constant times $e^{6|\phi|}$, one can see that for large $|\phi|$ the a^{-l} expansion coefficients for $l<0$ go as

$$F_l(\phi)\approx c_l e^{-l|\phi|}\ , \tag{42}$$

where the c_l's are constants that depend on $U(\phi)$ which could in principle be evaluated by solving Eqs. (29)–(31) numerically. Clearly both c_{-3} and c_{-2} are positive. Thus for large

$$a^2=y^2-x^2=uv \tag{43}$$

and large absolute value of

$$\phi=\operatorname{arctanh}\left[\frac{x}{y}\right]=\tfrac{1}{2}\ln\frac{v}{u}\ , \tag{44}$$

the action is roughly

$$\begin{aligned} I\approx c_{-3}a^3e^{3|\phi|}&=c_{-3}(y+|x|)^3\\ &=c_{-3}\max(u^3,v^3)\ , \end{aligned} \tag{45}$$

where the minisuperspace null coordinates

$$u=y-x=ae^{-\phi},\quad v=y+x=ae^{\phi} \tag{46}$$

each range over non-negative values. One can see the form of the Hamilton-Jacobi equation (27), rewritten in terms of these null coordinates,

$$\frac{\partial I}{\partial u}\frac{\partial I}{\partial v}=\tfrac{1}{4}uv-\tfrac{1}{2}u^2v^2U\left[\tfrac{1}{2}\ln\frac{v}{u}\right]\ , \tag{47}$$

that the asymptotic form (45) for the action is also valid near one of the null boundaries of the minisuperspace (say, $v=0$) as long as the other null coordinate (say, u) is large, even when $a^2=uv$ is taken to zero.

For the purpose of visualizing roughly how the action behaves over the whole minisuperspace, it may be useful to have an explicit expression that is a crude approximation everywhere. To get one, take the first two terms of (28) with $E=\frac{2}{9}m\sinh^2\frac{3}{2}\phi$, which is a simple function chosen to give the first term of (33) when $U=\frac{1}{2}m^2\phi^2$ and to have the correct asymptotic form (42), though with the wrong coefficient c_{-3}. This E would actually solve (29) if the scalar field potential were $U=\frac{2}{9}m^2\sinh^2\frac{3}{2}\phi$, so it is somewhat too large if the actual potential grows less rapidly, but for large ϕ, E'^2 and $9E^2$ will each go roughly as $9c_{-3}^2e^{6|\phi|}$ and be much larger than the $2U$ term if the potential does not rise this rapidly. The approximate E should be correct for all ϕ to within a factor of order unity (e.g., $18m^{-1}c_{-3}$ for large ϕ). By Eq. (30), it would give $F=\cosh^{4/3}\frac{3}{2}\phi$, which does match the first two terms of (34) and have the correct asymptotic form (42). These two approximate functions then give

$$\begin{aligned} I(a,\phi)&\sim\tfrac{2}{9}ma^3\sinh^2\tfrac{3}{2}\phi+\tfrac{1}{2}a^2\cosh^{4/3}\tfrac{3}{2}\phi\\ &=\tfrac{2}{9}m(\tfrac{1}{2}u^{3/2}-\tfrac{1}{2}v^{3/2})^2+\tfrac{1}{2}(\tfrac{1}{2}u^{3/2}+\tfrac{1}{2}v^{3/2})^{4/3}\ . \end{aligned} \tag{48}$$

If $U=\frac{1}{2}m^2\phi^2+O(\phi^4)$, this expression is correct to within a factor of $1+O(\phi^2)$ for $\phi^2\ll 1$ and is apparently correct to within a factor of order unity everywhere.

Once one has a solution of the Euclidean Hamilton-Jacobi equation (27) or (47) for the action I, the integral curves of the gradient vector field ∇I, with indices raised by the inverse of the minisuperspace metric

$$\begin{aligned} ds^2=-a\,da^2+a^3d\phi^2&=(y^2-x^2)^{1/2}(-dy^2+dx^2)\\ &=-u^{1/2}v^{1/2}du\,dv\ , \end{aligned} \tag{49}$$

give the trajectories of the classical Euclidean solutions.

The Euclidean time derivative along each of these curves is given by

$$\begin{aligned} \frac{d}{d\tau}=-\nabla I\cdot\nabla&=a^{-1}\frac{\partial I}{\partial a}\frac{\partial}{\partial a}-a^{-3}\frac{\partial I}{\partial\phi}\frac{\partial}{\partial\phi}\\ &=2u^{-1/2}v^{-1/2}\left[\frac{\partial I}{\partial u}\frac{\partial}{\partial v}+\frac{\partial I}{\partial v}\frac{\partial}{\partial u}\right]\ , \end{aligned} \tag{50}$$

or $df/d\tau=a^{-3}(\dot I\dot f-I'f')$ in the notation used above.

In terms of the Laplacian corresponding to the metric (49), the Wheeler-DeWitt equation (26) is proportional to

$$\tfrac{1}{2}a^{-3}W\Psi=(-\tfrac{1}{2}\nabla^2+V)\Psi=0\ , \tag{51}$$

$$V(a,\phi)=-\tfrac{1}{2}a+a^3U(\phi)\ , \tag{52}$$

and the Hamilton-Jacobi equation (27) or (47) has the form

$$\tfrac{1}{2}(\nabla I)^2=V\ . \tag{53}$$

If one then writes an exponential WKB wave function based on I as

$$\Psi=C(a,\phi)e^{-I}\equiv e^{h-I}\ , \tag{54}$$

then using Eq. (50) in the resulting Wheeler-DeWitt equation gives

$$h=\int[\tfrac{1}{2}\nabla^2I-\tfrac{1}{2}(\nabla h)^2-\tfrac{1}{2}\nabla^2h]d\tau\ , \tag{55}$$

where the integral is taken along each classical trajectory. In the WKB limit in which I varies much more rapidly than h, the first-order WKB approximation, which consists of dropping the terms involving h on the right-hand side of (55), is an accurate approximation. If one desires more accuracy, one can put the resulting left-hand side back into the right-hand side and iterate for higher-order WKB approximations.

Equation (55) has a constant of integration for each of the $(n-1)$-parameter set of trajectories tangent to ∇S in an n-dimensional minisuperspace. (Here $n=2$.) By specifying the prefactor $C=e^h$ to be a slowly varying but otherwise arbitrary function on a hypersurface of codimension 1 (here a line) that intersects the classical trajectories, one can get a family of WKB solutions all based on the same action I but that depends on this one function C of $n-1$ variables. Here the action is well behaved and is everywhere positive (except at $a=0$, ϕ finite, where it is zero) for a positive-semidefinite potential $U(\phi)$ that goes as $\frac{1}{2}m^2\phi^2$ for small ϕ. For large a the action tends to ∞ as $\frac{1}{2}a^2$ plus a^3 times a non-negative monotonically increasing function $E(\phi)$. Therefore, $\Psi=Ce^{-I}$ will be well behaved, bounded everywhere, and damped at infinity if C is well behaved and either stays bounded or else does not diverge as fast as e^I at infinity.

There do not appear to be any caustics in the trajectories generated by I, which stay well separated in the (u,v) plane as one follows them in from $a=\infty$ to $a=0$, so there appears to be no difficulty integrating Eq. (55) inward from some hypersurface where the prefactor is arbitrarily specified, and the resulting solution should be well behaved. However, in integrating (55) outward, one does have the difficulty that the trajectories all exponentially approach the line $\phi=0$ as a goes to ∞, assuming $U \propto \phi^2$ near $\phi=0$ as is necessary for I to be regular there. This means that the ∇h component across the converging trajectories will become large, eventually dominating over ∇I, so that the WKB approximation will break down and integrating (55) might even lead to a divergence. Hence one must use another scheme to get well-behaved wave functions near $\phi=0$ for a going to ∞. Once one has them there, Eq. (55) may be used to integrate them down to arbitrarily small a and large ϕ. By starting at sufficiently large a, one can also get to arbitrarily large ϕ at any finite a, i.e., to anywhere in the minisuperspace.

After some trial and error, a scheme was found that seems to work for $U=\frac{1}{2}m^2\phi^2$ (and which presumably could be modified to work for other potentials which also behave quadratically near $\phi=0$). The idea is to start with

$$\Psi_{n0}=\phi^n e^{-L_n(a,\phi)} \tag{56}$$

for each non-negative integer n, where L_n is a positive analytic function of a and ϕ which makes Ψ_{n0} solve the Wheeler-DeWitt equation (26) as nearly as possible. This form cannot solve (26) exactly, for the $-\Psi''_{n0}$ term includes a $-n(n-1)\phi^{-2}\Psi_{n0}$ term which cannot be canceled if L_n is analytic at $\phi=0$ (e.g., does not include a $\ln\phi$ term there). However, one can solve

$$\begin{aligned} W\Psi_{n0} &\equiv \ddot{\Psi}_{n0}-\Psi''_{n0}-a^4\Psi_{n0}+m^2a^6\phi^2\Psi_{n0} \\ &= -n(n-1)\phi^{-2}\Psi_{n0} , \end{aligned} \tag{57}$$

which is equivalent to

$$\dot{L}_n^2-L_n'^2-\ddot{L}_n+L_n''+2n\phi^{-1}L_n'-a^4+m^2a^6\phi^2=0 , \tag{58}$$

and which has an asymptotic series solution of the form

$$\begin{aligned} L_n &= \sum_{l=0}^{\infty} a^{3-l}L_{nl}(\phi)+c_n\ln a \\ &= a^3E(\phi)+\tfrac{1}{2}a^2F(\phi)+a[G(\phi)-(n+\tfrac{1}{2})mF^{1/2}(\phi)]+[\tfrac{1}{4}-\tfrac{3}{2}n-\tfrac{1}{8}(2n+1)^2m^2]\ln a+\sum_{l=3}^{\infty} a^{3-l}L_{nl}(\phi) \\ &= \tfrac{1}{2}ma^3\phi^2\left[1+\frac{3^2}{2^4}\phi^2+\frac{3^3}{2^7}\phi^4+\frac{3^5}{2^{13}}\phi^6+0\phi^8+\frac{3^7}{2^{20}}\phi^{10}-\frac{3^9}{2^{23}7}\phi^{12}+O(\phi^{14})\right] \\ &\quad +\tfrac{1}{2}a^2\left[1+\tfrac{3}{2}\phi^2+\frac{3^2 5}{2^6}\phi^4+\frac{3^2}{2^8}\phi^6-\frac{3^3 13}{2^{16}}\phi^8+\frac{3^4 41}{2^{15}5}\phi^{10}-\frac{3^4 2603}{2^{21}5}\phi^{12}-\frac{3^5 904\,417}{2^{24}7^2}\phi^{14}+O(\phi^{16})\right] \\ &\quad +\frac{3}{2^4}m^{-1}a\phi^2\left[1-\frac{3^2}{2^4}\phi^2+\frac{41}{2^8}\phi^4+\frac{345}{2^{15}}\phi^6-\frac{3\,698\,577}{2^{17}5^2}\phi^8-\frac{20\,872\,989}{2^{21}5^2}\phi^{10}+O(\phi^{12})\right] \\ &\quad -(n+\tfrac{1}{2})ma\left[1+\frac{3}{2^2}\phi^2+\frac{3^2}{2^7}\phi^4-\frac{3^2}{2^8}\phi^6+\frac{3^3 103}{2^{17}}\phi^8+\frac{3^4 221}{2^{15}5}\phi^{10}-\frac{3^5 72\,463}{2^{24}5}\phi^{12}+O(\phi^{14})\right] \\ &\quad -(\ln a)[\tfrac{3}{2}n-\tfrac{1}{4}+\tfrac{1}{8}(2n+1)^2m^2]+(\text{normalization constant arbitrarily set to zero}) \\ &\quad -\left[\frac{3}{2^4}m^{-2}+\frac{3^2}{2^4}n+\frac{3}{2^5}+\frac{3}{2^5}(2n+1)^2m^2\right]\phi^2-\left[\frac{3^2}{2^8}m^{-2}-\frac{3^4}{2^8}n-\frac{3^3 5}{2^9}-\frac{3^3}{2^{10}}(2n+1)^2m^2\right]\phi^4 \\ &\quad -\left[\frac{43}{2^9}m^{-2}+\frac{3^3 37}{2^{12}}n+\frac{3^5 7}{2^{13}}+\frac{3^3}{2^{12}}(2n+1)^2m^2\right]\phi^6 \\ &\quad +\left[\frac{3^3 26\,267}{2^{22}}m^{-2}+\frac{3^6 149}{2^{20}}n+\frac{3^7 79}{2^{21}}-\frac{3^6}{2^{20}}(2n+1)^2m^2\right]\phi^8 \\ &\quad -\left[\frac{3^4 1\,377\,457}{2^{19}5^2}m^{-2}-\frac{3^4 111\,481}{2^{21}5^2}n-\frac{3^4 119\,581}{2^{22}5^2}-\frac{3^9}{2^{21}5}(2n+1)^2m^2\right]\phi^{10}+O(\phi^{12})+O(a^{-1}) . \end{aligned} \tag{59}$$

Then for each non-negative integer n, one can get a well-behaved exponentially damped wave function of the asymptotic series form

$$\Psi_n=e^{-L_n}\sum_{k=0}^{\infty} a^{-k}f_{nk}(\phi) , \tag{60}$$

where $f_{n0}=\phi^n$ and where the f_{nk} with higher values of k are analytic functions of ϕ chosen so that Ψ_n satisfies the Wheeler-DeWitt equation (26) order by order in the expansion parameter a^{-1}. One can get the successive f_{nk}'s recursively from the indefinite integrals

$$f_{nk}=\phi^n F^{-k/2}\int \frac{d\phi}{2E'}\phi^{-n}F^{k/2}\sum_{l=1}^{k}\{\delta_{l3}f''_{n(k-l)}-2L'_{nl}f'_{n(k-l)}$$
$$+[2n\phi^{-1}L'_{nl}+2(k-l)(l-3)L_{nl}-(k-l)(2c_n+k-l)\delta_{l3}]f_{n(k-l)}\} . \quad (61)$$

In each successive integration, the constant of integration is to be chosen for f_{nk} so that no term with a nonanalytic $\ln\phi$ factor occurs in the following integration, for $f_{n(k+1)}$.

For example, evaluating (61) and avoiding $\ln\phi$ terms gives

$$f_{n0}=\phi^n , \quad (62)$$

$$f_{n1}=0 , \quad (63)$$

$$f_{n2}=\tfrac{9}{32}n(n-1)\phi^n F^{-1}$$
$$=\tfrac{9}{32}n(n-1)\phi^n(1-\tfrac{3}{2}\phi^2)+O(\phi^{n+4}) , \quad (64)$$

$$f_{n3}=-\frac{n(n-1)}{4m}\phi^{n-2}\left[1-\left[1+\frac{2n+1}{4}m^2\right]\phi^2\right]$$
$$+O(\phi^{n+2}) , \quad (65)$$

$$f_{n4}=-\frac{3n(n-1)}{2048}[24m^{-2}+277n^2-233n+60$$
$$-12(2n+1)^2m^2]\phi^n+O(\phi^{n+2}) , \quad (66)$$

$$f_{n5}=\frac{3n(n-1)}{128}[4m^{-3}+(n-1)(13n+8)m^{-1}]\phi^{n-2}$$
$$+O(\phi^n) , \quad (67)$$

$$f_{n6}=\frac{n(n-1)(n-2)(n-3)}{32m^2}\phi^{n-4}+O(\phi^{n-2}) , \quad (68)$$

$$f_{n(3j)}=\frac{n!}{j!(n-2j)!}\left[\frac{-1}{4m}\right]^j\phi^{n-2j}$$
$$+O\left[\frac{n!}{(n-2j+2)!}\phi^{n-2j+2}\right] , \quad (69)$$

$$f_{m(3j+1)}=O\left[\frac{n!}{(n-2j)!}\phi^{n-2j}\right] , \quad (70)$$

$$f_{n(3j+2)}=O\left[\frac{n!}{(n-2j)!}\phi^{n-2j}\right] . \quad (71)$$

It is clear from (61) that when the Laurent expansion of the integrand has no ϕ^{-1} term (which would integrate to a nonanalytic $\ln\phi$ term), each $f_{nk}(\phi)$ has a Laurent expansion with only non-negative powers of ϕ, so it is an analytic function of ϕ in a neighborhood of $\phi=0$ if the $L_{nl}(\phi)$ functions are analytic so that the power series in ϕ all converge. One can also readily see that Ψ_n is symmetric or antisymmetric in ϕ, depending on whether n is even or odd: $\Psi_n(a,-\phi)=(-1)^n\Psi_n(a,\phi)$.

One can see from (69)–(71) that for a large and ϕ small, where the series (60) may be evaluated to a good approximation, it gives

$$\Psi_n=\phi^n e^{-L_n}\sum_{k=0}^{[n/2]}\frac{n!}{k!(n-2k)!}\left[\frac{-1}{4ma^3\phi^2}\right]^k$$
$$\times\left[1+O\left[\frac{1}{a}\right]+O(\phi^2)\right] , \quad (72)$$

where $[n/2]\equiv$[largest integer $\leq(n/2)$]. For $4ma^3\phi^2 \gg n(n-1)$, the $k=0$ term dominates and gives

$$\Psi_n\approx\phi^n e^{-L_n}, \quad \tfrac{1}{4}n(n-1)m^{-1}a^{-3}\ll\phi^2\ll(n+1)^{-1} , \quad (73)$$

whereas for $4ma^3\phi^2\ll(n+1)^{-1}$, the $k=[n/2]$ term dominates. If n is even, say $n=2l$, the latter case gives

$$\Psi_{2l}\approx(2l-1)!!e^{-L_{2l}}(-2ma^3)^{-l} ,$$
$$\phi^2\ll\tfrac{1}{4}(2l+1)^{-1}m^{-1}a^{-3}\ll(2l+1)^{-1} , \quad (74)$$

but if n is odd, $n=2l+1$,

$$\Psi_{2l+1}\approx(2l+1)!!e^{-L_{2l+1}}(-2ma^3)^{-l}\phi ,$$
$$\phi^2\ll\tfrac{1}{8}(l+1)^{-1}m^{-1}a^{-3}\ll\tfrac{1}{2}(l+1)^{-1} . \quad (75)$$

For $(n+1)^{-1}\lesssim 4ma^3\phi^2\lesssim n(n-1)$, intermediate terms in k dominate, and the solution (72) is more complicated.

It would be interesting to try to construct Lorentzian WKB wave packets by superposing Ψ_n's with very large n. Perhaps one could then overcome the difficulty Kiefer[18] had in trying to construct wave packets following the classical trajectories near their turning points in a. However, this may require extending L_n inward from the region $a\gg m^{-1}+1+nm$ where the explicit asymptotic expression (72) applies, because that expression gives $\Psi_{2l}(a,\phi=0)$ monotonically decreasing in magnitude as a increases. To get a peak in the truncated asymptotic expression for $\Psi_{2l}(a,0)$ at large a requires $n\gtrsim m^{-1}$, but then the peak would occur roughly at $a\sim nm$, where the asymptotic expression breaks down by having the successive terms in (60) of comparable magnitude for $k>\frac{3}{2}n$.

One might also wish to expand the no-boundary wave function in terms of the Ψ_n. Because the no-boundary wave function goes as $e^{a^2/2}$ at large a when $\phi=0$, it would need to be expressed as an infinite sum of Ψ_n's, and the sum of the squares of the coefficients in an orthonormal basis of Ψ_n's would not converge. One can see that the exponentially growing part of the no-boundary wave function at large a and small ϕ, say as given in the asymptotic expression (11) of Ref. 19, is simply $\Psi_0 = e^{-L_0}$ with m replaced by im, a replaced by ia, and ϕ left alone.

Thus in minisuperspace models with both massless and massive scalar fields, there are, or at least appear to be, infinite discrete spectra of solutions of the Wheeler-DeWitt equation that are regular everywhere (including $a=0$, $\phi=\pm\infty$) and exponentially damped at $a=\infty$. These may be considered to be wormhole wave functions.

We do not know which other minisuperspace models allow such regular, bounded wormhole wave functions, but the fact that we have found them for the first two models we have investigated suggests that they may occur rather generally, thereby describing wormholes in theories with rather arbitrary matter content. Then one would not need a highly restrictive theory in order for wormholes to exist and perhaps mediate black-hole evaporation and/or set the effective cosmological constant to zero.

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*Present address.

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