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Observables, gauge invariance, and time in $(2+1)$-dimensional quantum gravity

S. Carlip
Institute for Advanced Study, Princeton, New Jersey 08540
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Two formulations of quantum gravity in 2+1 dimensions have been proposed: one based on Arnowitt-Deser-Misner variables and York's "extrinsic time," the other on diffeomorphism-invariant ISO(2,1) holonomy variables. In the former approach, the Hamiltonian is nonzero, and states are time dependent; in the latter, the Hamiltonian vanishes, and states are time independent but manifestly gauge invariant. This paper compares the resulting quantum theories in order to explore the role of time in quantum gravity. It is shown that the two theories are exactly equivalent for simple spatial topologies, and that gauge-invariant "time"-dependent operators can be constructed for arbitrary topologies.

I. INTRODUCTION

The "problem of time" has long plagued attempts to quantize gravity.[1] In its simplest manifestation, the difficulty is that the Hamiltonian of general relativity is a constraint. H thus annihilates physical states and commutes with observables, making it hard to describe, or even define, dynamics. Moreover, without a suitable definition of time—provided in most theories by a fixed background geometry—such basic features of quantization as the definition of an inner product and the normalization of wave functions become ambiguous.

These phenomena reflect the fact that the role of time in general relativity is quite different from its role in the rest of physics. Coordinate time is merely a parameter, which can be changed arbitrarily by a diffeomorphism; the Hamiltonian is the generator of such transformations. Information about physical time, on the other hand, is hidden in the field variables themselves:[2,3] the intrinsic geometry of a pair of spacelike hypersurfaces determines their location in spacetime, and thus the time interval which separates them.

Unfortunately, the extraction of information about physical time from the gravitational field has proven quite difficult. Various "clock" variables have been advocated;[3–6] modifications of general relativity to allow new definitions of time have been suggested;[7–9] and changes in quantum theory—from variations of the sum-over-histories prescription[10] to "third quantization"[11–13]—have been proposed. But gravity is hard enough even without the problem of time, and no approach is yet generally accepted.

In the past two years, it has become apparent that $(2+1)$-dimensional gravity can serve as a useful model for $(3+1)$-dimensional general relativity. In two spatial dimensions, the constraint equations can be solved exactly. The resulting physical phase space is finite dimensional, reducing the problem of quantization to one of quantum mechanics rather than field theory. At the same time, the conceptual problems of quantum gravity, including the problem of time, remain, but now in a context in which they can be much more easily studied.

Two versions of $(2+1)$-dimensional quantum gravity have been proposed. The first, due to Witten,[14] makes use of the analogy between $(2+1)$-dimensional gravity and Chern-Simons gauge theories. It involves no explicit gauge fixing; states and operators are built out of manifestly invariant quantities (holonomies of flat connections), and since time translation is a gauge transformation, the Hamiltonian on the physical phase space vanishes. The second version, due to Moncrief[15] and to Hosoya and Nakao,[16] use a particular gauge-fixing procedure based on York's "extrinsic time," in which the trace of the extrinsic curvature is used as a time variable. States and operators are defined in this gauge; the Hamiltonian is nonzero, and generally quite complicated.[17]

By comparing these two quantizations, we may hope to learn something about time in more realistic theories of quantum gravity. The goal of this paper is to make such a comparison as explicit as possible. In particular, I show that if space has the topology of a torus, one can construct a canonical transformation between Moncrief's and Witten's variables, and find an operator ordering which extends this transformation to the quantum theories. In principle, this procedure should apply to arbitrary spatial topologies, but in practice the problem of operator ordering becomes very difficult. I show, however, that it is possible to construct "time"-dependent operators acting on Witten's diffeomorphism-invariant Hilbert space for any spatial topology, making dynamics manifest even with a vanishing Hamiltonian.

II. TWO QUANTUM THEORIES

Let us begin with a brief review of the Moncrief-Hosoya-Nakao and Witten quantizations of $(2+1)$-dimensional gravity. We work on a spacetime manifold with the topology $\mathbb{R}\times\Sigma$, where Σ is a compact surface. In Arnowitt-Deser-Misner (ADM) variables, the metric is[20]

$$ds^2=N^2dt^2-g_{ij}(dx^i+N^idt)(dx^j+N^jdt) \tag{2.1}$$

and the Einstein action is[21]

$$S=\int d^3x\,(-{}^{(3)}g)^{1/2}\,{}^{(3)}R$$
$$=\int dt\int_\Sigma d^2x\,(\pi^{ij}\dot{g}_{ij}-N^i\mathcal{H}_i-N\mathcal{H})\ . \qquad (2.2)$$

The momentum π^{ij} conjugate to g_{ij} is essentially the extrinsic curvature; for solutions of the equations of motion,

$$\pi^{ij}=\sqrt{g}\,(K^{ij}-g^{ij}K)\ , \qquad (2.3)$$

where

$$K^{ij}=\frac{1}{2N}(\dot{g}_{kl}-\nabla_k N_l-\nabla_l N_k)g^{ik}g^{jl} \qquad (2.4)$$

is the extrinsic curvature of the surface $t=$const. The lapse and shift functions N and N^i act as Lagrange multipliers for the constraints

$$\mathcal{H}_i=-2\nabla_j\pi^j{}_i\ ,$$
$$\mathcal{H}=\frac{1}{\sqrt{g}}g_{ij}g_{kl}(\pi^{ik}\pi^{jl}-\pi^{ij}\pi^{kl})-\sqrt{g}R\ , \qquad (2.5)$$

which also generate the gauge transformations.

One can alteratively write the action (2.2) in first-order form, treating the local frame $e^a{}_\mu$ and the spin connection $\omega_{a\mu}=\frac{1}{2}\epsilon_{abc}\omega_\mu{}^{bc}$ as independent variables. The action is then

$$S=\int d^3x\,\epsilon^{\rho\mu\nu}e^a{}_\rho(\partial_\mu\omega_{a\nu}-\partial_\nu\omega_{a\mu}+\epsilon_{abc}\omega^b{}_\mu\omega^c{}_\nu)$$
$$=2\int dt\int_\Sigma d^2x\,(-\epsilon^{ij}e^a{}_i\dot{\omega}_{aj}+e^a{}_0\tilde{\Theta}_a+\omega_{a0}\Theta^a) \qquad (2.6)$$

with constraints

$$\Theta^a=\tfrac{1}{2}\epsilon^{ij}[\partial_i e^a{}_j-\partial_j e^a{}_i+\epsilon^{abc}(\omega_{bi}e_{cj}-\omega_{ci}e_{bj})]\ ,$$
$$\tilde{\Theta}^a=\tfrac{1}{2}\epsilon^{ij}(\partial_i\omega^a{}_j-\partial_j\omega^a{}_i+\epsilon^{abc}\omega_{bi}\omega_{cj})\ . \qquad (2.7)$$

The number of constraints has doubled, since the gauge group is now twice as large—the first-order action is invariant under local Lorentz transformations as well as diffeomorphisms—but both (2.2) and (2.6) are standard forms for the ordinary Einstein action in 2+1 dimensions.

A. Moncrief-Hosoya-Nakao quantization

To quantize the action (2.2) or (2.6), we first solve the constraints, and work directly on the physical phase space. Moncrief and Hosoya and Nakao begin with the second-order form (2.2), and choose coordinates in which the mean curvature

$$\tau=g^{-1/2}g_{ij}\pi^{ij} \qquad (2.8)$$

is constant on each surface of constant time. The curvature τ thus serves as a time coordinate, a use first suggested by York[5] in 3+1 dimensions. The effect of this choice of "extrinsic time" is to decouple the constraints $\mathcal{H}$ and $\mathcal{H}_i$. The momentum constraints $\mathcal{H}_i=0$ become

$$\nabla_j\tilde{\pi}^{ij}=0\ , \qquad (2.9)$$

where

$$\tilde{\pi}^{ij}=\pi^{ij}-\tfrac{1}{2}g^{ij}g_{kl}\pi^{kl} \qquad (2.10)$$

is the traceless part of the momentum. Equation (2.9) means that $\tilde{\pi}^{ij}$ is transverse, or, in complex coordinates, that it is a holomorphic quadratic differential on Σ.

To solve the remaining Hamiltonian constraint $\mathcal{H}=0$, Moncrief observes that any metric on a surface Σ of genus h is conformal to one of constant (intrinsic) curvature k, where $k=-1$ if $h>1$, $k=0$ if $h=1$, and $k=1$ if $h=0$. If we write

$$g_{ij}=e^{2\lambda}\tilde{g}_{ij}\ , \qquad (2.11)$$

where $\tilde{g}_{ij}$ is such a constant curvature metric, the Hamiltonian constraint becomes a differential equation for λ:

$$\Delta_{\tilde{g}}\lambda-\tfrac{1}{4}\tau^2e^{2\lambda}+\tfrac{1}{2}(\tilde{g}^{-1}\tilde{g}_{ij}\tilde{g}_{kl}\tilde{\pi}^{ik}\tilde{\pi}^{jl})e^{-2\lambda}-\frac{k}{2}=0\ . \qquad (2.12)$$

This equation has a unique solution, completely determining λ as a function of $\tilde{g}$ and $\tilde{\pi}$.

It is well known that up to spatial diffeomorphisms, the constant curvature metrics $\tilde{g}$ parametrize the moduli space $\mathcal{M}$ of Σ. Similarly, the quadratic differentials $\tilde{\pi}$ at $\tilde{g}$ parametrize the cotangent space to moduli space.[22] If we choose a set of coordinates $\{m_\alpha\}$ for $\mathcal{M}$, and define conjugate momenta by

$$p^\alpha=\int_\Sigma d^2x\,e^{2\lambda}\left[\tilde{\pi}^{ij}\frac{\partial}{\partial m_\alpha}\tilde{g}_{ij}\right]\ , \qquad (2.13)$$

the action (2.2) reduces to

$$S=\int d\tau\left[p^\alpha\frac{dm_\alpha}{d\tau}-H(m,p)\right]\ , \qquad (2.14)$$

where

$$H=\int_\Sigma d^2x\sqrt{\tilde{g}}\,e^{2\lambda(m,p,\tau)} \qquad (2.15)$$

with λ fixed by (2.12). The physical phase space is thus the cotangent bundle $T^*\mathcal{M}$, with coordinates m_α and p^α and dynamics determined by the Hamiltonian (2.15).

In principle, quantization is now straightforward. We promote the m_α and p^α to operators on the Hilbert space of square-integrable functions on $\mathcal{M}$, with canonical commutation relations

$$[\hat{m}_\alpha,\hat{p}^\beta]=i\delta_\alpha^\beta\ . \qquad (2.16)$$

The result is a quantum-mechanical system with a simple space of states, but with highly nontrivial dynamics. In practice, the Hamiltonian (2.15) will usually be a complicated nonpolynomial function of both coordinates and momenta, and there will be difficult operator-ordering problems. It is worth emphasizing, however, that the appearance of a nonzero Hamiltonian does not contradict the usual assertion that the gravitational Hamiltonian is a constraint. We have fixed a gauge and chosen a particular definition of time; the Hamiltonian (2.15) describes the evolution of this gauge choice as τ changes.

B. Witten's quantization

Witten's alternative approach to quantizing (2+1)-dimensional gravity starts with the first-order action (2.6). He observes that the constraints Θ^a and $\tilde{\Theta}^a$ generate the Lie algebra ISO(2,1), and that the frame $e^a{}_i$ and spin connection ω_{ai} together constitute an ISO(2,1) connection on Σ. Θ^a and $\tilde{\Theta}^a$ are the curvatures of this connection, and the conditions $\tilde{\Theta}^a=\Theta^a=0$ require the connection to be flat. At the same time, the constraints generate local SO(2,1) gauge transformations, thus requiring us to identify gauge-equivalent connections. The physical phase space is therefore the moduli space of equivalence classes of flat ISO(2,1) connections on Σ.

Like Moncrief's phase space, this moduli space is a cotangent bundle, in which SO(2,1) connections ω parametrize the base space and frames e which satisfy the constraints are cotangent vectors. Indeed, if we consider an infinitesimal variation $\omega=\omega_0+\delta\omega$ of a flat SO(2,1) connection, the constraints $\Theta^a=0$ for ω imply the remaining constraints $\tilde{\Theta}^a[\omega_0, e=\delta\omega]=0$ for the cotangent vector $\delta\omega$. The phase space is thus $T^*\mathcal{N}$, where $\mathcal{N}$ is the moduli space of flat SO(2,1) connections on Σ.

More concretely, a flat connection is determined, uniquely up to gauge transformations, by its holonomies around the nontrivial loops in Σ, that is, by a group homomorphism $\pi_1(\Sigma,*)\rightarrow SO(2,1)$. Under a gauge transformation $\mathcal{G}$: $\Sigma\rightarrow SO(2,1)$, holonomies are conjugated by $\mathcal{G}(*)$. Hence

$$\mathcal{N}=\mathrm{Hom}(\pi_1(\Sigma,*),SO(2,1))/\sim \tag{2.17}$$

where

$$f_1\sim f_2 \quad \text{if } f_2=h\cdot f_1\cdot h^{-1}, \quad h\in SO(2,1) \ . \tag{2.18}$$

This is not quite the whole story, however. Witten has shown that the constraints $\tilde{\Theta}^a$ generate transformations equivalent to those diffeomorphisms which can be deformed to the identity. But the gauge group of gravity is the full group of diffeomorphisms, including those which are not isotopic to the identity.[23] The group of equivalence classes of diffeomorphisms not isotopic to the identity—that is, the mapping class group $\mathcal{D}$ of Σ—acts on $\pi_1(\Sigma,*)$, and this action must still be factored from (2.17), leaving a physical phase space $T^*\mathcal{N}/\mathcal{D}$.

Once again, quantization is straightforward. Our Hilbert space is the space of L^2 functions on $\mathcal{N}/\mathcal{D}$. We can take as coordinates for $\mathcal{N}/\mathcal{D}$ a set of independent SO(2,1) holonomies $\tilde{m}_\alpha$ (the tilde distinguishes these coordinates from Moncrief's), which determine ω_{ai} up to gauge transformations. As in (2.13), we define canonical momenta

$$\tilde{p}^{\,\alpha}=-2\int_\Sigma d^2x\,\epsilon^{ij}e^a{}_i\frac{\partial}{\partial\tilde{m}_\alpha}\omega_{aj} \tag{2.19}$$

The action is then simply

$$S=\int dt\,\tilde{p}^{\,\alpha}\frac{d\tilde{m}_\alpha}{dt} \ . \tag{2.20}$$

Alternatively, we can define operators corresponding to the full ISO(2,1) holonomies of the connection $\{\omega_{ai},e^a{}_i\}$, involving both coordinates and momenta. Their commutation relations can be derived from the commutator

$$[\omega_{ai}(x),e^b{}_j(x')]=-\frac{i}{2}\delta^b_a\epsilon_{ij}\delta^2(x'-x) \tag{2.21}$$

and have been studied by Martin[24] and Nelson and Regge.[25]

At first sight, Witten's Hilbert space $L^2(\mathcal{N}/\mathcal{D})$ and Moncrief's Hilbert space $L^2(\mathcal{M})$ appear to be quite different. In fact, they are closely related. The moduli space $\mathcal{N}$ of flat SO(2,1) connections on Σ is not connected. One component $\mathcal{N}_0$ corresponds in (2.17) to the discrete embeddings of $\pi_1(\Sigma)$ into SO(2,1), and it can be shown that this component is homeomorphic to the Teichmüller space of Σ.[26–28] Moreover, the action of $\mathcal{D}$ on $\mathcal{N}_0$ is precisely the ordinary action of the mapping class group on Teichmüller space, so $\mathcal{N}_0/\mathcal{D}$ is the moduli space $\mathcal{M}$ of Σ.

We must still worry about the remaining components of $\mathcal{N}$, those which are not homeomorphic to any Teichmüller space. In principle, their existence implies that Witten's Hilbert space is larger than Moncrief's. The difference is probably not important, however. It is believed that the mapping class acts ergodically on these remaining components,[30] which implies, for instance, that any new operators on the Hilbert space $L^2(\mathcal{N}/\mathcal{D})$ would have to be constant.

But while the Witten and Moncrief Hilbert spaces are identical, the two descriptions of dynamics seem completely different. Moncrief finds a nontrivial, time-dependent, and generally very complicated Hamiltonian. Witten's Hamiltonian is zero, his states are manifestly diffeomorphism invariant, and the dynamics seems to have disappeared. Moreover, in Moncrief's quantization inner products are defined at fixed τ, while in Witten's quantization no such time dependence appears. It is far from obvious that these theories should describe the same physical system.

III. CLASSICAL EQUIVALENCE

To solve this paradox, let us begin by comparing the classical theories based on the actions (2.2) and (2.6). For simplicity, we first consider the simplest nontrivial spatial topology, that of a torus T^2.[31] Hosoya and Nakao have studied the evolution of the torus in the ADM formalism with the time variable τ. As expected from the general analysis, they find a two-dimensional configuration space homeomorphic to the usual torus moduli space. This moduli space can be parametrized by a single complex number $m=m_1+im_2$ lying in the upper half-plane; the mapping class group is generated by the two transformations

$$S: m\rightarrow -\frac{1}{m}, \quad T: m\rightarrow m+1 \ , \tag{3.1}$$

and a fundamental region is given by $-\frac{1}{2}\le m\le\frac{1}{2}$, $|m|\ge 1$.

Hosoya and Nakao find a nontrivial dynamics for m: $m(\tau)$ describes a semicircle centered on the real axis, i.e., a geodesic in moduli space with respect to the usual Poincaré metric for the upper half-plane. This result is not hard to derive from the action (2.14); for the torus, $\tilde{g}$ and $\tilde{\pi}$ are independent of x, and the unique solution of (2.12) is

$$\sqrt{\tilde{g}}\,e^{2\lambda}=\frac{1}{\tau}(2\tilde{g}_{ij}\tilde{g}_{kl}\tilde{\pi}^{ik}\tilde{\pi}^{jl})^{1/2}\ , \tag{3.2}$$

also independent of x, so the equations of motion become fairly simple.

A. Classical solutions from holonomies

To compare these results to Witten's formalism, we must first find the possible ISO(2,1) holonomies for the torus. The fundamental group of the torus is $\mathbb{Z}\oplus\mathbb{Z}$, so the physical phase space is parametrized by two commuting elements of ISO(2,1), that is, two commuting (2+1)-dimensional Poincaré transformations, up to overall conjugation. The SO(2,1) projections of these transformations stabilize either a null vector, a timelike vector, or a spacelike vector, but only in the latter case is the mapping from $\pi_1(T^2)$ into SO(2,1) a discrete embedding. For that case, we can conjugate the two elements of ISO(2,1) to the form

$$\begin{aligned}\Lambda_1&: (t,x,y)\to(t\cosh\lambda+x\sinh\lambda,x\cosh\lambda+t\sinh\lambda,y+a)\ ,\\ \Lambda_2&: (t,x,y)\to(t\cosh\mu+x\sinh\mu,x\cosh\mu+t\sinh\mu,y+b)\ .\end{aligned} \tag{3.3}$$

It is easy enough to write down a flat connection with these holonomies: for instance,

$$e^{(2)}=a\,dx+b\,dy,\quad \omega^{(2)}=\lambda\,dx+\mu\,dy\ , \tag{3.4}$$

where x and y are periodic coordinates with period 1. The metric coming from this choice of local frame is singular, however; we should instead find a gauge-equivalent connection for which $\det e^a{}_\mu\neq0$. Such a connection is

$$\begin{aligned}&e^{(0)}=-\dot\beta(t)dt\ , && \omega^{(0)}=0\ ,\\ &e^{(1)}=\beta(t)(\lambda\,dx+\mu\,dy)\ , && \omega^{(1)}=0\ ,\\ &e^{(2)}=a\,dx+b\,dy\ , && \omega^{(2)}=\lambda\,dx+\mu\,dy\ ,\end{aligned} \tag{3.5}$$

where $\beta(t)$ is an arbitrary function of time. The metric is then[32]

$$\begin{aligned}ds^2=\dot\beta^2dt^2&-(a^2+\beta^2\lambda^2)dx^2\\ &-2(ab+\beta^2\lambda\mu)dx\,dy-(b^2+\beta^2\mu^2)dy^2\ ,\end{aligned} \tag{3.6}$$

and the mean curvature of a slice of constant t is

$$K=\frac{1}{2N}g^{ij}\dot g_{ij}=\frac{1}{\beta}\ , \tag{3.7}$$

so to reach Moncrief's coordinates, we should choose

$$\beta=\frac{1}{\tau}\ . \tag{3.8}$$

At fixed τ, the spatial part of the metric (3.6) can be diagonalized by the transformations

$$\begin{aligned}x'&=x+\left[a^2+\frac{\lambda^2}{\tau^2}\right]^{-1}\left[ab+\frac{\lambda\mu}{\tau^2}\right]y\ ,\\ y'&=\left[a^2+\frac{\lambda^2}{\tau^2}\right]^{-1}\frac{a\mu-\lambda b}{\tau}y\ ,\end{aligned} \tag{3.9}$$

giving

$$d\sigma^2=\left[a^2+\frac{\lambda^2}{\tau^2}\right](dx'^2+dy'^2)\ , \tag{3.10}$$

which is periodic under the shifts

$$\begin{aligned}&(x',y')\to(x'+1,y')\ ,\\ &(x',y')\to\left[x'+\left[a^2+\frac{\lambda^2}{\tau^2}\right]^{-1}\left[ab+\frac{\lambda\mu}{\tau^2}\right],\ y'+\left[a^2+\frac{\lambda^2}{\tau^2}\right]^{-1}\frac{a\mu-\lambda b}{\tau}\right]\ .\end{aligned} \tag{3.11}$$

From the definition of the modulus m, $d\sigma^2$ is therefore the metric of a torus with

$$\begin{aligned}m_1&=\left[a^2+\frac{\lambda^2}{\tau^2}\right]^{-1}\left[ab+\frac{\lambda\mu}{\tau^2}\right]\ ,\\ m_2&=\left[a^2+\frac{\lambda^2}{\tau^2}\right]^{-1}\frac{a\mu-\lambda b}{\tau}\ ,\end{aligned} \tag{3.12}$$

i.e.,

$$m=m_1+im_2=\left[a+\frac{i\lambda}{\tau}\right]^{-1}\left[b+\frac{i\mu}{\tau}\right]\ . \tag{3.13}$$

We will later need the momenta conjugate to $\{m_1,m_2\}$, which can be derived from the definition (2.13), using (2.3) and (2.4). We find that

$$p^1=-2a\lambda,\quad p^2=-\tau\left[a^2-\frac{\lambda^2}{\tau^2}\right]\ , \tag{3.14}$$

i.e.,

$$p=p_1+ip_2=-i\tau\left[a-\frac{i\lambda}{\tau}\right]^2\ . \tag{3.15}$$

The modulus m and its conjugate p have precisely the properties we desire. In particular, a simple calculation shows that

$$\left[m_1-\tfrac{1}{2}\left[\frac{\mu}{\lambda}+\frac{b}{a}\right]\right]^2+m_2{}^2=\tfrac{1}{4}\left[\frac{\mu}{\lambda}-\frac{b}{a}\right]^2\ , \tag{3.16}$$

so m describes a semicircle, reproducing the results of Hosoya and Nakao. Further, under the modular transformations of the holonomies,

$$\begin{aligned}S&: (a,\lambda)\to(b,\mu),\quad (b,\mu)\to(-a,-\lambda)\\ T&: (a,\lambda)\to(a,\lambda),\quad (b,\mu)\to(b+a,\mu+\lambda)\ ,\end{aligned} \tag{3.17}$$

m transforms according to (3.1); moreover, the momentum p is fixed by T and transforms as $p\to\bar m^2p$ under S, so the symplectic form $dm\,d\bar p+d\bar m\,dp$ is invariant.

Witten's coordinates and momenta $\{a,b,\lambda,\mu\}$ can thus be viewed as parametrizing Moncrief's phase space $\{m,p\}$. Indeed, the two sets of variables are equivalent under a (time-dependent) canonical transformation

$$p_1dm_1+p_2dm_2=2a\,d\mu-2b\,d\lambda+H\,d\tau+dF\ , \tag{3.18}$$

where

$$H=\sqrt{g}=\frac{a\mu-\lambda b}{\tau} \tag{3.19}$$

is Moncrief's Hamiltonian and

$$F(m_1,m_2,\mu,\lambda)=-\frac{1}{m_2\tau}[(\mu-m_1\lambda)^2+m_2{}^2\lambda^2] \tag{3.20}$$

generates the transformation. The passage from Moncrief's to Witten's variables thus represents a standard procedure in classical mechanics: we solve the equations of motion by finding a time-dependent canonical transformation to new time-independent coordinates and momenta.

B. Classical solutions as quotient spaces

It is instructive to consider an alternative derivation of the metric (3.6). As elements of the Poincaré group, the holonomies Λ_1 and Λ_2 are isometries of the Minkowski metric. Together, they generate a subgroup H of ISO(2,1). If we can find a fundamental region $\mathcal{F}$ of Minkowski space upon which the action of H is properly discontinuous, the metric and spin connection induced on the quotient $\mathcal{F}/H$ will have exactly the right holonomies. This is a variation of the construction described by Deser, Jackiw, and 't Hooft[21] for point particles in $2+1$ dimensions, in which elements of the Poincaré group are used to glue together flat coordinate patches to form conical spacetimes.

To find $\mathcal{F}$, let us define new coordinates

$$t=\frac{1}{\tau}\cosh u,\quad x=\frac{1}{\tau}\sinh u\ . \tag{3.21}$$

The Minkowski metric is then

$$ds^2=\frac{1}{\tau^4}d\tau^2-\frac{1}{\tau^2}du^2-dy^2 \tag{3.22}$$

and a simple calculation shows that the surfaces of constant τ have mean curvature τ, so τ is Moncrief's time coordinate. In the new coordinates, the transformations (3.3) are

$$\begin{aligned}&\Lambda_1\colon (\tau,u,y)\to(\tau,u+\lambda,y+a)\ ,\\ &\Lambda_2\colon (\tau,u,y)\to(\tau,u+\mu,y+b)\ .\end{aligned} \tag{3.23}$$

Hence on a surface of constant τ, a fundamental region $\mathcal{F}$ for the action of $\langle\Lambda_1,\Lambda_2\rangle$ is simply the torus $(u,y)\sim(u+\lambda,y+a)\sim(u+\mu,y+b)$. To put this in a more standard form, we can define new spatial coordinates (at fixed τ)

$$\begin{aligned}x'&=\left[a^2+\frac{\lambda^2}{\tau^2}\right]^{-1}\left[ay+\frac{\lambda}{\tau^2}u\right],\\ y'&=\left[a^2+\frac{\lambda^2}{\tau^2}\right]^{-1}\left[\frac{\lambda y-au}{\tau}\right].\end{aligned} \tag{3.24}$$

It is easily checked that the spatial metric is again (3.10), with the periodicities (3.11)

The advantage of this approach is that it may be generalized to surfaces of arbitrary genus, allowing powerful techniques of hyperbolic geometry to be brought to bear. Using such techniques, Mess[33] has shown that for any topology of Σ, each equivalence class of discrete embeddings $\pi_1(\Sigma)\to$ ISO(2,1) gives rise to a unique maximal Einstein spacetime. While the metric of such a spacetime cannot ordinarily be written explicitly, many of its properties, such as the existence of an initial or final singularity and the behavior near that singularity, can be studied.

IV. COMPARING QUANTUM THEORIES

It remains for us to see whether the classical equivalence of these two approaches to $(2+1)$-dimensional gravity extends to an equivalence of quantum theories. To investigate this question, let us begin with Witten's Hilbert space, and try to define operators $\hat{m}$ and $\hat{p}$ to represent Moncrief's coordinates and momenta.

Witten's Hilbert space is characterized by self-adjoint operators $\hat{a}$, $\hat{b}$, $\hat{\lambda}$, and $\hat{\mu}$, with commutators, derived from (2.21), of the form

$$[\hat{a},\hat{\mu}]=[\hat{\lambda},\hat{b}]=\frac{i}{2}\ . \tag{4.1}$$

We can try to construct operators $\hat{m}_1$, $\hat{m}_2$, $\hat{p}_1$, and $\hat{p}_2$ by replacing the holonomies in the classical expressions (3.12) and (3.14) with the corresponding operators. The results are not self-adjoint, however, and the reordering of operators to make $\hat{m}$ and $\hat{p}$ self-adjoint is not unique; we need some principle to fix an operator ordering.

Modular invariance provides such a principle. If the two theories are to be equivalent, the modular transformations (3.17) of the holonomies must induce modular transformations (3.1) of $\hat{m}_1$ and $\hat{m}_2$. To achieve this, we can start with the complex expressions (3.13) and (3.15), and define

$$\begin{aligned}\hat{m}&=\left[\hat{a}+\frac{i\hat{\lambda}}{\tau}\right]^{-1}\left[\hat{b}+\frac{i\hat{\mu}}{\tau}\right],\\ \hat{p}&=-i\tau\left[\hat{a}-\frac{i\hat{\lambda}}{\tau}\right]^2,\end{aligned} \tag{4.2}$$

which are easily seen to transform correctly. $\hat{m}$ and $\hat{p}$ are not self-adjoint, of course, but they should not be, since the classical variables m and p are not real; the "real" and "imaginary" parts

$$\begin{aligned}&\hat{m}_1=\tfrac{1}{2}(\hat{m}+\hat{m}^\dagger),\quad \hat{p}_1=\tfrac{1}{2}(\hat{p}+\hat{p}^\dagger)\ ,\\ &\hat{m}_2=\frac{1}{2i}(\hat{m}-\hat{m}^\dagger),\quad \hat{p}_2=\frac{1}{2i}(\hat{p}-\hat{p}^\dagger)\ ,\end{aligned} \tag{4.3}$$

are manifestly self-adjoint.

It follows from (4.1) that $\hat{m}$ and $\hat{p}$ satisfy the standard canonical commutation relations

$$[\hat{m}_1,\hat{p}_1]=[\hat{m}_2,\hat{p}_2]=i \tag{4.4}$$

with all other commutators vanishing. Moreover, if we define the Hamiltonian operator

$$\hat{H}=\frac{\hat{a}\,\hat{\mu}-\hat{\lambda}\,\hat{b}}{\tau} \tag{4.5}$$

we find that

$$i\frac{d\hat{m}}{d\tau}=[\hat{m},\hat{H}],\quad i\frac{d\hat{p}}{d\tau}=[\hat{p},\hat{H}]\ . \tag{4.6}$$

Moncrief's moduli $\hat{m}$ and their conjugate momenta can therefore be viewed as "time"-dependent operators in Witten's Hilbert space, satisfying the appropriate Heisenberg equations of motion. Alternatively, we can pass to a Schrödinger picture by diagonalizing $\hat{m}_1(\tau)$ and $\hat{m}_2(\tau)$; the equations of motion (4.6) then guarantee that states built from linear combinations of such basis elements with τ-independent coefficients will satisfy the gravitational Schrödinger equation. Observe also that the time-independent inner product of Witten's Hilbert space is equivalent to the inner product of such Schrödinger wave functions at fixed τ. Inner products and norms can thus be defined without ever referring to the choice of time variable.

From this point of view, τ is simply a new parameter introduced into Witten's quantization to label certain operators and states. No new physics is added in this process. "Time" can be "measured" only by solving for τ in terms of the moduli, and this is equivalent to eliminating τ and looking at correlations among operators built purely out of holonomies. Nevertheless, τ is a *useful* parameter. Its physical interpretation comes from examining the behavior of solutions; for instance, by constructing wave packets which approximate classical solutions, we can recover the interpretation of τ as a mean curvature. This viewpoint is close to that of Rovelli,[34] who argues that quantum mechanics can be defined in terms of variables which are constant on each classical trajectory, without any explicit reference to time; a similar approach follows from the covariant phase-space methods of Ashtekar and Magnon[35] and Crnkovic and Witten.[36]

V. TIME-DEPENDENT OPERATORS

On a two-torus, of course, gravity is simple. If we try to generalize these arguments to more complicated topologies, we quickly encounter technical difficulties. For higher-genus surfaces, no explicit representation of the metric analogous to (3.6) is known; the work of Mess on the construction of metrics from holonomies is the closest thing we have to an exact characterization of the classical phase space. Nor is the solution of Moncrief's Hamiltonian constraint (2.12) known, although it is known to exist classically.

Nevertheless, we can try to generalize the construction of τ-dependent operators to more complicated topologies. The key ingredient in the construction of the moduli $\hat{m}$ and $\hat{p}$ was the identification of the τ dependence of the metric (3.6). Although this identification was phrased as a coordinate choice, it can be expressed more invariantly. As Tsamis and Woodard have stressed,[37] a quantity defined in a particular gauge or coordinate system can always be "invariantized" and viewed as a (usually nonlocal) gauge-invariant object. In our case, τ can be defined as follows: it is the value of the scalar function $T(x)$ which is given at x by the trace of the extrinsic curvature of the unique surface of constant mean curvature through x.

This definition of $T(x)$ holds for a spacetime of arbitrary spatial topology, and takes a rather simple form in terms of Witten's variables. If we define one-forms $e^a=e^a{}_\mu dx^\mu$ and $\omega^a=\omega^a{}_\mu dx^\mu$, $T(x)$ satisfies

$$dT=\rho n_a e^a,\quad \epsilon^{abc}e_a e_b(Dn_c-\tfrac{1}{3}Te_c)=0\ , \tag{5.1}$$

where D is the covariant exterior derivative,

$$Dn^a=dn^a+\epsilon^{abc}\omega_b n_c\ , \tag{5.2}$$

and the zero-form ρ is determined by the condition that n_a have unit norm:

$$\rho=({}^{(3)}g^{\mu\nu}\partial_\mu T\partial_\nu T)^{1/2}\ . \tag{5.3}$$

Given the function $T(x)$, we can define the Hamiltonian H_τ as the area of the surface $T(x)=\tau$. The area element $d\mu$ is determined by the condition

$$(n_a e^a)d\mu=\rho^{-1}dT\,d\mu=dV\ , \tag{5.4}$$

where $dV=(\det e^a{}_\mu)d^3x$ is the spacetime volume element, so

$$H_\tau=\int_{T=\tau}d\mu=\int dV\rho\delta(T-\tau)\ . \tag{5.5}$$

Similarly, for any two-form α which commutes with H_τ, we can define a "time"-dependent operator

$$A(\tau)=\int_{T=\tau}\alpha=\int\alpha\,dT\,\delta(T-\tau)\ . \tag{5.6}$$

It may now be checked that such an operator obeys the Heisenberg equations of motion, at least up to possible ambiguities in operator ordering. We need one basic commutator,

$$[H_\tau,T(x)]\delta(T(x)-\tau)=i\delta(T(x)-\tau)\ , \tag{5.7}$$

which follows from quite general considerations: for any spacelike surface, the extrinsic curvature and the area element are always canonically conjugate. This result can also be verified directly from the definitions (5.1) and (5.5) and the commutator (2.21). We then have

$$\begin{aligned}[A(\tau),H_\tau]&=\int\alpha(x)[dT(x)\delta(T(x)-\tau),H_\tau]\\&=\int\alpha(x)\{d([T(x),H_\tau])\delta(T(x)-\tau)+dT(x)[\delta(T(x)-\tau),H_\tau]\}\\&=\int\alpha(x)d\{[T(x),H_\tau]\delta(T(x)-\tau)\}+\int\alpha(x)\{dT(x)[\delta(T(x)-\tau),H_\tau]-[T(x),H_\tau]d\delta(T(x)-\tau)\}\ .\end{aligned} \tag{5.8}$$

The first term in (5.8) is

$$\int \alpha(x) d\{[T(x),H_\tau]\delta(T(x)-\tau)\} = i\int d\alpha(x)\delta(T(x)-\tau) \ . \qquad (5.9)$$

In the classical limit, in which we replace commutators with Poisson brackets and operators with commuting classical observables, this becomes

$$-i\int \alpha(x) dT(x)\delta'(T(x)-\tau) = i\frac{d}{d\tau}A(\tau) \ . \qquad (5.10)$$

The second term in (5.8) vanishes in the classical limit, since in that limit

$$\begin{aligned} dT(x)[\delta(T(x)-\tau),H_\tau] &= dT(x)\delta'(T(x)-\tau) \\ &\quad \times [T(x),H_\tau] \\ &= d\delta(T(x)-\tau)[T(x),H_\tau] \ . \end{aligned} \qquad (5.11)$$

The quantum analog of these relations will depend on the operator ordering in $T(x)$ and H_τ, but if we can find a suitable ordering, $A(\tau)$ will obey the Heisenberg equations of motion. In particular, this means that the problem of finding operators to represent Moncrief's moduli for an arbitrary spatial topology reduces to one of finding a complete set of commuting operator-valued two-forms $\alpha(x)$ which also commute with H_τ. In this regard, it is worth noting that the torus modulus (4.2) can be written without any explicit τ dependence, since, by (3.5),

$$\begin{aligned} a+\frac{i\lambda}{\tau} &= (e^{(2)}+ie^{(1)})_x \ , \\ b+\frac{i\mu}{\tau} &= (e^{(2)}+ie^{(1)})_y \ . \end{aligned} \qquad (5.12)$$

VI. DISCUSSION

We have seen that, up to possible problems of operator ordering for genus greater than one, the time-dependent quantization of (2+1)-dimensional gravity proposed by Moncrief and Hosoya and Nakao is equivalent to Witten's time-independent quantization. Despite the fact that Witten's Hamiltonian vanishes, "time"-dependent operators in his Hilbert space can be constructed, and dynamical questions can be asked and answered. Further, at least for the case of the torus, where the Hilbert spaces can be written down in detail, the natural time-independent inner product in Witten's variables is equivalent to the fixed-time inner product of Schrödinger wave functions in Moncrief's variables. No modification of general relativity is needed in order to formulate a sensible quantum theory.

The important question, of course, is whether these conclusions can be generalized to 3+1 dimensions. The details certainly cannot. We have no explicit characterization of the space of solutions of the constraints, and hence cannot work directly on the physical phase space. In particular, holonomy variables cannot be used to parametrize solutions of the constraints, although the loop variables of Rovelli and Smolin[38] may come close. Furthermore, operator ordering, already a problem in 2+1 dimensions, becomes much more difficult when the physical phase space is infinite dimensional.

Nonetheless, some of the lessons of (2+1)-dimensions should still apply. We have seen that dynamics can be described even when the Hamiltonian vanishes, and that an inner product can be defined without reference to time. The problem of time does not force us to reformulate general relativity or quantum mechanics; there is no need to alter classical gravity to introduce a physical time variable, and "third quantization" of the Wheeler-DeWitt equation, while an interesting idea, is not required. The (2+1)-dimensional model strongly suggests that while the problem of time is a difficult technical problem—that of solving the constraints—it need not be anything more.

Note added. After completing this paper, I received a paper from Moncrief[39] which advocates a related but somewhat different view of the relationship between holonomy and ADM variables.

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