Quantum mechanics without time: A model

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The quantization of a simple dynamical system in which a unitary time evolution appears only within a certain approximation is studied in detail. The probabilistic interpretation of quantum mechanics in the regimes in which time is not defined is discussed and shown to be consistent.

I. INTRODUCTION

The idea that quantum mechanics is well defined even if time evolution is an approximate concept, which emerges within a certain approximation, has received a certain amount of attention.[1] The motivations for considering quantum systems without time come from research in quantum gravity and in nonperturbative string theory. Even though a complete quantum theory of the gravitational field is still lacking, there are certain inferences that can be made just from the basic principles of quantum theory and of gravitational physics (general covariance). Among these inferences, one of the most interesting is the possibility that at a fundamental level time is, in a precise technical sense, not well defined. We will recall the precise statement of this hypothesis and its motivations in the next section.

If time is not well defined at the fundamental level, then the following natural question arises: Does the absence of time imply that standard quantum mechanics cannot hold at the fundamental level, or does quantum mechanics still hold even in the absence of a well-defined time evolution?

The idea that standard quantum mechanics can be directly applied to physics without time evolution has been considered and analyzed in detail in Ref. 2. In this paper we consider a simple model in which this idea may be implemented. This application is meant to give concreteness to the theoretical discussion in Ref. 2. The model we consider here is complex enough not to be trivial (in particular, it is not linear[3]); but it is simple enough to be completely solved (and in more than one way) both classically and quantum mechanically. This allows us to discuss the features of the formalism in detail, and, what is crucial, the possibility of giving a completely coherent interpretation to the formalism (in the sense of quantum mechanics).

We will show that a quantum-mechanical system can be constructed and that the system admits a consistent interpretation even if time is not well defined. Moreover, we will show a way in which a standard time evolution may appear within a certain approximation of the theory.

The paper is organized as follows. Section II recalls what we mean by a dynamical system with no time, and the reasons for considering these systems. The model is introduced in Sec. III and quantized in Sec. IV. In Sec. V we discuss how a time evolution appears in a certain approximation, and in Sec. VI we discuss the interpretation of the system beyond this approximation. Section VII contains the conclusions and some relevant considerations.

II. DYNAMICAL SYSTEMS WITHOUT TIME: DEFINITION AND MOTIVATIONS

A Hamiltonian dynamical system is defined by a phase space S, which is a symplectic manifold, and by a Hamiltonian H, which is a smooth function on S. If σ is the symplectic two-form on S (which defines the Poisson brackets structure), then the dynamics is given by the Hamiltonian equations

$$i_X\sigma = -dH \tag{1}$$

(below, we translate in coordinate language), where the motions $s(t)$ in S are given by the integral lines of the vector field X:

$$X = \frac{\partial}{\partial t} . \tag{2}$$

Many dynamical systems considered in fundamental physics can be put in this form, but not all of them. Certain systems admit only a slightly more general formulation, called presymplectic formulation. Any Hamiltonian system can be put in presymplectic form as follows. Let R be the real line and t (the time) be in R. Then the manifold $P = S \times R$, equipped with the two-form

$$\pi = \sigma - dH \wedge dt , \tag{3}$$

is a presymplectic manifold, and the Hamiltonian equations (1) can be expressed on S by

$$i_Y\pi = 0 . \tag{4}$$

The relation is given by the fact that the integral curves of y in P are the graphs of the motions $s(t)$ defined by the integral curves of X in S by Eq. (2). Note that Y is defined by Eq. (4) only up to rescaling, and therefore its integral lines are defined only up to reparametrization, which of course does not affect the corresponding motions $s(t)$.

Now, there are in physics certain dynamical systems that admit a presymplectic formulation, but not a Hamiltonian (symplectic) formulation. We call these systems *dynamical systems without time*. General relativity, as well as any generally covariant (nonperturbative) version of string theory, are, most likely, systems of this kind. [More precisely, a Hamiltonian formulation of general relativity (in a strict sense) is not known.]

In coordinate language, if q_i and p^i are canonical coordinates, then the presymplectic formulation is given in terms of the coordinates (q_i, p^i, t). The presymplectic form (3) is

$$\pi = dp^i \wedge dq_i - H(p,q) \wedge dt \tag{5}$$

and Eq. (4) defines an evolution in a fictitious time parameter, generally denoted τ:

$$\frac{dq_i}{d\tau} = \frac{dt}{d\tau}\frac{\partial H}{\partial p^i}, \quad \frac{dp^i}{d\tau} = -\frac{dt}{d\tau}\frac{\partial H}{\partial q_i} . \tag{6}$$

These equations are invariant under reparametrization of τ. The physical motions $q(t)$ are obtained from the solutions of the equations (6) by eliminating τ, namely, by inverting $t(\tau)$ and by

$$q_i(t) \equiv q_i(\tau(t)) . \tag{7}$$

A common form in which presymplectic systems appear in fundamental physics is as constrained Hamiltonian systems with vanishing canonical Hamiltonian. These systems correspond to reparametrization-invariant Lagrangian systems. The constraint surface of these systems, supplemented by the restriction of the symplectic form on it, is the presymplectic space.

One reason for using the presymplectic formulation, for example, is that it provides the only form of a manifestly Lorentz-invariant canonical theory. Indeed, any Hamiltonian formulation always corresponds to a specific Lorentz time, and therefore breaks Lorentz invariance. A presymplectic formulation, on the contrary, may correspond to different Hamiltonian formulations, one for every Lorentz time. For instance, a free particle is represented on the phase space with canonical coordinates, x^μ and p_μ by the constraint

$$C = p_\mu p^\mu - m^2 \sim 0 . \tag{8}$$

The constraint surface is the presymplectic manifold, and the orbits defined by the integral lines of the induced presymplectic two-forms are of course

$$x^\mu = p^\mu \tau + x_0^\mu . \tag{9}$$

One may identify the Hamiltonian time t of Eq. (5) with x^0,

$$t = x^0 , \tag{10}$$

and obtain a Hamiltonian formulation; but one can also identify t with a different Lorentz time, say,

$$t = \left[1 - \frac{v^2}{c^2} \right]^{-1/2} x^0 + \frac{v}{c^2} x^1 , \tag{11}$$

and obtain a different (equivalent) Hamiltonian system that represents the same physics in terms of a different *choice of time*. (In this case, of course, different times are measured by the clocks of observers in different inertial motions.) Thus, the relativistic particle dynamics admits many equivalent nonmanifestly Lorentz-invariant Hamiltonian formulations and one Lorentz-invariant presymplectic formulation.

As we said, there are systems that can be formulated in the presymplectic formulation, but which do not have any Hamiltonian formulation. Among these a prominent one is general relativity. The canonical formulation of classical general relativity can be given by the Arnowitt-Deser-Misner (ADM) constraints surface equipped with the induced presymplectic two-form. This formulation is perfectly generally covariant. [It is often stated that the ADM formulation breaks general covariance, or breaks the relativistic symmetry between time and space. This is wrong. It is a consequence of the unfortunate original form in which the ADM canonical theory was developed. The phase space of general relativity (and its structures) can be introduced as the space of the solutions of Einstein equations, without any reference to spacelike hypersurfaces or something similar.[4]] The absence of a preferred Hamiltonian formulation reflects the fact that there is no perferred choice of time in a generally covariant theory.

Moreover, there are reasons to believe that there is no Hamiltonian formulation of general relativity. This is suggested first by the fact that no such formulation has been found up to now, and second by the fact that cosmological models tend to have this property.

Physically, nothing really strange is going on. Suppose, as an example, that the geometry of the Universe had just one degree of freedom, its radius R, and that the only matter field was a constant scalar field ϕ. Suppose the dynamics is such that, for every classical trajectory, the value of the scalar field is bounded (oscillates) and the value of the radius reaches a maximum and then decreases, so that any classical trajectory turns out to be closed. Then the evolution cannot be described by a Hamiltonian system, essentially because neither the field ϕ nor the radius R can be chosen as good clocks for the entire history of the system. Thus the classical physics of these "systems with no time" presents no difficulty for the intuition.

But is there a quantum version of such a model? Note that no Schrödinger equation can hold for this system, because any quantum system described by a Schrödinger equation has a classical limit that is a Hamiltonian system.

It is an old idea that time is a property of macroscopic objects, which loses its meaning at the Planck length. (The relation between very short distances and global properties of the classical orbits will be discussed later.) General relativity teaches us that the role of the Poincaré group and the fundamental role played by the Hamiltonian are just consequences of a particular vacuum solution of the equations, and have no fundamental significance. Thus, it is natural to inquire whether quantum mechanics can deal with systems in which a Hamiltonian plays no fundamental role. It is generally accepted that, at a fundamental nonperturbative level, any theory that includes gravitation, for instance nonperturbative string theory, should be generally covariant; therefore, not only quantum general relativity, but any theory that claims to include high-energy gravitational physics, has to face problems of this type. The class of the presymplectic dynamical systems was singled out in Ref. 2 as a natural class of systems for dealing with the absence of time. Indeed, this class includes general relativity, it contains systems which may have no time, but the presymplectic systems are still close enough to Hamiltonian systems so that quantum mechanics can be applied. In the following sections we will introduce a simple presymplectic system that does not admit a Hamiltonian formulation and study its quantization.

III. THE MODEL AND ITS CLASSICAL DYNAMICS

We consider the phase space with canonical coordinates

$$q_1,q_2,p_1,p_2, \quad \{q_i,p_j\}=\delta_{ij} \tag{12}$$

and the constraint

$$C=\tfrac{1}{2}(p_1^2+p_2^2+q_1^2+q_2^2)-M\sim 0 . \tag{13}$$

The constraint surface $C=0$ is a presymplectic space. No Hamiltonian system can correspond to this dynamical system, since the presymplectic space is compact and therefore cannot contain any $S\times R$ structure. It is straightforward to integrate the equations of motion, namely, to find the integrals lines of Y [see Eq. (4)]. These are

$$q_i=a_i\sin(\tau+\phi_i) . \tag{14}$$

The orbits on the constraint surface are the ones for which

$$a_i^2+a_j^2=2M , \tag{15}$$

and since orbits with the same support describe, by assumption, the same physics, we should restrict ourselves to the orbits

$$\begin{aligned} q_1&=\sqrt{2A}\ \sin(\tau) ,\\ q_2&=\sqrt{2M-2A}\ \sin(\tau+\phi) , \end{aligned} \tag{16}$$

where the range of the integration constants is

$$0\le A\le M, \quad 0\le\phi<2\pi . \tag{17}$$

Any other orbit described by the parametric equations (14) can be obtained by a τ reparametrization. Note that for $\pi\le\phi<2\pi$ we have the same orbits as for $0\le\phi<\pi$, but with reverse orientation.

This set of orbits is given by the manifold of the ellipses that are inscribed in a rectangle with diagonal equal to M. This manifold constitutes the physical phase space $S_{\rm ph}$ of the system. For $A=0$ and $A=M$ the orbit does not depend on ϕ, so that the topology of the physical phase space is the one of a sphere.

The parameter evolution (16) is not, by itself, observable. In other words, it has no physical meaning. What does have physical meaning is, for instance, the evolution of one of the coordinates in terms of the other one. By eliminating τ we have

$$q_2(q_1)=\sqrt{M/A-1}[\cos\phi q_1\pm\sin\phi(2A-q_1^2)^{1/2}] . \tag{18}$$

The function $q_2(\cdot)$ given by Eq. (18) defines a one-parameter set of physical observables: For every real number t, $q_2(t)$ gives a classical physical observable which is interpreted as "the value of q_2 when q_1 has the value t." In other words, we may see q_1 precisely as a time variable, but with two differences. First, the choice of it (rather than any other function of q_1 and q_2) as the time variable is arbitrary; second, not likely a true time, the range of q_1 is bounded and depends on the orbit. For certain orbits the question "where is q_2 when q_1 has the value t" has no answer, because q_1 never reaches the value t in that orbit.

Two other observables are simply A and ϕ. They are given in terms of the original coordinates by

$$\begin{aligned} 4A&=2M+p_1^2-p_2^2+q_1^2-q_2^2 ,\\ \tan\phi&=\frac{p_1q_2-p_2q_1}{p_1p_2+q_2q_1} . \end{aligned} \tag{19}$$

A and ϕ (as well as, of course, any function of them) are constant along the orbits and therefore are good coordinates on the physical phase space $S_{\rm ph}$: They are physical observables. Note, in fact, that they have vanishing Poisson brackets with the constraint, as required for physical observables by the general theory of constrained systems. It is interesting to note here that also the observable $q_2(t)$, for every real t, has vanishing Poisson brackets with the constraint, and is therefore well defined on $S_{\rm ph}$ (the space of the orbits). (It is sometimes stated that if gauge invariance, as here, is related to evolution, then also non-gauge-invariant objects are physical observables. We believe that it is essentially a matter of terminology. See Ref. 2 for a detailed discussion of this point.) The Poisson brackets between A and ϕ are easily computed

$$\{\phi,A\}=1 . \tag{20}$$

In the quantum theory the operators corresponding to A and ϕ must have, up to $\hbar$ terms, this commutator algebra, and they must have a spectrum included in the rectangle (17).

It will be useful for what follows to consider also the observables

$$\begin{aligned} L_x&=\tfrac{1}{2}(p_1p_2+q_2q_1) ,\\ L_y&=\tfrac{1}{2}(p_1q_2-p_2q_1) ,\\ L_z&=\tfrac{1}{4}(p_1^2-p_2^2+q_1^2-q_2^2) . \end{aligned} \tag{21}$$

They are related by

$$L_x^2+L_y^2+L_z^2=\frac{M^2}{4} . \tag{22}$$

They are related to A and ϕ by

$$\begin{aligned} L_z&=A-\frac{M}{2} ,\\ L_x&=\sqrt{A(M-A)}\cos\phi , \end{aligned} \tag{23}$$

$$L_y=\sqrt{A(M-A)}\sin\phi\ ,$$

and their Poisson brackets are

$$\{L_x,L_y\}=L_z,\quad \{L_y,L_z\}=L_x,\quad \{L_z,L_x\}=L_y\ . \tag{24}$$

In terms of the L observables, $q_2(t)$ is given by

$$q_2(t)=(M/2+L_z)^{-1}[L_yt\pm L_x(M+2L_z-t^2)^{1/2}]\ . \tag{25}$$

IV. QUANTIZATION

Let us assume that a quantum version of the system exists. Then the observables L should be represented by self-adjoint operators $\hat{L}$ with commutation relations given by $i\hbar$ times the Poisson brackets (24).[5,6] Moreover, in order that Eq. (22) be preserved in the classical limit, we must have

$$\hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2=\frac{M^2}{4}\ . \tag{26}$$

It follows, using the well-known representation theory of the su(2) algebra, that the theory can be quantized only if

$$\frac{M^2}{4}=\hbar^2 j(j+1)\ , \tag{27}$$

where j is integer or half-integer. Then the Hilbert space of the quantum theory has dimension $2j+1$ and on the basis vectors $|m\rangle$, $m=-j,\ldots,+j$, defined by

$$\hat{L}_z|m\rangle=m|m\rangle\ , \tag{28}$$

the operators $\hat{L}$ are given by the standard expressions that can be found in any textbook of quantum mechanics. It follows that

$$\hat{A}|m\rangle=(M/2+m)|m\rangle \tag{29}$$

and

$$\widehat{\sin\phi}|m\rangle=(M^2/4-\hat{L}_z^2)^{-1/4}\hat{L}_x(M^2/4-\hat{L}_z^2)^{-1/4}\ ,$$
$$\widehat{\cos\phi}|m\rangle=(M^2/4-\hat{L}_z^2)^{-1/4}\hat{L}_y(M^2/4-\hat{L}_z^2)^{-1/4}\ . \tag{30}$$

These operators are well defined, and self-adjoint; they have the correct commutation relations (which reproduce, up to $\hbar$ terms, the classical Poisson algebra), and in the $\hbar\to 0$ limit their spectrum goes to the classical range of variability of the corresponding classical observables.

This provides a linear space, a scalar product, and a complete set of self-adjoint operators that represent observables. In any given state, we may measure any of the elementary observables, and the standard rules of quantum mechanics will provide us with the probability distributions of the outcome of the measurements. Since all the operators that we have introduced are self-adjoint, the properties of their expectation value required by the probabilistic interpretation are necessarily satisfied.

However, the most intuitive set of observables is given by the "evolutionlike" observables $q_2(t)$. Let us introduce on the Hilbert space H the two symmetric operators

$$\hat{q}_2^{\pm}(t)=f(\hat{L}_z)[\hat{L}_yt\pm(M+2\hat{L}_z-t^2)^{1/4}\hat{L}_x\times(M+2\hat{L}_z-t^2)^{1/4}]f(\hat{L}_z)\ ,$$
$$f(\hat{L}_z)=\sqrt{M/2-L_z}\ , \tag{31}$$

which are well defined, because functions of self-adjoint operators are well defined. These operators can be used to describe "evolution" in the quantum context. We will discuss these operators in detail later.

V. THE EMERGENCE OF TIME

In the next section, we will discuss the physical interpretation of the theory in the general case. In this section we discuss how classical evolution is recovered from the quantum theory. Indeed, is it possible that the dynamics of the classical system, in which we see two coordinates (q_1 and q_2) evolving one with respect to the other, may emerge from the $2j+1$ states $|m\rangle$ and from the $\hat{L}$ operators?

We start by casting our system in a more intuitive form. We define the Fock coordinates

$$a_i=\frac{1}{\sqrt{2}}(p_i-iq_i),\quad a_i^{\dagger}=\frac{1}{\sqrt{2}}(p_i+iq_i),\quad N_i=a_i^{\dagger}a_i\ . \tag{32}$$

Note that these functions on the phase space do not commute with the constraint; therefore they are not physical observables, nor do we expect that they will be well-defined observables on H. We have

$$\{a_i^{\dagger},a_i\}=i \tag{33}$$

and

$$L_x=\tfrac{1}{2}(a_1^{\dagger}a_2+a_2^{\dagger}a_1)\ , \tag{34}$$

$$L_y=\frac{i}{2}(a_1^{\dagger}a_2-a_2^{\dagger}a_1)\ , \tag{35}$$

$$L_z=\tfrac{1}{2}(a_2^{\dagger}a_2-a_1^{\dagger}a_1)\ . \tag{36}$$

This suggests that we may consider a Fock-space quantization of $a_i^{\dagger}$ and a_i:

$$\hat{a}_1^{\dagger}|n,m\rangle=\sqrt{n+1}|n+1,m\rangle\ , \tag{37}$$

$$\hat{a}_2^{\dagger}|n,m\rangle=\sqrt{m+1}|n,m+1\rangle\ , \tag{38}$$

$$\hat{a}_1|n,m\rangle=\sqrt{n}\,|n-1,m\rangle\ , \tag{39}$$

$$\hat{a}_2|n,m\rangle=\sqrt{m}\,|n,m-1\rangle\ . \tag{40}$$

It is easy to check that with the ordering given in Eqs. (34)–(36) the correct commutation relations of the $\hat{L}$ operators are preserved. In terms of the L operators, the constraint is given by Eq. (22), so that the quantum Dirac constraint equation is

$$(\hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2-M^2/4)|n,m\rangle=0\ . \tag{41}$$

Since

$$\hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2=\left[\frac{\hat{N}_1+\hat{N}_2}{2}\right]^2+\frac{\hat{N}_1+\hat{N}_2}{2}\ , \tag{42}$$

we have that the Dirac quantum constraint (38) is solved by the states with

$$(\hat{N}_1+\hat{N}_2)|n,m\rangle=2j|n,m\rangle\ , \tag{43}$$

where, as before,

$$j(j+1)=M^2/4\ . \tag{44}$$

The states that solve Eq. (43) have $n+m=2j$. There are precisely $2j+1$ of these states, so we obtain again the previous result: The subspace of the Fock space spanned by these vectors is unitarily equivalent to H, the relation being given by

$$|m\rangle\leftrightarrow|2j-m,2j+m\rangle\ . \tag{45}$$

This reformulation [which is a standard technique for building representations of SU(N) groups in terms of Fock operators] is interesting here because it allows us to represent the states in an intuitive way. Indeed, let $\psi_n(q)$ be the nth eigenfunction of the harmonic oscillator; then we may represent the states $|n,m\rangle$ in terms of the $L_2[R^2]$ functions $\psi(q_1,q_2)$ as

$$|n,m\rangle\leftrightarrow\psi(q_1,q_2)=\psi_n(q_1)\psi_m(q_2)\ . \tag{46}$$

Our physical states $|m\rangle$ are represented by the functions

$$|m\rangle\leftrightarrow\psi_m(q_1,q_2)=\psi_{2j-m}(q_1)\psi_{2j+m}(q_2)\ , \tag{47}$$

so that a generic state in the "coordinate representation" has the form

$$\psi(q_1,q_2)=\sum_{m=-j}^{+j}c_m\psi_{2j-m}(q_1)\psi_{2j+m}(q_2)\ ,\qquad \sum_{m=-j}^{+j}|c_m|^2=1\ . \tag{48}$$

Note that these functions satisfy the differential equation

$$\left[-\hbar^2\frac{\partial^2}{\partial q_1^2}-\hbar^2\frac{\partial^2}{\partial q_2^2}+q_1^2+q_2^2-(2j+1)\right]\psi(q_1,q_2)=0\ , \tag{49}$$

which is (up to the crucial issue of ordering) the coordinate-picture Dirac constraint equation. The representation (48) of H allows us to visualize the quantum states in the same q_1,q_2 space in which the classical orbits were considered, and they will also allow us to visualize the classical limit of the theory.

Consider for a moment a fictitious Hamiltonian system with Hamiltonian given by the constraint C. Let $\psi(q_1,q_2,t)$ be a solution of the time-dependent Schrödinger equation

$$-i\hbar\frac{\partial}{\partial t}\psi(q_1,q_2,t)=\hat{C}\psi(q_1,q_2,t)\ . \tag{50}$$

Given any of these solutions, we may obtain a state of our constrained system by a time average. Indeed, the function

$$\psi(q_1,q_2)=\frac{1}{2\pi}\int_0^{2\pi}\psi(q_1,q_2,t)dt \tag{51}$$

is in H, because the integral projects out the zero energy component. Note that since everything is periodic in t with period 2π, it is sufficient to integrate in the bounded interval. Now, the time-dependent system is a double oscillator on an additional constant negative potential. Let us assume from now on that M is very large compared to $\hbar$. Consider a coherent state of the double oscillator which follows a classical trajectory. Its time average has support on the smeared support of the classical trajectory. Therefore, for big enough M, there are states of our system that are described by a wave function $\psi(q_1,q_2)$ which looks like the classical trajectory, but a bit smeared (see Fig. 1). Let the state

$$\psi^{(A,\phi)}(q_1,q_2)=\sum_{m=-j}^{+j}c_m^{(A,\phi)}\psi_{2j-m}(q_1)\psi_{2j+m}(q_2) \tag{52}$$

be this coherent state that has support around the classical trajectory (A,ϕ). On such a coherent state, the expectation value of $\hat{A}$ will be given by A, and the spread is small (recall that $j\gg1$). Therefore,

$$c_m\sim0\quad\text{unless }(M/2+m)\sim A\ . \tag{53}$$

Let us consider a measuring apparatus that is able to perform the following measurement. It measures whether, at a given value t of q_1, the coordinate q_2 is or is not in the interval $[q_2,q_2+\Delta q]$ (see Fig. 1). The apparatus is also able to perform a sequence of measurements for a sequence of successive t. Let us assume that the interval Δq is big enough compared with the quantum spread of the wave function around the classical trajectory, so that the projection of the wave function is negligible. It is clear that the sequence of measurements cannot be distinguished from a sequence of classical measurements of a system evolving in the time t.

Can such an apparatus be described in the quantum

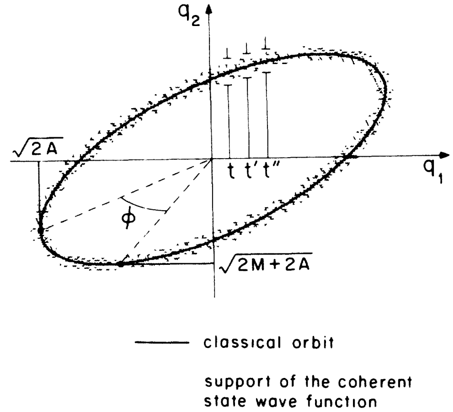

FIG. 1. A classical orbit and a coherent-state wave-function support in the extended configuration space.

formalism that we constructed? Consider the operators $\hat{q}_2^{\pm}(t)$ defined at the end of the preceding section. Since they are symmetric they can be diagonalized. Let $P^{\pm}_{[q_2,q_2+\Delta q]}(t)$ be the projection operator that projects on the eigenspace of $\hat{q}_2^{\pm}(t)$ with real eigenvalues contained in the interval $[q_2,q_2+\Delta q]$, and

$$P_{[q_2,q_2+\Delta q]}(t)=P^{+}_{[q_2,q_2+\Delta q]}(t)+P^{-}_{[q_2,q_2+\Delta q]}(t) . \tag{54}$$

It is clear that this operator represents exactly the measurement that we described. Note that the projection remains in H, namely within the finite-dimensional subspace defined by Eq. (45).

Let us assume that the system is in one of the coherent states described above. As long as t is smaller than $\sqrt{2A}$ [see Eq. (16)], we have

$$\langle \psi | P_{[0,+\infty]}(t) | \psi \rangle \sim 1 . \tag{55}$$

As long as the interval $[q_2,q_2+\Delta q]$ is large compared to the width of the coherent state, the measurements do not disturb the quantum state. The evolution in t observed by a sequence of measurements described by $P_{[q_2,q_2+\Delta q]}(t)$ is precisely the evolution that we expect from the classical dynamics.

If we decrease the width Δq of the interval, the wave-function collapse begins to become nontrivial. In the collapse, components other than the ones that satisfy Eq. (53) become excited. As long as the c_m components that we excite are still in the range $m > M/2-t^2/2$, i.e., as long as after the wave-function collapse

$$m \geq M/2-t^2/2 \Longrightarrow c_m \sim 0 , \tag{56}$$

Eq. (55) continues to hold. The quantum properties of the system appear, but the evolution in t is well described by a regular unitary evolution. Within this approximation, the operator, say, $\hat{q}_2^{+}(t)$ is the precise analog of the Heisenberg operators

$$\hat{q}(t)=e^{i\hat{H}t}\hat{q}e^{-i\hat{H}t} \tag{57}$$

of standard quantum mechanics in the Heisenberg picture.

The nonstandard behavior appears if we further decrease the width, or if we reach a value of t such that Eq. (56) does not hold anymore. At this point we have

$$\langle \psi | P_{[0,+\infty]}(t) | \psi \rangle < 1 . \tag{58}$$

Therefore, the evolution in t becomes nonunitary.

Thus, there is a certain range of states and a certain range of measurements within which the system behaves, in the first approximation, precisely as a standard quantum Hamiltonian system. This regime is defined by Eq. (56). We shall refer to this regime as the *Schrödinger regime.* Therefore, there is no contradiction between the quantum mechanics of the systems that we are considering and the experimental evidence of the flow of time. In the next section, we inquire about the behavior of the system beyond this Schrödinger regime.

VI. INTERPRETATION OF THE SYSTEM IN THE ABSENCE OF TIME: THE OPERATORS $\hat{q}_2^{\pm}(t)$

In the preceding section we have shown that, in a certain class of states, there is a class of observables that have an "evolution" which reproduces the standard unitary time evolution and the standard wave-function collapse of Hamiltonian quantum mechanics. This conclusion is not new, since it has been suggested several times that a standard Schrödinger quantum mechanics may appear as the approximation of a Wheeler–DeWitt-like theory such as ours. The crucial question, however, is whether or not the theory admits an interpretation also beyond this approximation.

Let us consider the problem in two steps. First of all, we note that the quantum system and its interpretation are well defined. For instance, the observables $\hat{A}$, $\widehat{\sin\phi}$, and $\widehat{\cos\phi}$ form a complete algebra of observables. Any classical measurement on the system can be described by a function of A and ϕ [as we saw is possible for $q_2(t)$]; any quantum measurement, according to the standard interpretation of quantum mechanics, corresponds to a classical measurement, and it therefore can be described by a suitable function of the operators $\hat{A}$, $\widehat{\sin\phi}$, and $\widehat{\cos\phi}$, or of the operators $\hat{L}$.

The outcome of the measurements on any state will be given by the rules of the standard probabilistic interpretation of quantum mechanics. Since the basic operators are self-adjoint operators, no contradiction will emerge as far as probability amplitudes are concerned. Thus, we may conclude that the quantum system is well defined both within and outside the Schrödinger approximation.

However, there still is an open question. How do we relate the Schrödinger behavior in the approximation and the apparent no-time behavior beyond the approximation? We introduced two different sets of operators for describing the system in the two different regimes. How are they related? Of course there is no problem in measuring A and ϕ also in the Schrödinger regime. The problem is when we want to measure the $q_2(t)$ observables outside the Schrödinger regime.

Mathematically the problem is that the operators $\hat{q}_2^{\pm}(t)$ are not self-adjoint. They are self-adjoint if restricted to a suitable subspace of the Hilbert space [defined by Eq. (56)]. The Schrödinger regime is precisely the regime in which the state is restricted to live in this subspace, and we do not perform measurements precise enough to project the wave function out of this subspace. The problem comes from the term

$$(M+2\hat{L}_z-t^2)^{1/2} \tag{59}$$

in Eq. (31), which becomes imaginary for large t. Therefore the eigenvalues of the symmetric operator $\hat{q}_2^{\pm}(t)$ are complex for states outside the Schrödinger regime.

Physically, it is easy to see what is going on. The operator $\hat{q}_2^{\pm}(t)$ measures the value of q_2 on a certain state $|\psi\rangle$ when q_1 is t. The imaginary term obtained by operating on certain states simply means the state does not arrive at t. Precisely the same effect happens classically: On an orbit singled out by a given A and ϕ, the observable is well defined only if t is within a certain range; otherwise it is imaginary.

The physics and the mathematics of the situation is quite clear. The problem is how to interpret this effect in the quantum system. In the classical system, if we want to measure $q_2(t)$ for an incorrect t, the measurement simply cannot be performed. (For instance, the prescription for the measurement may be "wait till the clock q_1 reaches t, and then do such and such," and clearly this prescription could never be carried out until the completion.) But a quantum system may be in a superposition of a state in which the measurement could be completed and a state in which the measurement cannot be completed. What would happen if we try to measure $q_2(t)$ in such a state?

This question, we believe, is a physically meaningful question. But it is not particularly deep. In everyday physics there are plenty of situations of this kind: In high-energy physics we may measure the angle θ at which an electron comes out from a scattering reaction, but there may be out states of the reaction in which there is no electron at all. Thus there is no reason to assume that the electron "must have gone somewhere": if $P_{[\theta]}$ is the projector on the state in which the electron is emitted at an angle θ, then

$$\int_0^{2\pi} d\theta\, P_{[\theta]} < 1 , \tag{60}$$

and this does not contradict the probabilistic interpretation of the theory. (Of course, in the electron example there is an underlying unitary theory.)

Similarly, if we measure $q_2(t)$ for our system we may obtain a certain real value q_2, or we may obtain nothing at all, precisely as in the classical situation. The operators $\hat{q}_2^{\pm}(t)$ have real eigenvalues only on a subspace of H, precisely as an operator $\hat{\theta}$ for the electron would have.[7,8] Everything can be formulated in terms of well-defined self-adjoint operators by using the projection operators (54).

Note, however, that these projection operators are completely different from the projection operators introduced in the context of Wheeler–DeWitt quantum mechanics in certain earlier works,[1] because, unlike those, they do not take the state out of the physical Hilbert space of the theory.

VII. CONCLUSIONS AND REFLECTIONS

In this paper we have addressed the question of whether or not the systems in which the evolution is described by a constraint, rather than by a Hamiltonian, are inconsistent with quantum mechanics and its probabilistic interpretation.

We have displayed a simple model which does not admit a Schrödinger unitary evolution, but has the following characteristics.

(a) It can be completely quantized: The Hilbert space (including the scalar product), a set of self-adjoint operators, and a set of t-dependent projection operators can be defined.

(b) In a certain regime, given by certain states and certain observables, or, more precisely, in the regime defined by the relation (56) between the states and the t parameter, the system behaves as a standard quantum-mechanical system with a unitary evolution in t.

(c) Outside this regime unitarity in t is lost, but the probabilistic interpretation, as well as the entire structure of the theory, still holds.

To what extent can we generalize these conclusions? And how are they relevant to the problem of defining a quantum theory of the gravitational field? The model that we have considered is certainly very simple, and it may be risky to extrapolate conclusions to infinite-dimensional theories such as quantum general relativity. However, certain tentative conclusions can be drawn.

First of all, the model shows that it makes sense to construct a quantum theory without a well-defined time evolution, and that the problem of recovering the time evolution can be faced at a second stage, once the theory has been constructed. Therefore it is not strictly necessary to work hard to extract an exact internal time variable from general relativity, in order to face the quantization problem: Quantum theory does not necessarily require a Hamiltonian operator.

Second, there is the issue of the observables. The quantization of our model shows that on the physical Hilbert space we should define only the observables that commute with the constraint. It is extremely important to emphasize that this does not wash away the evolution. Indeed, the observables $q_2(t)$ that we defined are perfectly gauge invariant and commute with the constraint, and still they represent evolution.

One may object that the game was too easy in the model, since we were able to solve the equation of motion, and therefore to write $q_2(t)$ explicitly as a function of the constants A and ϕ. But this is not the point. The point is that, even if we were not able to write them down explicitly, the quantum operators $\hat{q}_2(t)^{\pm}$ nevertheless do exist and are gauge invariant. It is therefore not true that one has to introduce non-gauge-invariant objects in order to recover an evolution. In a situation in which the equation of motion cannot be solved one may construct a perturbation expansion for $q_2(t)$, or find some other approximation method, but this should always produce gauge-invariant objects, namely, objects that commute with the constraint that defines the evolution.

Accordingly, there is no necessity in general relativity of drawing a distinction between the constraints that generate gauge symmetries and the constraints that generate evolution. The model shows that a constraint that generates evolution can be treated precisely in the same way as we treat a standard gauge constraint.

Another conclusion that one can draw from the model is the importance of the global properties of the physical phase space. Unfortunately, very little is known about these in general relativity. A systematic way for quantizing a presymplectic system taking into account the global properties of the phase space was developed in Ref. 9 along the lines of Ref. 6. Again, the poor understanding that we have of the dynamics of general relativity probably prevents us from using these precise methods. However, once more, one should distinguish between the impossibility of doing something for reasons of principle, which would force us to abandon a project, and the technical difficulty of finding certain solutions, which simply forces us to develop some approximation scheme.

In our model there is an approximation within which the variable q_1 behaves as a time t, and the evolution is unitary in t. This approximation is defined by a condition on the precision of the measurement and also by a condition on the "time" t. We expect this feature to be quite general.[10,11] Let us speculate how it may apply to a realistic model of the Universe. Let us assume that the Universe may be described in some approximation, by a coherent state around a cosmological solution of the Einstein equations. Then we expect that the time parameter t loses its standard evolution properties at particular points like initial singularity. Far away from these points the evolution is described by a Schrödinger equation, provided that measurements are not accurate enough to project the wave function also in regions of the phase space where also at the present t some nonunitarity may appear. These measurements, of course, may be only very-high-energy measurements; thus, we are led to recover the relation between small space time distances and the global properties of the orbits. Since this is just a vague speculation, we do not want to engage here in any naive numerology.

Finally, let us mention, for clarity, the problems that have *not* been addressed in this paper. First, we have made use of quantum mechanics and its standard (Copenhagen) probabilistic interpretation. We did not question this theory. We do not see any reason for the problem of the absence of an exact time in the fundamental theory to be related to the old difficulties of quantum mechanics. Perhaps it is related, but we believe, with Heisenberg, that a good physicist has to be conservative, and change the rules only if forced by experiment. The absence of a Hamiltonian in gravitational physics forces us to consider quantum systems without a well-defined time; nothing, up to now, suggests that quantum mechanics itself is wrong.

In this paper we have noted that systems without a Hamiltonian are well described within the framework of presymplectic dynamics, and we have discussed a simple model in order to show that they admit a complete and coherent quantization. A more extended theoretical discussion of the general case can be found in Ref. 2.

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[2] C. Rovelli, University of Pittsburgh Report No. PITT in 90-02, 1989 (unpublished). The same viewpoint is discussed also in C. Rovelli, in *Conceptual Problems in Quantum Gravity* (Ref. 1).

[3] Simple models of evolution with no time that were previously considered had linear motions [C. Rovelli, in *Conceptual Problems in Quantum Gravity* (Ref. 1)]. I thank Ted Jacobson for pointing out to me that because of this linearity these models were too simple to be taken as examples for the general case.

[4] A. Ashtekar, L. Bombelli, and O. Reula, in *Mechanics, Analysis and Geometry: 200 years after Lagrange,* edited by M. Francaviglia and D. Holm (North-Holland, Amsterdam, 1990); C. Crnkovic and E. Witten, Princeton University report (unpublished); J. Frauendiener and G. A. J. Sparling (unpublished).

[5] The choice of the physical observables algebra to be represented in the quantum theory is a nontrivial choice, because, as is well known, it is not possible to preserve the entire classical Poisson algebra of all the physical observables as a quantum commutator algebra. It is precisely in this choice that the global properties of the physical phase space matter. In fact, suppose we had chosen to reproduce the canonical A,ϕ algebra (20) exactly in the quantum theory; then the Stone–von Neumann theorem would state that the spectrum of A would be the real line, thus preventing us from implementing the global conditions (17) in the quantum theory. The precise condition that the observables whose Poisson algebra is exactly reproduced in the quantum theory have to satisfy in order to preserve the global properties of the spectrum is studied in Ref. 6. Essentially, this condition is that the Hamiltonian motions generated by these observables on S_{ph} be complete. Our L observables satisfy this requirement, and this is the reason for which the quantization defined by them works, in the sense that all the quantum operators have spectra included in the correct intervals, so that the classical limit of the quantum theory turns out to be the classical system that we want.

[6] C. J. Isham, in *Relativity Groups and Topology II,* proceedings of the Les Houches Summer School, Les Houches, France, 1983, edited by B. S. DeWitt and R. Stora (Les Houches Summer School Proceedings Vol. 40) (North-Holland, Amster-

dam, 1984).
[7]In other words, unitarity is not required by the probabilistic interpretation of quantum mechanics. In a field theory one may argue that the probabilistic interpretation plus locality may imply unitarity (see Ref. 8). However, in order to even present this argument for the general covariant theories that we are interested in, one should first find a satisfactory (diffeomorphism-invariant) definition of locality for a generally covariant theory, a problem which is completely unsolved.
[8]T. Jacobson, in *Conceptual Problems in Quantum Gravity* (Ref. 1).
[9]C. Rovelli, Nuovo Cimento **B100**, 343 (1987).
[10]Of course it is highly nontrivial to show that in a realistic model there is indeed a variable that plays the role of a good unitary time within some approximation. An old idea for obtaining this variable is to introduce "very massive clocks." A more realistic proposal is made in Ref. 11, where it is suggested that the unitary evolution emerges in a time variable given by the expanding ensemble of galaxies themselves, and the conditions of validity of the approximation are studied.
[11]B. Bertotti and C. Rovelli (unpublished).

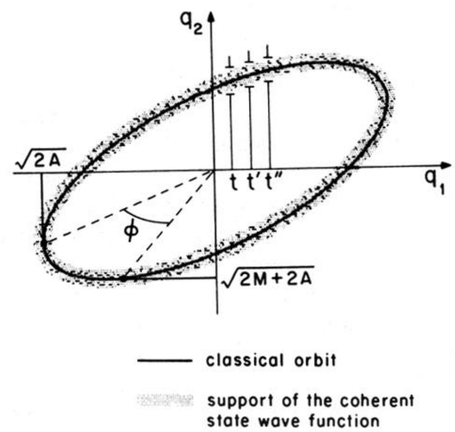

FIG. 1. A classical orbit and a coherent-state wave-function support in the extended configuration space.