

Cosmological perturbations in generalized gravity theories: Solutions

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We have derived second-order differential equations for cosmological perturbations, using gauge-invariant variables in a Friedmann-Lemaître-Robertson-Walker background, for each of the following gravity theories: $f(R)$ gravity, generalized scalar-tensor gravity, gravity with nonminimally coupled scalar field, and induced gravity. Assuming a spatially flat background, each equation is put into a form suggested by Mukhanov, and asymptotic solutions are derived for the large- and small-scale limits. We also present conservation quantities which remain constant in the large-scale limit.

I. INTRODUCTION

Recently, there has been growing interest in modified gravity theories. Many models have been constructed that have favorable inflationary scenarios based on modified theories such as R^2 gravity, Brans-Dicke theory, induced gravity, etc. The motivation for these gravity theories is usually based on the argument that similar corrections to general relativity (GR) can be found in the following theories: supergravity, superstring, and Kaluza-Klein models, etc. In constructing these inflationary models, the density and gravitational-wave perturbations generated during inflation act as strong constraints. In many of these models based on modified gravity theories, it is usual to simply adopt the results derived from exponential inflation in GR with a minimally coupled scalar field (MSF). As a step in clarifying the situation, in this paper we show that even in a wide variety of gravity theories one can *directly solve the perturbation equations* in analogy with the usual GR treatment.

In a recent paper, we presented a simple way of deriving cosmological perturbation equations in a Friedmann-Lemaître-Robertson-Walker (FLRW) background universe in generalized gravity theories, using gauge-invariant (GI) metric and matter variables.¹ This approach was applied to generalized $f(\phi, R)$ gravity theory, which includes the following theories as special cases: $f(R)$ gravity, R^2 gravity, generalized scalar-tensor (GST) theory, Einstein gravity with nonminimally coupled scalar field (NMSF), and induced gravity. Background and perturbation equations were derived for these specialized gravity theories. In this paper, we present solutions for these equations derived from specific gravity theories. We will first show that the perturbation equations can be put into a *second-order differential equation* using a pair of conformally transformed GI potential variables for each theory. (However, we will not use the conformal transformation method in this paper.) By changing variables we can transform these equations into a form suggested by Mukhanov and co-workers.^{2,3} One can then proceed following Ref. 3 and find *asymptotic solutions for both large- and small-scale limits*. We have also derived

conservation quantities for each gravity theory when the scale lies in the large-scale limit. Einstein gravity with MSF and R^2 gravity cases have been investigated in Refs. 2 and 3. In this paper we will generalize those results to the generalized gravity theories mentioned above.

In Sec. II we introduce a Lagrangian which includes these theories as special cases, and present equations for both the background and the perturbations. In Sec. III we solve the perturbed equations for individual gravity theories. Section IV is a brief discussion about the result.

II. GENERALIZED $f(\phi, R)$ GRAVITY

In Ref. 1 a theory with the following Lagrangian was called “generalized $f(\phi, R)$ gravity” theory:

$$\beta L = \frac{1}{2} f(\phi, R) - \frac{\omega(\phi)}{2} \phi_{,c} \phi^{;c} + \beta L_M, \quad (1)$$

where ϕ and R are the scalar field and scalar curvature, respectively, L_M is an additional matter component of the Lagrangian, and β is a constant needed to fix units. Since we will consider a single-component field, we can neglect L_M in this paper.

For the background, we can use the two equations⁴

$$H^2 = \frac{\mu}{3} - \frac{K}{a^2}, \quad \dot{H} = -\frac{\mu+p}{2} + \frac{K}{a^2}, \quad (2)$$

where a is a scale factor, H is a Hubble parameter ($H \equiv \dot{a}/a$), K is a spatial curvature, and μ and p are the “effective” energy density and pressure, respectively. For a gravity theory of the form considered in Eq. (1), the effective background fluid quantities are derived in Ref. 1 as

$$\begin{aligned} \mu &= \frac{\omega}{2F} \dot{\phi}^2 + \frac{FR-f}{2F} - \theta \frac{\dot{F}}{F}, \\ p &= \frac{\omega}{2F} \dot{\phi}^2 - \frac{FR-f}{2F} + \frac{\ddot{F}}{F} + \frac{2}{3} \theta \frac{\dot{F}}{F}, \end{aligned} \quad (3)$$

where $F \equiv \partial f / \partial R$ and θ is the expansion scalar which is $3H$ in the background. (We neglect the term “effective” in the following.)

Perturbed gravitational field equations (GFE’s) are derived for this theory using the GI metric variables Φ_H

and Φ_A in Ref. 1. From these potentials we can reconstruct the behavior of the fluid and kinematic variables. The GI energy-density perturbation in the energy frame and the GI velocity variable, respectively, are

$$\epsilon_m^E = 2 \frac{k^2 - 3K}{a^2 \mu} \Phi_H, \quad v_s^E = \frac{2k}{a(\mu + p)} \left[-\dot{\Phi}_H + \frac{\dot{a}}{a} \Phi_A \right]. \quad (4)$$

The electric part of the conformal tensor is $E_C = -(k^2/a^2)\Phi_H$, whereas the magnetic part just vanishes for scalar perturbations. For more details and conventions, see Refs. 1 and 5.

The perturbation equations can be simplified if we use the conformally transformed potential variables introduced in Ref. 1:

$$\hat{\Phi}_H = \Phi_H + \frac{\Delta F}{2F}, \quad \hat{\Phi}_A = \Phi_A + \frac{\Delta F}{2F}. \quad (5)$$

ΔF is a GI and frame-independent perturbation variable for F introduced in Ref. 1. Using $\hat{\Phi}_H$ as the variable, the perturbed GFE can be written as

$$\hat{\Phi}_A + \hat{\Phi}_H = 0, \quad (6)$$

$$\dot{\hat{\Phi}}_H + \left[\frac{\dot{a}}{a} + \frac{\dot{F}}{2F} \right] \hat{\Phi}_H = -\frac{3}{4} \frac{\dot{F} \Delta F}{F^2} - \frac{1}{2} \frac{\omega}{F} \dot{\phi} \Delta \phi, \quad (7)$$

$$\begin{aligned} \ddot{\hat{\Phi}}_H + \left[4 \frac{\dot{a}}{a} + \frac{3}{2} \frac{\dot{F}}{F} \right] \dot{\hat{\Phi}}_H \\ + \left[\frac{k^2 - 5K}{a^2} - \frac{\omega}{2F} \dot{\phi}^2 - \frac{3}{4} \frac{\dot{F}^2}{F^2} + \frac{FR - f}{2F} \right] \hat{\Phi}_H \\ = \frac{3}{4} \left[-2 \frac{\ddot{F}}{F} - \theta \frac{\dot{F}}{F} + \frac{3}{2} \frac{\dot{F}^2}{F^2} - \frac{2}{3} \frac{\omega}{F} \dot{\phi}^2 \right] \frac{\Delta F}{F} \\ + \frac{1}{2} \left[-f_{,\phi} + 3\omega \dot{\phi} \left[\frac{\dot{a}}{a} + \frac{\dot{F}}{2F} \right] \right] \frac{\Delta \phi}{F}. \end{aligned} \quad (8)$$

These follow from Eqs. (35), (34), and (36) of Ref. 1, respectively. In these equations $\Delta \phi$ is a GI and frame-independent perturbation variable for ϕ , and k is a wave number. Using Eqs. (7) and (8), we can derive single second-order differential equation for $\hat{\Phi}_H$ for each of the specific gravity theories.

III. SOLUTIONS IN SPECIFIC GRAVITY THEORIES

A. $f(R)$ gravity

This is a special case of Eq. (1) with $f = f(R)$, $\omega = 0$, $\phi = 0$, and $\beta = 1$. The background fluid quantities become

$$\mu = -\theta \frac{\dot{F}}{F} + \frac{RF - f}{2F}, \quad p = \frac{\ddot{F}}{F} + \frac{2}{3} \theta \frac{\dot{F}}{F} - \frac{RF - f}{2F}. \quad (9)$$

The conformally transformed potential variables are

$$\hat{\Phi}_H = \Phi_H + \frac{\Delta F}{2F}, \quad \hat{\Phi}_A = \Phi_A + \frac{\Delta F}{2F}. \quad (10)$$

From Eq. (7) and Eqs. (6) and (10), we have

$$\frac{\Delta F}{F} = -\frac{4\sqrt{F}}{3a\dot{F}} (a\sqrt{F} \hat{\Phi}_H)' = -(\Phi_A + \Phi_H). \quad (11)$$

After some manipulation of the GFE [Eqs. (7) and (8)], we can derive the following second-order differential equation for $\hat{\Phi}_H$:

$$\begin{aligned} \ddot{\hat{\Phi}}_H + \left[\frac{\dot{a}}{a} + 3 \frac{\dot{F}}{F} - 2 \frac{\ddot{F}}{\dot{F}} \right] \dot{\hat{\Phi}}_H \\ + \left[\frac{k^2 - 2K}{a^2} + 3 \frac{\dot{a}}{a} \frac{\dot{F}}{F} - \frac{\ddot{F}}{F} - 2 \frac{\dot{a}}{a} \frac{\ddot{F}}{\dot{F}} \right] \hat{\Phi}_H = 0. \end{aligned} \quad (12)$$

Defining

$$u \equiv \frac{aF^{3/2}}{F'} \hat{\Phi}_H, \quad z \equiv \frac{(a\sqrt{F})'}{a^2 F'}, \quad (13)$$

where a prime denotes a derivative with respect to a conformal time ($dt = a d\eta$), Eq. (12) can be written as

$$u'' + \left[k^2 - \frac{z''}{z} - 2K \frac{(a^2 F')' / (a^2 F')}{(a\sqrt{F})' / (a\sqrt{F})} \right] u = 0.$$

For $K = 0$, this equation becomes

$$u'' + \left[k^2 - \frac{z''}{z} \right] u = 0. \quad (14)$$

This is a form of the equation derived in Refs. 2 and 3 in their study of MSF and R^2 gravity.⁶ Following their treatment, we can derive asymptotic solutions in the large- and small-scale limits.⁷ (For all asymptotic solutions, we will assume $K = 0$.)

(i) $k^2 \ll z''/z$. In the large-scale limit, Eq. (14) has the solution

$$u = c_d z + c_g z \int \frac{d\eta}{z^2}, \quad (15)$$

where c_d and c_g are constants indicating a decaying and a growing mode, respectively. Using Eq. (13), it follows that

$$u = \frac{(a\sqrt{F})'}{a^2 \dot{F}} \left[c_d - \frac{4}{3} c_g \int aF dt \right] + \frac{4}{3} c_g \frac{F^{3/2}}{\dot{F}}, \quad (16)$$

where we have used the identity, $\dot{F}^2 = -\frac{4}{3} F^{5/2} [(a\sqrt{F})' / (aF)]$, derived from background equations [Eqs. (2) and (9)]. From Eqs. (11), (13), and (16), we can show that

$$\frac{\Delta F}{F} = \frac{\dot{F}}{aF^2} \left[c_d - \frac{4}{3} c_g \int aF dt \right], \quad (17)$$

Using Eqs. (10), (13), (16), and (17), we find the following solutions for the original potential variables:

$$\begin{aligned} \Phi_A &= \left[\frac{1}{aF} \right] \left[c_d - \frac{4}{3} c_g \int aF dt \right] - \frac{4}{3} c_g, \\ \Phi_H &= -\frac{\dot{a}}{a^2 F} \left[c_d - \frac{4}{3} c_g \int aF dt \right] + \frac{4}{3} c_g. \end{aligned} \quad (18)$$

From these solutions we can derive the following GI conservation quantities in the large-scale limit:

$$\begin{aligned} \xi &\equiv -\Phi_A + \left[\frac{(aF)'}{aF} \right]^2 \frac{1}{[(aF)'/aF]} \left[\Phi_A + \frac{aF}{(aF)'} \dot{\Phi}_A \right] \\ &= \Phi_H - \frac{\dot{a}^2}{a^2 F} \frac{1}{[\dot{a}/(aF)]'} \left[\Phi_H + \frac{a}{\dot{a}} \dot{\Phi}_H \right] = \frac{4}{3} c_g . \end{aligned} \quad (19)$$

These conservation quantities do not depend on a change of background scale factor and equation of state, etc., as long as the scale remains large. Note that the decaying mode does not affect these conserved quantities. (These comments will also apply to the other gravity theories analyzed later.)

(ii) $k^2 \gg z''/z$. In the small-scale limit, Eq. (14) has the oscillatory solution

$$u = c_1 e^{ik\eta} + c_2 e^{-ik\eta} = c_1 e^{ik\eta} + c_2 \times \text{c.c.} , \quad (20)$$

where c.c. denotes complex conjugation. Using Eqs. (10), (11), (13), and (20), we have

$$\begin{aligned} \Phi_A &= \frac{2c_1}{3\sqrt{F}} \left[\frac{ik}{a} + \frac{(aF^{-5/2}\dot{F})'}{aF^{-5/2}\dot{F}} \right] e^{ik\eta} + c_2 \times \text{c.c.} , \\ \Phi_H &= \frac{2c_1}{3\sqrt{F}} \left[\frac{ik}{a} + \frac{(a\sqrt{F}\dot{F})'}{a\sqrt{F}\dot{F}} \right] e^{ik\eta} + c_2 \times \text{c.c.} \end{aligned} \quad (21)$$

B. Generalized scalar-tensor theory

This is a case of Eq. (1) with $f = 2\phi R - 2V(\phi)$, $\beta = 16\pi$, and $\omega \rightarrow 2\omega(\phi)/\phi$. The background fluid quantities are

$$\mu = \frac{V}{2\phi} + \frac{\omega}{2} \frac{\dot{\phi}^2}{\phi^2} - \theta \frac{\dot{\phi}}{\phi} , \quad p = -\frac{V}{2\phi} + \frac{\omega}{2} \frac{\dot{\phi}^2}{\phi^2} + \frac{\ddot{\phi}}{\phi} + \frac{2}{3} \theta \frac{\dot{\phi}}{\phi} . \quad (22)$$

The conformally transformed potential variables become

$$\hat{\Phi}_H = \Phi_H + \frac{\Delta\phi}{2\phi} , \quad \hat{\Phi}_A = \Phi_A + \frac{\Delta\phi}{2\phi} . \quad (23)$$

From Eq. (7) we have

$$\frac{\Delta\phi}{\phi} = -\frac{2\sqrt{\phi}}{a\dot{\phi}w} (a\sqrt{\phi}\dot{\Phi}_H)' . \quad (24)$$

From Eqs. (7) and (8) one can derive a second-order differential equation for $\hat{\Phi}_H$:

$$\begin{aligned} \ddot{\hat{\Phi}}_H + \left[\frac{\dot{a}}{a} + 3\frac{\dot{\phi}}{\phi} - 2\frac{\ddot{\phi}}{\dot{\phi}} - \frac{\dot{w}}{w} \right] \dot{\hat{\Phi}}_H \\ + \left[\frac{k^2 - 2K}{a^2} + 3\frac{\dot{a}}{a} \frac{\dot{\phi}}{\phi} - \omega \frac{\dot{\phi}^2}{\phi^2} \right. \\ \left. - \left[\frac{\dot{a}}{a} + \frac{\dot{\phi}}{2\phi} \right] \left[\frac{\dot{w}}{w} + 2\frac{\ddot{\phi}}{\dot{\phi}} \right] \right] \hat{\Phi}_H = 0 , \end{aligned} \quad (25)$$

where $w \equiv \omega + \frac{3}{2}$. Defining

$$u \equiv \frac{a\phi^{3/2}}{\dot{\phi}'\sqrt{w}} \hat{\Phi}_H , \quad z \equiv \frac{(a\sqrt{\phi})'}{a^2\dot{\phi}'\sqrt{w}} , \quad (26)$$

Eq. (25) can be written as

$$u'' + \left[k^2 - \frac{z''}{z} - 2K \frac{(a^2\dot{\phi}'\sqrt{w})'/(a^2\dot{\phi}'\sqrt{w})}{(a\sqrt{\phi})'/(a\sqrt{\phi})} \right] u = 0 .$$

For $K=0$, this equation can be written as Eq. (14) and we can find asymptotic solutions as in the $f(R)$ gravity case.

(i) $k^2 \ll z''/z$. From Eqs. (15) and (26) we have

$$u = \frac{(a\sqrt{\phi})'}{a^2\dot{\phi}\sqrt{w}} \left[c_d - 2c_g \int a\phi dt \right] + 2c_g \frac{\phi^{3/2}}{\dot{\phi}\sqrt{w}} , \quad (27)$$

where we have used the identity $\dot{\phi}^2 = -2\phi^{5/2}w^{-1}[(a\sqrt{\phi})'/(a\phi)]'$, derived from Eqs. (2) and (22). Using Eqs. (24), (26), and (27), we have

$$\frac{\Delta\phi}{\phi} = \frac{\dot{\phi}}{a\phi^2} \left[c_d - 2c_g \int a\phi dt \right] . \quad (28)$$

Finally, using Eqs. (23), (26), (27), and (28), we can find the potential variables

$$\begin{aligned} \Phi_A &= \left[\frac{1}{a\phi} \right] \left[c_d - 2c_g \int a\phi dt \right] - 2c_g , \\ \Phi_H &= \frac{\dot{a}}{a^2\phi} \left[c_d - 2c_g \int a\phi dt \right] + 2c_g . \end{aligned} \quad (29)$$

Compared to $f(R)$ gravity case [Eq. (18)], F has been changed into ϕ (in fact, 2ϕ) and $c_g \rightarrow \frac{3}{2}c_g$. From this we can derive conservation quantities

$$\begin{aligned} \xi &\equiv -\Phi_A + \left[\frac{(a\phi)'}{a\phi} \right]^2 \frac{1}{[(a\phi)'/a\phi]} \\ &\quad \times \left[\Phi_A + \frac{a\phi}{(a\phi)'} \dot{\Phi}_A \right] \\ &= \Phi_H - \frac{\dot{a}^2}{a^2\phi} \frac{1}{[\dot{a}/(a\phi)]'} \left[\Phi_H + \frac{a}{\dot{a}} \dot{\Phi}_H \right] = 2c_g . \end{aligned} \quad (30)$$

(ii) $k^2 \gg z''/z$. Using Eqs. (20), (23), (24), and (26), we find the solutions

$$\begin{aligned} \Phi_A &= \frac{c_1}{\sqrt{\phi}w} \left[\frac{ik}{a} + \frac{(a\phi^{-1}\dot{\phi}\sqrt{w})'}{a\phi^{-1}\dot{\phi}\sqrt{w}} - w \frac{\dot{\phi}}{\phi} \right] e^{ik\eta} \\ &\quad + c_2 \times \text{c.c.} , \\ \Phi_H &= \frac{c_1}{\sqrt{\phi}w} \left[\frac{ik}{a} + \frac{(a\phi^{-1}\dot{\phi}\sqrt{w})'}{a\phi^{-1}\dot{\phi}\sqrt{w}} + w \frac{\dot{\phi}}{\phi} \right] e^{ik\eta} \\ &\quad + c_2 \times \text{c.c.} \end{aligned} \quad (31)$$

C. Nonminimally coupled scalar field

This is a case of Eq. (1) with $f = \alpha R - \xi\phi^2 R - 2V(\phi)$, where $\alpha=1$, $\omega=1$, and $\beta=1$. (For generality, we keep α in the following equations.) The background fluid quantities are

$$\mu = \frac{1}{\alpha - \xi\phi^2} \left[\frac{\dot{\phi}^2}{2} + V + 2\xi\theta\phi\dot{\phi} \right], \quad (32)$$

$$p = \frac{1}{\alpha - \xi\phi^2} \left[\frac{\dot{\phi}^2}{2} - V - 2\xi \left[\phi\ddot{\phi} + \frac{2}{3}\theta\phi\dot{\phi} + \dot{\phi}^2 \right] \right].$$

The conformally transformed potential variables become

$$\hat{\Phi}_H \equiv \Phi_H - \frac{\xi\phi\Delta\phi}{\alpha - \xi\phi^2}, \quad \hat{\Phi}_A \equiv \Phi_A - \frac{\xi\phi\Delta\phi}{\alpha - \xi\phi^2}. \quad (33)$$

From Eq. (7) we have

$$\frac{\Delta F}{F} = \frac{-2\xi\phi\Delta\phi}{\alpha - \xi\phi^2} = -\frac{8\xi^2\phi^2\sqrt{F}}{aE\dot{F}} (a\sqrt{F}\hat{\Phi}_H)'. \quad (34)$$

From Eqs. (7) and (8) one can derive a second-order differential equation for $\hat{\Phi}_H$:

$$\begin{aligned} \ddot{\hat{\Phi}}_H + \left[\frac{\dot{a}}{a} - 2\frac{\ddot{\phi}}{\dot{\phi}} - \frac{6\xi\phi\dot{\phi}}{\alpha - \xi\phi^2} - \frac{2\xi(6\xi - 1)\phi\dot{\phi}}{\alpha + \xi(6\xi - 1)\phi^2} \right] \dot{\hat{\Phi}}_H \\ + \left[\frac{k^2 - 2K}{a^2} - 2\frac{\dot{a}}{a}\frac{\ddot{\phi}}{\dot{\phi}} + \frac{1}{\alpha - \xi\phi^2} [-\dot{\phi}^2 + 2\xi\phi(\ddot{\phi} - \theta\dot{\phi})] \right. \\ \left. - \frac{2\xi(6\xi - 1)\phi\dot{\phi}}{\alpha + \xi(6\xi - 1)\phi^2} \left[\frac{\dot{a}}{a} - \frac{\xi\phi\dot{\phi}}{\alpha - \xi\phi^2} \right] \right] \hat{\Phi}_H = 0. \end{aligned} \quad (35)$$

This equation becomes simplified in both the minimally coupled case and in the conformally coupled case where $\xi=0$ and $\frac{1}{6}$, respectively. Defining

$$u \equiv \frac{aF^{3/2}}{\phi'\sqrt{E}} \hat{\Phi}_H, \quad z \equiv \frac{(a\sqrt{F})'}{a^2\phi'\sqrt{E}}, \quad (36)$$

where $F = \alpha - \xi\phi^2$ and $E \equiv \alpha + \xi(6\xi - 1)\phi^2$, this equation can be written as

$$u'' + \left[k^2 - \frac{z''}{z} - 2K \frac{(a^2\phi'\sqrt{E})'/(a^2\phi'\sqrt{E})}{(a\sqrt{F})'/(a\sqrt{F})} \right] u = 0.$$

For $K=0$ this has the same form as Eq. (14).

(i) $k^2 \ll z''/z$. From Eqs. (15) and (36) we can derive

$$u = \frac{(a\sqrt{F})'}{a^2\phi'\sqrt{E}} \left[c_d - 2c_g \int aF dt \right] + 2c_g \frac{F^{3/2}}{\phi'\sqrt{E}}, \quad (37)$$

where we have used the identity $E\dot{\phi}^2 = -2F^{5/2}[(a\sqrt{F})'/(aF)]$, derived from Eqs. (2) and (32). Using Eqs. (36), (34), and (37), we have

$$\frac{\Delta F}{F} = \frac{\dot{F}}{aF^2} \left[c_d - 2c_g \int aF dt \right]. \quad (38)$$

Finally using Eqs. (33), (36), (37), and (38), we find that

$$\begin{aligned} \Phi_A &= \left[\frac{1}{aF} \right] \left[c_d - 2c_g \int aF dt \right] - 2c_g, \\ \Phi_H &= \frac{\dot{a}}{a^2F} \left[c_d - 2c_g \int aF dt \right] + 2c_g. \end{aligned} \quad (39)$$

These equations have the same forms as in the $f(R)$ gravity case [Eq. (18)] with $c_g \rightarrow \frac{3}{2}c_g$. Thus the conservation quantities can be written exactly the same form as in

Eq. (19) with $c_g \rightarrow \frac{3}{2}c_g$.

(ii) $k^2 \gg z''/z$. Using Eqs. (20), (33), (36), and (34), we can derive

$$\begin{aligned} \frac{\Delta F}{F} &= -\frac{2c_1}{\sqrt{EF}} \frac{\dot{F}}{\dot{\phi}} \left[\frac{ik}{a} + \frac{(a\dot{\phi}\sqrt{EF}^{-1})'}{a\dot{\phi}\sqrt{EF}^{-1}} \right] e^{ik\eta} \\ &\quad + c_2 \times \text{c.c.}, \\ \Phi_A &= -\frac{\sqrt{E}}{F^{3/2}} \dot{\phi} (c_1 e^{ik\eta} + c_2 e^{-ik\eta}) - \frac{\Delta F}{2F}, \\ \Phi_H &= \frac{\sqrt{E}}{F^{3/2}} \dot{\phi} (c_1 e^{ik\eta} + c_2 e^{-ik\eta}) - \frac{\Delta F}{2F}. \end{aligned} \quad (40)$$

D. Minimal coupling

This is a case of NMSF with $E=F=\alpha=1$. Although this case has been studied in detail in Refs. 2, 8, and 9, we will present an analysis here to show its similarity to the other gravity cases. From Eqs. (5)–(7) we have

$$\Phi_H = -\Phi_A, \quad \Delta\phi = -\frac{2}{a\dot{\phi}} (a\Phi_H)'. \quad (41)$$

Thus Eq. (35) can be written as

$$\ddot{\Phi}_H + \left[\frac{\dot{a}}{a} - 2\frac{\ddot{\phi}}{\dot{\phi}} \right] \dot{\Phi}_H + \left[\frac{k^2 - 2K}{a^2} - 2\frac{\dot{a}}{a}\frac{\ddot{\phi}}{\dot{\phi}} - \dot{\phi}^2 \right] \Phi_H = 0.$$

This equation was derived in Refs. 2 and 8 for the $K=0$ case. From Eq. (36) we have

$$u = \frac{a}{\phi'} \Phi_H, \quad z = \frac{a'}{a^2\phi'}. \quad (42)$$

Directly calculating or reducing the results derived in the NMSF case, we can find the asymptotic solutions on large scales:

$$\Phi_H = - \left[\frac{1}{a} \right] \left[c_d - 2c_g \int a dt \right] + 2c_g, \quad (43)$$

$$\Delta\phi = \frac{\dot{\phi}}{a} \left[c_d - 2c_g \int a dt \right].$$

From this we have the conservation quantity

$$\xi \equiv \Phi_H - \frac{\dot{a}^2}{a^2} \frac{1}{(\dot{a}/a)'} \left[\Phi_H + \frac{a}{\dot{a}} \dot{\Phi}_H \right] = 2c_g. \quad (44)$$

This is the large-scale limiting case of the same variable used in Ref. 9. This equation can also be written as

$$\xi = \Phi_H + \frac{2}{3} \frac{1}{1+p/\mu} \left[\Phi_H + \frac{1}{H} \dot{\Phi}_H \right], \quad (45)$$

which is also the case for a perfect-fluid medium.^{9,10} In the small-scale limit, we have

$$\Phi_H = \dot{\phi} c_1 e^{ik\eta} + c_2 \times \text{c.c.}, \quad (46)$$

$$\Delta\phi = 2c_1 \left[\frac{ik}{a} + \frac{(a\dot{\phi})'}{a\dot{\phi}} \right] e^{ik\eta} + c_2 \times \text{c.c.}$$

E. Conformal coupling

This is the case of a NMSF with $E = \alpha = 1$ and $F = 1 - \frac{1}{6}\phi^2$; thus,

$$u = \frac{aF^{3/2}}{\phi'} \hat{\Phi}_H, \quad z = \frac{(a\sqrt{F})'}{a^2\phi'}. \quad (47)$$

F. Induced gravity

This gravity model can be considered as the case of a NMSF with $\alpha = 0$. The equation becomes

$$\begin{aligned} \ddot{\hat{\Phi}}_H + \left[\frac{\dot{a}}{a} - 2\frac{\ddot{\phi}}{\phi} + 4\frac{\dot{\phi}}{\phi} \right] \hat{\Phi}_H \\ + \left[\frac{k^2 - 2K}{a^2} - 2\frac{\dot{a}}{a} \frac{\ddot{\phi}}{\phi} \right. \\ \left. + \left[\frac{1}{\xi} - 2 \right] \frac{\dot{\phi}^2}{\phi^2} - 2\frac{\ddot{\phi}}{\phi} + 4\frac{\dot{a}}{a} \frac{\dot{\phi}}{\phi} \right] \hat{\Phi}_H = 0. \end{aligned} \quad (48)$$

One can easily check that this can also be derived from GST theory by substituting¹¹

$$\phi \rightarrow \frac{1}{2}\xi\phi^2, \quad \omega \rightarrow -\frac{1}{4\xi}.$$

Defining

$$u \equiv \frac{a\phi^2}{\phi'} \hat{\Phi}_H, \quad z = \frac{(a\phi)'}{a^2\phi\phi'}, \quad (49)$$

Eq. (48) can be written as

$$u'' + \left[k^2 - \frac{z''}{z} - 2K \frac{(\phi\phi'^{-1})' / (\phi\phi'^{-1})}{(a\phi)' / (a\phi)} \right] u = 0.$$

Since this case is a special case of the other gravity theories analyzed above, the asymptotic solutions can be derived easily and we will omit them.

IV. DISCUSSION

Although the gravity theories we have considered may be quite different in nature from each other, we have seen that the perturbed equations in a FLRW background can be managed very similarly. This may be due to their conformal transformation (CT) properties discussed in Ref. 1.

The asymptotic solutions and conserved quantities we derived in the large-scale limit deserve particular attention. The comoving scales we are interested in at the present epoch go outside the Hubble radius, defined as $a/k \equiv H^{-1}$, in the early stages of the evolution of the Universe. If we consider the inflationary scenario, most interesting scales are pushed outside the Hubble radius during the inflationary epoch and come back inside the Hubble radius during the radiation- or matter-dominated epochs. Since we want to know about the perturbation spectrum, of the energy density or the potential, measured at the second Hubble radius crossing time, usually all we have to know are the *initial spectrum calculated at the first Hubble radius crossing time and a transfer function coding the evolution of the spectrum while the scales*

stay outside the Hubble radius. In gravity theories with MSF, as we have seen in Eqs. (41) and (43), there is one independent perturbed potential and ζ is the conserved (usually called ‘‘frozen’’) variable on a large scale. [Φ_H is directly related to a GI density variable, and at the Hubble radius crossing, from Eq. (4), we have $\epsilon_m^E = \frac{2}{3}\Phi_H$. For a detailed discussion, see Refs. 2 and 9.] We have shown that there also exist GI conservation quantities ζ expressible in two ways even in the more complicated gravity theories considered in this paper.

There have been many studies of cosmological perturbations in generalized gravity theories, e.g., R^2 gravity,^{3,12} Brans-Dicke theory,^{13,14} NMSF theory,^{14,15} and induced gravity theory.^{14,16} However, many of these studies of the last three gravity cases were approximate. As we have seen, although one can treat these gravity theories in a similar way, it is not necessarily true that one can naively adopt a result [Eq. (45)] obtained for the Einstein gravity case with the perfect-fluid assumption. From Eq. (19), which is similarly true for the other gravity cases as in Eq. (30), we have

$$\zeta = \Phi_H - \frac{H^2}{\dot{H} - H \frac{F}{F'}} \left[\Phi_H + \frac{1}{H} \dot{\Phi}_H \right]. \quad (50)$$

Comparing this to the conventional result for perfect-fluid Einstein gravity [Eq. (45)], we notice two differences. First, there is an additional term in the denominator. Although $H^2 \gg -\dot{H}$ [$= \frac{1}{2}(\mu + p)$] during the usual inflationary epoch, one needs to check the importance of this additional term in each gravity theory. Second, in the conventional case, if $p \propto \mu$, one can see that the growing mode part of Φ_H is a constant and we have

$$\left[1 + \frac{2}{3} \frac{1}{1+p/\mu} \right] \Phi_H = \zeta. \quad (51)$$

However, depending on the background evolution, it is not obvious that one can treat Φ_H as a constant in the corresponding limit in all generalized gravity theories. Φ_H is a constant if a and F evolve as power laws in t . However, *if the background evolution is known, the evolution of Φ_H in the large scale limit can be determined from the integral form of solutions we have in Eqs. (18), (29), and (39).* In some cases people have employed the conformal transformation properties of these gravity theories to the conventional Einstein theory with a minimally coupled scalar field with a special potential.

In this paper we have shown that one can treat the perturbation equations in a rigorous manner even in these complicated gravity theories and solve them in a simple way. The connection between these results and previous work, and the physical implications of the solutions derived in this paper, especially during some realizable inflationary models, will be presented elsewhere.¹⁷

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- ⁴The main idea of Ref. 1 was to treat a given generalized gravity as a formal Einstein gravity theory and identify all additional terms as effective (imperfect) fluid contributions by using $G_{ab} = T_{ab}$.
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- ⁷If $z''/z = n/\eta^2$, where n is a constant, using $x \equiv k\eta$ and $y \equiv \eta^{-1/2}u$, Eq. (14) can be written in the form of the Bessel equation. The solution can be written in terms of Bessel or Hankel functions. In terms of the Hankel function the solution becomes
- $$u = \eta^{1/2} [c_1 H_\nu^{(1)}(k\eta) + c_2 H_\nu^{(2)}(k\eta)] ,$$
- where $\nu \equiv \sqrt{1/4 + n}$ and $c_{1,2}$ are the constant coefficients.
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