Classical limit in quantum cosmology: Quantum mechanics and the Wigner function

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We study the classical limit of quantum mechanics as applied to quantum cosmology. Conventional wisdom regards the peaking of the Wigner function of the Universe around a classical trajectory as being a quantum prediction of that trajectory. We show that, with quantum interference correctly taken into account, the hoped for "classical correlation" does not exist. There are, therefore, serious difficulties with the notion that a pure quantum state has a classical limit relevant to the description of our world. Some form of quantum decoherence appears necessary for a strict classical limit to exist. This alternative is briefly discussed.

I. INTRODUCTION

The nonexistence of time in quantum general relativity is a major conceptual obstacle confronting the quantization of the theory. This is because a background time appears impossible to avoid in conventional quantization procedures.[1] On the other hand, the approach of Hartle and Hawking[2] and of Vilenkin[3] in their quantum cosmology program envisages a transition from a "timeless" Euclidean theory to a semiclassical Lorentzian limit, where a classical time parameter is supposed to appear.

The appearance of this classical time is a specific instance of a quite general problem: When does a quantum system become *effectively* classical? Several competing ways of addressing this question exist. For pure states, we may implement either of the large-N,[4] WKB,[5] or wave-packet[6] (coherent-state) limits. The idea of "dephasing" or "decohering" the quantum system by an external environment[7] is relevant when treating open systems. In the case of quantum cosmology, this last idea has been implemented as a certain coarse-graining procedure involving a "trace over unobservables" (TOU).[8]

In this paper we will not attempt to tackle the deep problem of time, although recently this has been the main motivation for studying the classical limit in quantum cosmology. What we will do is to ask whether the WKB limit by itself is formally enough to provide a classical limit of the sort desired. Coherent states and the large-N limit will not be explored in any detail.

At present, it appears that two viewpoints exist with regard to the question of the classical limit of quantum cosmology. The first involves taking the "one-universe Everett" or "quantum mechanics of individual systems" (QMIS) interpretation seriously. Supporters of this view regard the peaking of the wave function or the corresponding Wigner function as being tantamount to a prediction. Thus, in order for a classical limit to be realized, they would want this peaking to occur around a classical trajectory. It has been argued that it is enough to have a WKB or a Gaussian state for an effectively classical description to be valid.[9] The second, and perhaps more complete, description is to argue that by some decoherence process the initial pure-state wave function goes over to a density matrix describing a mixed state, quantum interference terms vanish, and the system follows a classical trajectory. This seems to be more in line with a conventional interpretation of quantum mechanics. However, this latter approach has not been fully developed as yet. A tentative beginning has been made,[8] but major conceptual and technical questions remain to be answered. Prominent among them is the lack of a strong justification for TOU and of reasons for selecting one particular scheme over another. The efficacy of the usual procedures in removing quantum interference and producing an acceptable classical limit is, in our opinion, still an open question.[10]

In principle, there is a distinction between coarse graining and decoherence arrived at via TOU. The first procedure is usually justified by arguing that since only "coarse-grained" measurements are possible, our information is incomplete and possibly of a rather poor kind.[11] In the second case one asserts that a TOU is equivalent to some "quantum measurement" process that results in an effectively classical state. Both procedures lead to a consideration of reduced density matrices with an associated gain in entropy. Therefore, we shall treat them as being equivalent at this level. In this paper "decoherence" is taken to mean a suppression of quantum interference. Precisely how this is done, i.e., via coarse graining or tracing out of "irrelevant" degrees of freedom, is not considered important. On the other hand, the underlying bias here is that the classical limit is not obtained because of the observer doing something to the wave function, but rather that the system becomes classical on its own. This is the philosophy behind most of the work on quantum decoherence in quantum cosmology. While we will deal with minisuperspace models in this paper, decoherence procedures are usually implemented by starting in a midisuperspace and then tracing over the extra degrees of freedom to get to a minisuperspace. One of the points being made here is that one cannot begin with a minisuperspace and hope to obtain a classical limit within a QMIS interpretation.

The notion of a classical limit is not definitive since, depending on the questions we ask of a system and the experiments we perform on it, both classical and quantum answers can be obtained. A weak definition of this limit is the following. We define the classical limit as arising when no quantum interference effects are observable. Interference is certainly not the only quantum-mechanical effect, and it is possible for the system to possess other nonclassical properties. In this paper the Wigner function will serve as a diagnostic tool for the existence of interference. In what follows, the WKB form of the wave function will be often encountered. As will be shown, this "semiclassical" limiting form does not satisfy our definition of classicality.

The Wigner function has featured prominently in recent work on quantum cosmology.[9,12,13] The motivation has been twofold: Proponents of the "correlation" approach have attempted to use it as a "pointer" delineating the classical limit, whereas the authors of Ref. 13 wish to promote a more general approach which encompasses the density matrix of the Universe. There are other differences as well: in Ref. 9 the idea is to Wigner transform the wave function of the Universe (in the WKB limit), while in Refs. 12 and 13 the Wigner function has been found by directly solving the Wheeler-DeWitt equation. Whatever the approach, all authors appear to be in agreement on the classical limit of their respective Wigner functions, viz., that there should be a δ-function peak on the classical trajectory and very little elsewhere. This is a surprising result; one knows in ordinary quantum mechanics that the Wigner function is in general *not* peaked only on the classical trajectory (this is true in the WKB limit as well). In this paper we will resolve this apparent contradiction. It will be found that the authors of Ref. 9 have incorrectly evaluated the Wigner transform in the WKB limit, whereas the authors of Refs. 12 and 13 have implemented an approximation that implicitly *assumes* quantum interference effects to be absent. Once these oversights are corrected and the nature of the assumptions clarified, quantum interference effects return and prevent the pure-state Wigner function from being interpreted classically.

Our analysis will consist of a careful treatment of the Wigner transform for WKB wave functions. The method is not new; indeed, it has been used extensively in chemical physics (see, e.g, Refs. 6 and 14, and citations therein) and in the study of the classical limit of quantum mechanics.[15] This refinement in technique over previous studies in quantum cosmology will enable us to obtain two key insights: (1) The WKB Wigner function *already contains* the quantum spreading that other authors have sought to obtain via adiabatic expansions and (2) the difference between Hartle-Hawking and Vilenkin boundary conditions in phase space. This difference is best studied starting from the corresponding wave functions. Somewhat surprisingly, we find that the implementation of the Vilenkin boundary condition is not quite obvious in phase space. In this paper, all calculations will be for Hartle-Hawking-like wave functions; the Vilenkin prescription will be taken up in a future communication.

All of our conclusions derive directly from quantum mechanics and are independent of the details of the minisuperspace models that were considered in earlier work.[9,12,13] One may therefore confidently anticipate that quantum interference will show up in all quantum cosmology models. This leads directly to another central message of this paper: In quantum cosmology there can be no classical limit without quantum decoherence. This remark appears to be rather obvious, if not trivial. To put it in perspective we have to contrast the situations arising in quantum mechanics against those in quantum cosmology. Quantum interference shows up in the guise of positive- and negative-going oscillations in the Wigner function. In the $\hbar \rightarrow 0$ limit these oscillations tend to become quite rapid, and whenever expectation values are computed they do not contribute. As far as quantum mechanics is concerned, the oscillations are harmless in almost all applications. However, in quantum cosmology expectation values cannot be defined because the wave functions are not normalizable. In the WKB limit, the Wigner function does indeed vary rapidly in certain regions, but given the QMIS interpretation, it cannot be neglected there because of the occurrence of strong peaks.

The attempt here is also to set the stage for an improved analysis of the semiclassical limit with quantum decoherence being a key ingredient. In the present work quantum cosmological considerations have been deliberately deemphasized in order to keep the discussion as general as possible. A detailed treatment of minisuperspace examples and a quantitative discussion of decoherence will be presented elsewhere.[10]

The plan of the paper is as follows. First, the QMIS "correlation" arguments of Ref. 9 will be scrutinized and the improved WKB analysis presented. Next, we will consider the approach of Kodama[12] and of Calzetta and Hu.[13] It will be shown that the adiabatic approximation used by these authors presupposes that quantum interference terms be almost negligible. Furthermore, various obscurities in the treatment of the Wigner function in quantum cosmology will be discussed. This will be followed by a consideration of concrete examples where the preceding theoretical arguments will be illustrated. Finally, we will end with a somewhat general discussion of the quantum decoherence problem; this will serve as the motivation for future communications.

II. WIGNER FUNCTIONS AND THE WKB LIMIT

A useful semiclassical formalism in quantum mechanics is the WKB approximation.[5] In the one-dimensional, time-independent case, the Schrödinger equation for a particle in a potential $V(x)$ is

$$\frac{\partial^2\psi}{\partial x^2}+\alpha^2[E-V(x)]\psi=0 \ , \tag{1}$$

where

$$\alpha^2=\frac{2m}{\hbar^2} \ . \tag{2}$$

This equation is formally identical to the Wheeler-DeWitt equation of quantum cosmology for effectively one-dimensional minisuperspace models. The reader is directed to Ref. 13 for details.

The idea behind WKB is to assume the ansatz

$$\psi = e^{i\alpha S(x)} , \tag{3}$$

with $S(x)$ complex, and expanded in the formal power series,

$$S(x) = S_0(x) + \left[\frac{1}{i\alpha}\right] S_1(x) + \left[\frac{1}{i\alpha}\right]^2 S_2(x) + \left[\frac{1}{i\alpha}\right]^3 S_3(x) + \cdots , \tag{4}$$

where all the $S_i(x)$ are real. In order for (4) to be a reasonable expansion, one has to show that relatively few terms have to be kept. This imposes the condition that the de Broglie wavelength of the particle must vary little over the "size" of the particle. Moreover, the straightforward approach fails near the classical turning points of $V(x)$. As shown later, the WKB Wigner function does not suffer from this deficiency.

In the case of a confining potential, the primitive (in contrast with the more accurate uniform approximation[15]) WKB approximation to the true energy eigenfunction is

$$\psi = C\Delta^{1/2} e^{iS_{\rm cl}(x,I)/\hbar} , \tag{5}$$

where $S_{\rm cl}(x,I)$ is the position-dependent classical action, C is a constant, Δ is the one-dimensional Van Vleck determinant,[16]

$$\Delta(x,I) = \left| \frac{\partial^2 S_{\rm cl}(x,I)}{\partial x\, \partial I} \right| , \tag{6}$$

and the classical action variable

$$I = \frac{1}{2\pi} \oint p(x) dx . \tag{7}$$

The integral is over a phase-space orbit and the classical momentum $p(x) = \sqrt{2m[E - V(x)]}$. When the orbits are unbounded, as is often the case in quantum cosmology, we will use the energy E in place of the action variable I in the Van Vleck determinant.

The WKB form (5) is obtained by substituting the semiclassical power series of (4) in the Schrödinger equation and solving order by order in $(1/i\alpha)$. The expansion is semiclassical since $m \to \infty$ or $\hbar \to 0$ (hence $\alpha \to \infty$) defines a "classical limit." In the present situation it does not matter how this particular limit is defined ($m \to \infty$ or $\hbar \to 0$), and so from now on we will stick with the conventional choice of $\hbar \to 0$.

The question is now to see what the WKB limit (5) implies for the trajectory of the particle in phase space. The wave function itself is not localized and possesses multiple peaks. Therefore, it is not a useful object with which to implement a classical limit.[17] In order to obtain a satisfactory phase-space description, it is customary to take the Wigner transform[18] of the wave function. This involves shifting to the new variables

$$X = \tfrac{1}{2}(x_1 + x_2) , \quad x = x_1 - x_2 , \tag{8}$$

and then taking the Fourier transform of the pure-state density matrix $\Psi(x_1)\Psi^*(x_2)$ over x,

$$f(X,k) = \int_{-\infty}^{+\infty} \frac{dx}{(2\pi\hbar)} e^{-ikx/\hbar} \Psi\left[X + \frac{x}{2}\right] \Psi^*\left[X - \frac{x}{2}\right] . \tag{9}$$

For the primitive WKB form of the wave function, the Wigner function (9) becomes

$$f_{\rm WKB}(X,k) = C^2 \int_{-\infty}^{+\infty} \frac{dx}{(2\pi\hbar)} e^{-ikx/\hbar} \times \exp\{ i[S_{\rm cl}(X + x/2) - S^*_{\rm cl}(X - x/2)]/\hbar \} \times \Delta_+^{1/2} \Delta_-^{1/2} , \tag{10}$$

where

$$\Delta_\pm \equiv \Delta(X \pm x/2, I) . \tag{11}$$

The integrand varies rapidly in the semiclassical limit. Therefore, (10) may be evaluated by the method of stationary phase (later to be supplemented by the method of uniform approximation). In Ref. 9 an alternative approach has been attempted by introducing a new variable $x = \hbar u$ and formally expanding $S_{\rm cl}$ around X inside the integral. Higher-order terms in $\hbar$ are then dropped and the integral found. However, this procedure is illegitimate and leads to incorrect results. It amounts to unjustifiably switching the $u \to \pm\infty$ limit of the integration with the WKB limit $\hbar \to 0$.

In a fascinating paper,[15] on which we will rely heavily, Berry has obtained the correct expressions for $f_{\rm WKB}$ (see also Ref. 6). We now outline his calculational procedure. The stationary phase condition imposed on (10) leads to

$$p(X + x/2) + p(X - x/2) = k . \tag{12}$$

For every phase-space point (X,k) this equation yields two solutions $\pm x_0(X,k)$ which define the chord P_1P_2 (Fig. 1). The magnitude of the phase of the integrand in (10) is just the area between the chord and the energy shell $\mathcal{E}$. Then, for (X,k) inside $\mathcal{E}$ and not close to the turning points, a simple quadratic expansion of the phase yields

$$f_{\rm WKB}(X,k) = \frac{2\cos[A(X,k)/\hbar - \pi/4]}{\pi\sqrt{\hbar}\,[I_X(P_2)I_k(P_1) - I_X(P_1)I_k(P_2)]^{1/2}} , \tag{13}$$

where $A(X,k)$ is the area defined above, $I_X \equiv \partial I/\partial X$, $I_k \equiv \partial I/\partial k$, and $C^2 = 1/2\pi$. This form is not entirely satisfactory; it is divergent on $\mathcal{E}$ (where P_1 and P_2 coalesce) and is not normalized to unity. However, it is square integrable (for bound states) and clearly displays an oscillating "fringe" structure within $\mathcal{E}$.

When the phase-space point (X,k) approaches the energy shell, the two stationary points coalesce into one and

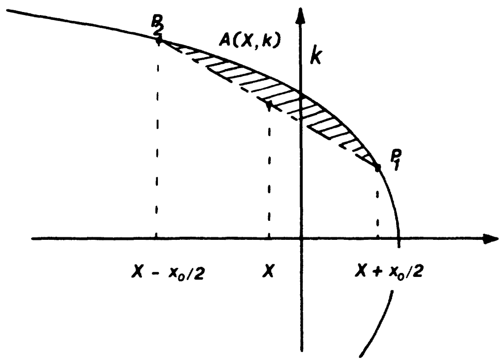

FIG. 1. Stationary phase points $\pm x_0/2$ determine the chord $\overline{P_1P_2}$. $A(X,k)$ is the area of the shaded region defined by $\overline{P_1P_2}$ and the constant energy surface $\mathcal{E}$.

(13) diverges. The integral can now be evaluated by expanding the phase to cubic order, this giving the "transitional approximation"

$$f_{\rm WKB}=\frac{1}{\pi}[\hbar^2 B(X,k)]^{-1/3}\times {\rm Ai}(2[I(X,k)-I_E][\hbar^2 B(X,k)]^{-1/3}) , \quad (14)$$

where Ai denotes the Airy function and

$$B(X,k)=I_X^2 I_{kk}+I_k^2 I_{XX}-2I_X I_k I_{kX} . \quad (15)$$

Note that the "classical" Airy peak height is of $O(\hbar^{-2/3})$ and, further, that this peak is *not* on the classical trajectory, but shifted from it by

$$\Delta I \sim \tfrac{1}{2}\hbar^{2/3}B^{1/3} , \quad (16)$$

where $\Delta I = I(X,k)-I_E$. The width of this peak is not representation invariant. For example, Berry has shown[15] that the "action-angle" Wigner function has a δ-function peak on I_E and is zero everywhere else. The Wigner function for the eigenfunction of an operator attains a minimal spread in the representation in which that operator is diagonal. This is illustrative of the fact that the Wigner function is *not* invariant under canonical transformations in phase space.[19] The quantum spread of the classical peak of $O(\hbar^{2/3})$ is due solely to the fact that we have evaluated $f_{\rm WKB}$ in the (q,p) representation, in which the Hamiltonian is not diagonal.

Uniformization (see, e.g., Refs. 6, 14, and 20) enables a single expression to be written for the two limiting cases found above and leads to Berry's main result:

$$f_{\rm WKB}=\frac{\sqrt{2}[3A(X,k)/2]^{1/6}{\rm Ai}(-[3A(X,k)/2\hbar]^{2/3})}{\pi\hbar^{2/3}[I_X(P_2)I_k(P_1)-I_X(P_1)I_k(P_2)]^{1/2}} . \quad (17)$$

This form shares with the less general expressions (13) and (14) a remarkable property. It is valid near turning points despite the fact that the WKB wave function (5) is not; this is a well-known advantage of phase-space methods.[21] The improved $f_{\rm WKB}$ is correctly normalized and square integrable for bound states. It also possesses the characteristic oscillations inside $\mathcal{E}$. With (X,k) outside $\mathcal{E}$, $f_{\rm WKB}$ is exponentially damped. Note that (13) is obtained as an asymptotic limit of (17) when (X,k) is inside $\mathcal{E}$.

The stationary phase evaluation fails at "catastrophes" of $f_{\rm WKB}$.[15] While it is possible to fix this by using another analytic extension, we will not bother to do so. As a result, the specific examples considered in Sec. IV will have artificial "blow-ups" in $f_{\rm WKB}$ very close to $k=0$.

When the phase-space trajectory is unbounded, the Wigner function is not strictly normalizable. Also, as remarked earlier, the action variable is now to be replaced by the energy. This step is essentially trivial and, modulo constants, amounts to replacing I by E in (17). In most situations wave functions in quantum cosmology correspond to those for the unbounded case in quantum mechanics. Apart from the normalization there is no fundamental difficulty in defining the Wigner transform. This point is worth emphasizing. The fact that the phase-space trajectory is not bounded does not at all imply that the Wigner function cannot be defined. All it means is that the usual normalization and square-integrability requirements no longer hold.

Let us now attempt to make contact with the results of Ref. 9. Contrary to their claims, we see immediately that $f_{\rm WKB}$ is *not* positive definite, nor is the classical delta function realized as $\hbar\to 0$. Both (13) and (17) are highly oscillatory for small $\hbar$ with no accumulation on the classical energy shell $\mathcal{E}$. This feature has previously been noted by Heller[6] and is characteristic of Wigner functions in this limit. Such behavior arises from the fact that eigenfunctions of Hermitian operators do not localize as $\hbar$ goes to zero and represents the persistence of quantum interference.

Given that $f_{\rm WKB}$ is not positive definite, one might ask whether it is fair to compare it to a classical distribution function. It would seem that any insistence on the positivity of a pure-state Wigner function is misplaced; we have only to recall a theorem of Hudson,[22] which states that the *only* pure state with $f\geq 0$ is the Gaussian (with a,b,c complex and Re$a<0$):

$$\Psi \sim e^{ax^2+bx+c} , \quad (18)$$

a highly restrictive condition if we were to take such arguments seriously. In actual fact the positivity of the pure-state Wigner function may not be a real concern. Assuming decoherence arguments to be valid, one expects the effective coarse graining so induced to destroy the quantum interference effects that lead to negativity. The classical limit may then be regained, but the Wigner function will be that for a *mixed* state.

The classical distribution function $f_{\rm cl}$ representing the orbit of a mass point in a strictly confining potential is

$$f_{\rm cl}=\frac{1}{2\pi}\delta(I(X,k)-I_E) , \quad (19)$$

and is identical to the form claimed for $f_{\rm WKB}$ in Ref. 9. Clearly, in this case (19) is *unacceptable* as a quantum distribution function because it is not square integrable. This condition follows from the requirement that the underlying wave functions be normalized, and this is certainly true for the corresponding uniformized WKB case in quantum mechanics. Even in the unbounded case $f_{\rm WKB}$ will not have the form (19) for the reasons given previously. There is little difference between bounded and unbounded cases as far as the stationary phase evalution of $f_{\rm WKB}$ is concerned. Since this evaluation is approximately local, it is insensitive to the global nature of the phase-space trajectory (except perhaps very close to $\mathcal{E}$) and depends only on its local curvature.

With (19) as the classical limit, how can we show that $f_{\rm WKB}$ will ever tend to it? This can be done, provided we introduce an explicit "smoothing" or "averaging" scheme to remove the quantum interference *by hand.* First, observe that for wave functions $\psi(x)$ and $\phi(x)$ with corresponding Wigner functions $f^{(\psi)}(X,k)$ and $f^{(\phi)}(X,k)$,

$$\left|\int_{-\infty}^{+\infty} dX\, \psi^*(X)\phi(X)\right|^2 = (2\pi\hbar)\int_{-\infty}^{+\infty} dX\, dk\, f^{(\psi)}(X,k) f^{(\phi)}(X,k)\ . \quad (20)$$

Since the left-hand side of (20) is manifestly positive, it follows that on tracing $f_{\rm WKB}$ times any other Wigner function over phase space, a non-negative object is obtained. If, furthermore, one assumes that the "smearing" Wigner function is smooth in the classical limit, it can be shown[6] that the classical delta function is indeed realized (but only as a limit). A common choice for the "smearing" function is a Gaussian, this defining the well-known Husimi distribution.[23] While this averaging procedure can perhaps be justified for a laboratory system, it is a highly nontrivial assumption in quantum cosmology, especially for minisuperspace models with few degrees of freedom. In any case, it is easy to show that this is not the sort of smoothing so far envisaged in quantum cosmology (see Sec. V). Yet another point should be made: the "smoothed" objects of (20) are not guaranteed to be Wigner functions themselves[24] (this arises from the constraint that the corresponding density matrix be positive semidefinite). Therefore, there is, in principle, a restriction on the sorts of coarse grainings that are to be allowed.

As is already clear, the literal interpretation of the Wigner function as a correlation function is problematic. This is also reflected in the occurrence of large interference peaks, many more prominent than the peak near the classical trajectory. However, the negative-going oscillations do not show up in the probability distributions

$$P(X)=\int_{-\infty}^{+\infty} dk\, f(X,K)\ ,$$
$$P(k)=\int_{-\infty}^{+\infty} dX\, f(X,k)\ , \quad (21)$$

as they are removed by the integration. Of course, $P(X)$, and $P(k)$ are just $|\psi(X)|^2$ and $|\psi(k)|^2$, and no more information is obtained than that contained in the wave function in the first place.

Now we turn to the issue to Hartle-Hawking and Vilenkin boundary conditions for the wave function. Although nothing concrete was said about these choices, the previous calculations include *both* signs of the classical momenta and thus correspond to the Wigner transform of

$$\psi \sim e^{iS/\hbar} + e^{-iS/\hbar}\ , \quad (22)$$

which corresponds in quantum cosmology to a Hartle-Hawking wave function. The Vilenkin wave function has only *one* sign for the classical momentum. This can mean one of two things: (1) Half of the phase space simply does not exist, or (2) only one branch of the classical trajectory exists. In either case, the stationary phase evaluation of the Wigner transform will be complicated by having to separate the phase space into different regions, each corresponding to the number of stationary phase points for each (X,k) in that region (see Fig. 2). At present, the computation of $f_{\rm WKB}$ for this case remains somewhat obscure, but we can make the following remarks. Close to the classical trajectory both Vilenkin and Hartle-Hawking Wigner functions will be very similar since the dominant contribution will come from stationary phase points on that trajectory. However, away from the classical trajectory this will no longer be true. Since in Vilenkin's case only one branch of the trajectory is utilized, quantum interference effects (as seen by oscillations of $f_{\rm WKB}$) will be smaller. This implies that the Hartle-Hawking and Vilenkin wave functions will decohere on different scales.

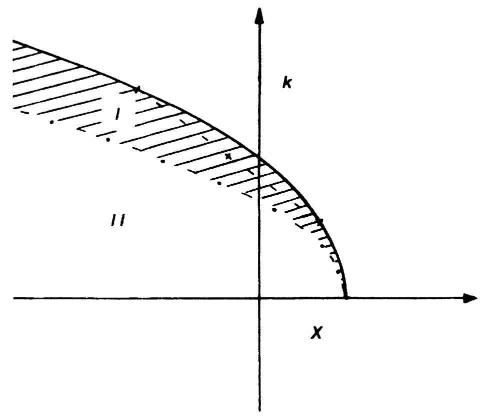

FIG. 2. Phase-space points in region I are associated with two stationary phase points. This is not true for points in region II. The dividing line between the two areas follows from an obvious geometric construction.

A final cautionary note on the use of Wigner functions in quantum cosmology is that such objects may not even be defined. This is because (1) the wave functions there are not necessarily normalizable, (2) the range of the argument of the wave function is not the complete real line, and (3) minisuperspaces are not flat in general. A more extended discussion of these problems is given in Ref. 10.

III. ADIABATIC APPROXIMATION

We have demonstrated that the Wigner transform of the WKB wave function does not have a strict classical limit if interference is not averaged out. On the other hand, in Refs. 12 and 13 the Wigner function was found by directly solving the Wheeler-DeWitt equation, and a relatively sharp classical limit was obtained. In this section we will show that this result is an artifact of the adiabatic approximation which ignores interference effects.

Some previous quantum cosmology literature on the Wigner function gives the dangerous impression that the WKB result is a δ function and that higher-derivative terms in the quantum Liouville equation are needed to find the quantum spread around the classical trajectory.[9,12] This is false. In fact, the WKB approximation we have implemented is more accurate than an adiabatic expansion and *does* account for the quantum softening of the classical peak. As we saw earlier, the Airy "head" of $f_{\rm WKB}$ on the classical trajectory occurs not right on top of it, but rather at an action distance of $O(\hbar^{2/3})$.

In Ref. 12, Kodama defined and evaluated the Wigner function for a Robertson-Walker minisuperspace model. His definition of the Wigner transform (as also those of Ref. 9) is valid only for a flat minisuperspace. This shortcoming was rectified by Calzetta and Hu[13] who also evaluated their Wigner function for a few minisuperspace examples.

These authors relied on an expansion in derivatives of the potential. Such an approximation can, of course, be used in quantum mechanics and we shall do so here. Our aim will be to contrast it with the Wigner transform of the WKB approximation for the wave function.

The philosophy behind the adiabatic expansion is to solve directly for the Wigner function rather than to first obtain the wave function. As pointed out by Calzetta and Hu, this is a more general approach since it enables the consideration of mixed states in quantum cosmology for which no underlying wave function exists. Derivative expansions are commonly used in statistical mechanics and in quantum field theory in curved spacetime[25,26] with the tacit assumption that there are no "fast" terms to spoil the formal adiabatic series. Such terms do not exist for decohered quantum systems, but they are present, quite generically, for pure states.

To begin, we rewrite the time-independent Schrödinger equation in the language of density matrices:

$$H(x_1)\rho(x_1,x_2)=E\rho(x_1,x_2) . \tag{23}$$

$H(x)$ will be taken to be the Hamiltonian for a particle in a one-dimensional potential. Next, we reexpress (23) in Wigner variables (X,x), expand $H(x_1)$ around X, and then Fourier transform with respect to x. This procedure yields two equations corresponding to real and imaginary parts, respectively:

$$L_S f(X,k)=0 , \tag{24}$$

$$L_A f(X,k)=0 , \tag{25}$$

where

$$L_S=-\frac{\hbar^2}{8m}\frac{\partial^2}{\partial X^2}+\frac{k^2}{2m}+V(X) -E+\sum_{n\ \rm even}\left[\frac{i\hbar}{2}\right]^n\frac{1}{n!}\frac{\partial^n V(X)}{\partial X^n}\frac{\partial^n}{\partial k^n} , \tag{26}$$

$$L_A=-\frac{k}{m}\frac{\partial}{\partial X}+\frac{\partial V(X)}{\partial X}\frac{\partial}{\partial k} +\sum_{\lambda\ \rm odd}\left[\frac{\hbar}{2i}\right]^{\lambda-1}\frac{1}{\lambda!}\frac{\partial^\lambda V(X)}{\partial X^\lambda}\frac{\partial^\lambda}{\partial k^\lambda} . \tag{27}$$

Note that (27) is just the time development operator of the quantum Liouville equation

$$\frac{\partial}{\partial t}f(X,k)=L_A f(X,k) =L_{\rm cl}f(X,k)+L_Q f(X,k) , \tag{28}$$

where $L_{\rm cl}$ is the first two terms of L_A, and L_Q is the sum involving odd-order momentum derivatives. It is clear that all that (25) says is that $\partial f(X,k)/\partial t=0$. In general, solving (24) and (25) together is a formidable enterprise. However, there are at least two potentials where this can be done exactly: the linear potential and the harmonic oscillator. The reason is simple. When $V(X)=aX+bX^2$, there are no "quantum" terms in (25) as L_Q is identically zero. This means that $f(X,k)$ is a function only of $H=k^2/2m+V(X)$, reducing (24) to an ordinary differential equation, which can then be solved by elementary means. The solutions for the linear potential and the nth eigenstate of the harmonic oscillator are, respectively,

$$f_{\rm lin}(X,k)=\frac{1}{\pi\hbar}\left[\frac{m}{a^2\hbar^2}\right]^{1/3} \times {\rm Ai}\left[2\left[\frac{m}{a^2\hbar^2}\right]^{1/3}(H-E)\right] , \tag{29}$$

$$f_{\rm LHO}(X,k)=\frac{(-1)^n}{\pi\hbar}L_n\left[\frac{4H}{\hbar\omega}\right]e^{-2H/\hbar\omega} , \tag{30}$$

where L_n are the Laguerre polynomials. These solutions illustrate the fact that even if (28) is just the classical Liouville equation, quantum interference effects are unavoidably present, manifest in the oscillations of the Laguerre and Airy functions. These are the same oscillations already seen to be present in $f_{\rm WKB}$.

The adiabatic expansion aims to solve (24) and (25) by keeping terms only up to some derivative or "adiabatic" order. For (24) and (25) this amounts to assuming that the sums in these equations are expansions in increasing powers of $\hbar$. The initial impression of having a convergent power series in $\hbar$ is mistaken (such an erroneous impression is conveyed in Ref. 9). This important point has been emphasized by Heller.[27] The reason is simply that the Wigner function may involve an $\hbar$ dependence ("fast" momentum behavior) such that when derivatives with respect to momentum are taken, terms $O(0)$ and $O(1/\hbar)$ appear in every individual term of the sums in (24) and (25). Such terms lead to L_S and L_A being $O(0)$ and $O(1/\hbar)$, formally divergent in the $\hbar\to 0$ limit. This is precisely the reason that accounts for the failure of derivative expansions. There one begins by assuming that

$$f(X,k)=\sum_{n=0}^{\infty}f_n(X,k)\delta^{(n)}(H) , \tag{31}$$

where

$$\delta^{(n)}(y) \equiv \frac{d^n \delta(y)}{dy^n} . \tag{32}$$

Provided that a derivative expansion is sensible, it is possible to show that up to a given order in the expansion only a finite number of the f_n's exist.[26] Thus, to some order, one obtains the quantum spreading of the classical delta function. However, the occurrence of the $O(1/\hbar)$ terms render the adiabatic expansion invalid and (31) is no longer a correct ansatz.

It is easy to see that $f_{\rm WKB}$ and the two exact solutions mentioned above produce $O(1/\hbar)$ terms. This is a warning that adiabatic expansions will fail when solving for pure states. However, for states where the interference has been "quenched," the adiabatic expansion makes sense. As an example of a state that is free from the dangerous "fast" terms, consider the Wigner function for an ensemble of harmonic oscillators at finite temperature:

$$f_T(X,k) = \frac{1}{\pi\hbar} \tanh(\hbar\omega\beta/2) \times \exp[-(2/\hbar\omega)\tanh(\hbar\omega\beta/2) H(X,k)] . \tag{33}$$

One sees immediately that momentum derivatives do not produce any dangerous terms.

At present there does not exist any unambiguous way to decide when the adiabatic approximation may be used. However, the following general remarks are in order. First, the "classical" region of phase space (the Airy "head" of $f_{\rm WKB}$) may be reasonably well approximated because there are no oscillations to take into account. Second, since this method is really suited to the study of mixed states, it is useful when the quantum Liouville equation becomes a master equation. This means that the approximation can safely be used for studying the approach of a system toward equilibrium as in this case decoherence is extremely rapid.[28] Therefore, it appears that while the adiabatic approximation is perfectly acceptable for studying statistical systems, it is not adequate for an extensive analysis of the Wigner function of the Universe, at least not until it has been decohered.

As a final aside, note that in quantum field theory particle creation by strong fields may be studied in the WKB approximation. However, the neglect of "fast" terms inherent in the adiabatic expansion does not allow its use to treat the same problem.[29]

IV. EXAMPLES

In this section we will study some explicit examples of Wigner functions. In the case of the harmonic oscillator and the linear potential, the exact and WKB answers will be compared. Two examples that are of some relevance to quantum cosmology are the exponential potential and the upside-down harmonic oscillator $[V(x) \sim -x^2]$; the WKB Wigner function for these cases will be calculated. As remarked in Sec. II, $f_{\rm WKB}$ will have artificial divergences very close to $k=0$.

The linear harmonic oscillator is the only bound-state problem we consider, mainly for illustrative purposes, and to show how remarkably accurate $f_{\rm WKB}$ can be. Not only will the WKB calculations be straightforward, but the exact answer is also known[30] (and see Sec. III):

$$f(X,k) = \frac{(-1)^n}{\pi\hbar} e^{-2H/\hbar\omega} L_n(4H/\hbar\omega) . \tag{34}$$

The L_n are Laguerre polynomials and the Hamiltonian, $H = k^2/2m + \frac{1}{2}m\omega^2X^2$. The energy of the nth state is $E = (n+1/2)\hbar\omega$. Note that the WKB approximation gives these energy eigenvalues correctly.

The WKB Wigner function is found easily by simple substitution in the general formulas of Sec. II. The stationary phase condition (12) yields

$$x_0 = \frac{2k}{m\omega} \left[\frac{E - H(X,k)}{H(X,k)} \right]^{1/2} . \tag{35}$$

The action variable $I = E/\omega$ and

$$I_X(P_2)I_k(P_1) - I_X(P_1)I_k(P_2) = (X + x_0/2)p(X - x_0/2) - (X - x_0/2)p(X + x_0/2) . \tag{36}$$

The area $A(X,k)$ will be split into three pieces:

$$A_1 = kx_0 , \quad A_2 = \int_{X - x_0/2}^{X + x_0/2} dx' p(x') , \quad A_3 = 2\int_{X_c}^{X - x_0/2} dx' p(x') , \tag{37}$$

where X_c is the classical turning point. Then,

$$A(X,k) = |A_2| + \theta(k_{\max} - k)|A_3| - |A_1| , \tag{38}$$

where, for a given X,

$$k_{\max} = \tfrac{1}{2} p(2X - X_c) . \tag{39}$$

This is a cumbersome way of writing out the area; nevertheless, it is convenient computationally and we will use it in all the examples that follow. For the harmonic oscillator,

$$\begin{aligned} A_2 &= \tfrac{1}{2}(X + x_0/2)p(X + x_0/2) - \tfrac{1}{2}(X - x_0/2)p(X - x_0/2) \\ &\quad + \frac{E}{\omega}\{\arcsin[(X + x_0/2)\omega\sqrt{m/2E}\,] - \arcsin[(X - x_0/2)\omega\sqrt{m/2E}\,]\} , \\ A_3 &= (X - x_0/2)p(X - x_0/2) - \frac{E\pi}{\omega} + \frac{2E}{\omega}\arcsin[(X - x_0/2)\omega\sqrt{m/2E}\,] . \end{aligned} \tag{40}$$

Substitution of these equations (17) gives the required $f_{\rm WKB}$. The exact and WKB Wigner functions are plotted in Figs. 3 and 4 for different choices of n and $\hbar$, but with constant E. Furthermore, m and ω have been set to unity so that $\mathcal{E}$ is a circle. The oscillations and large interference peaks clearly stand out.

The correspondence with quantum cosmology involves

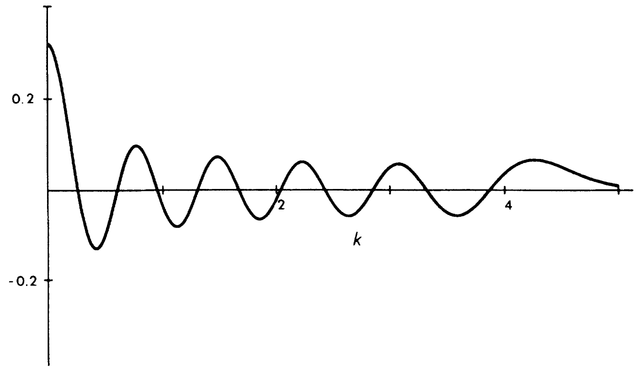

FIG. 3. Wigner function for a linear harmonic oscillator; on this scale f_{WKB} is indistinguishable from the exact solution. Parameter values are $\hbar=1$, $m=\omega=1$, and $E=10.5$. Shown is a $X=0$ slice of phase space. The Wigner function is rotationally invariant around the origin for this particular choice of the parameters.

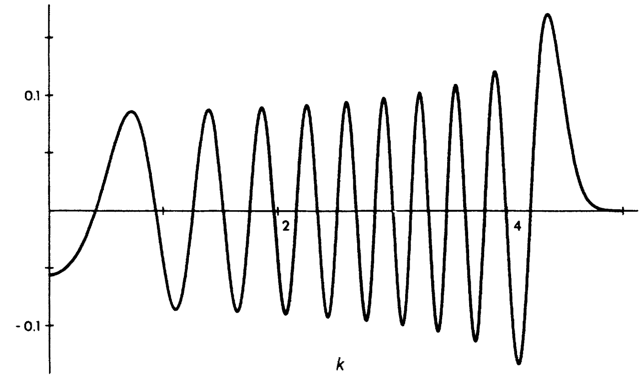

FIG. 5. Linear potential; the transitional approximation is exact in this case. Shown is a $X=0$ slice of phase space. The Wigner function is invariant under momentum reflection. Parameter choices are $\hbar=1$, $\alpha=-1$, and $E=10$.

studying "unstable" potentials. The simplest is the linear potential

$$V(x)=-\alpha x \ . \tag{41}$$

A minisuperspace example where such a potential appears is a Friedmann-Robertson-Walker (FRW) universe with a cosmological constant.[10] In this case,

$$x_0=\frac{k\sqrt{2m}}{\alpha}\sqrt{H(X,k)-E} \ , \tag{42}$$

and

$$\begin{aligned} A_2&=\frac{2\sqrt{2m}}{3\alpha}\{[E+\alpha(X+x_0/2)]^{3/2} \\ &\qquad -[E+\alpha(X-x_0/2)]^{3/2}\} \ , \\ A_3&=\frac{4\sqrt{2m}}{3\alpha}[E+\alpha(X-x_0/2)]^{3/2} \ . \end{aligned} \tag{43}$$

The solution is plotted in Figs. 5 and 6. The transitional approximation is identical to the exact solution. Note that the classical peak is now the biggest feature.

The upside-down harmonic oscillator is an important example since it is relevant to inflation and to questions regarding the quantum to classical transition. The potential is

$$V(x)=-\tfrac{1}{2}m\omega^2x^2 \ . \tag{44}$$

The stationary phase points are given by

$$x_0=\frac{2k}{m\omega}\left[1-\frac{E}{H(X,k)}\right]^{1/2} , \tag{45}$$

and the areas (for $E<0$)

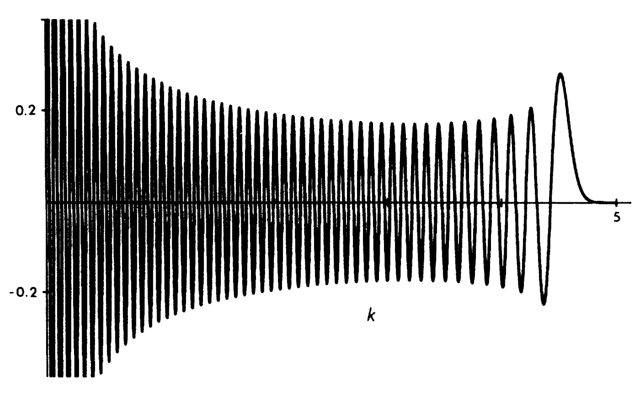

FIG. 4. Same as for Fig. 3, except that $\hbar=\frac{7}{67}$.

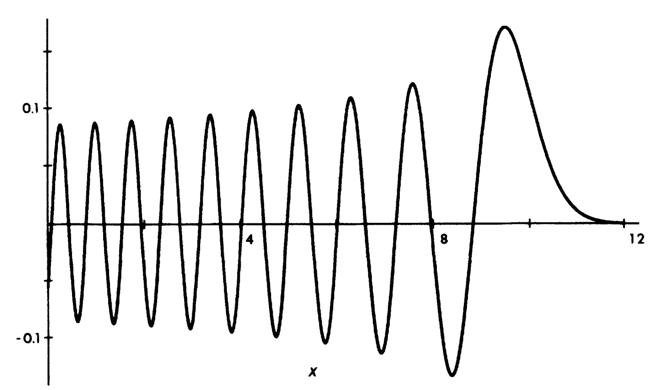

FIG. 6. Parameter values are those of Fig. 5. A $k=0$ slice is displayed. The oscillations of the Wigner function fall off smoothly in amplitude for values of X in the left half-plane.

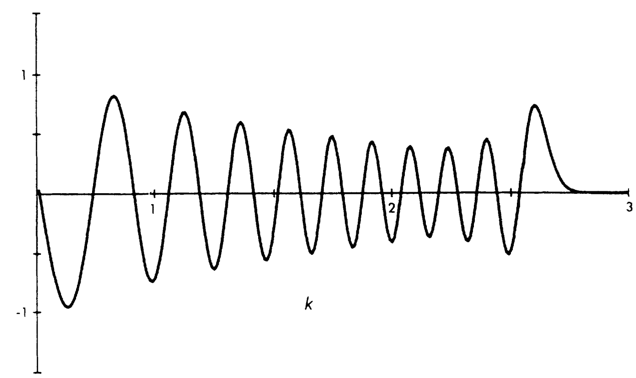

FIG. 7. Upside-down harmonic oscillator; a $X=3$ slicing. Values of constants are $h=0.5$, $m=\omega=1$, and $E=-1$.

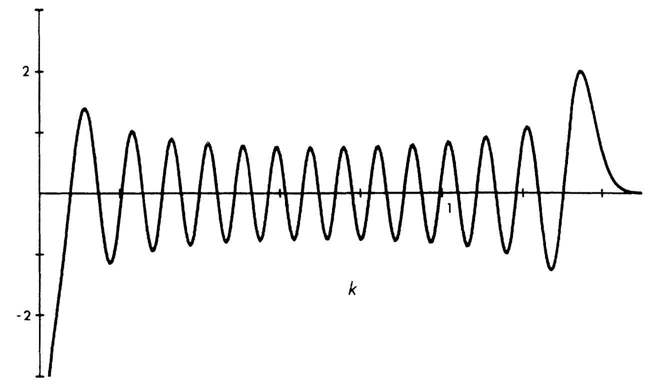

FIG. 8. Exponential potential; another $X=3$ slice. The parameters are $h=0.5$, $m=\alpha=\beta=1$, and $E=1$.

$$A_2=\tfrac{1}{2}(X+x_0/2)p(X+x_0/2)-\tfrac{1}{2}(X-x_0/2)p(X-x_0/2)-\frac{E}{\omega}\left[\ln\left[\frac{X+x_0/2+(m\omega)^{-1}p(X+x_0/2)}{X-x_0/2+(m\omega)^{-1}p(X-x_0/2)}\right]\right],$$
$$A_3=(X-x_0/2)p(X-x_0/2)-\frac{2E}{\omega}\left[\ln\left[\frac{X-x_0/2+(m\omega)^{-1}p(X-x_0/2)}{\sqrt{-2E/m\omega^2}}\right]\right]. \tag{46}$$

The corresponding Wigner function is shown in Fig. 7. Extrapolating from our experience with the linear potential, we might again expect the classical peak to be dominant. In fact, quite the reverse happens. Just as for the normal harmonic oscillator, we find large interference peaks near the middle of the two "arms" of the classical trajectory. This behavior is therefore not peculiar to bound states alone and creates further difficulties for the QMIS interpretation in the quantum cosmological context.

Finally, we consider the exponential potential, which has as minisuperspace analog, the Kantowski-Sachs universe. With

$$V(x)=\alpha e^{-\beta x}, \tag{47}$$

we find

$$x_0=\frac{2}{\beta}\operatorname{arccosh}\{\tfrac{1}{2}[-\mathcal{A}+(\mathcal{A}^2+4\mathcal{D})^{1/2}]\}, \tag{48}$$

where

$$\mathcal{A}=\frac{2k^2e^{\beta X}}{m\alpha},$$
$$\mathcal{D}=k^2e^{2\beta X}\left[\frac{k^2-2mE}{m^2\alpha^2}\right]-1. \tag{49}$$

The areas are given by somewhat complicated expressions. Defining

$$\Theta_{\pm}\equiv(E-\alpha e^{-\beta(X\pm x_0/2)})^{1/2}, \tag{50}$$

we find

$$A_2=-\frac{\sqrt{2m}}{\beta}\left\{2(\Theta_+-\Theta_-)+\sqrt{E}\times\ln\left[\left[\frac{\Theta_+-\sqrt{E}}{\Theta_++\sqrt{E}}\right]\times\left[\frac{\Theta_-+\sqrt{E}}{\Theta_--\sqrt{E}}\right]\right]\right\},$$
$$A_3=-\frac{2\sqrt{2m}}{\beta}\left[2\Theta_-+\sqrt{E}\ln\left[\frac{\Theta_--\sqrt{E}}{\sqrt{E}-\Theta_-}\right]\right]. \tag{51}$$

The resulting Wigner function is displayed in Fig. 8. Again, we note the existence of relatively large interference terms.

V. DECOHERENCE AND THE CLASSICAL LIMIT

Decohering the WKB Wigner distribution function so as to yield a classical distribution is the goal of a program undertaken by several authors.[8] However, this holy grail remains elusive because of certain subtle and some not-so-subtle issues. Some of these points will now be raised and discussed in the light of results set out in previous sections.

The first and most obvious difficulty is that there is no natural scale or parameter with which to define a coarse graining in quantum cosmology. Because of this problem, several different TOU schemes have been implemented, some of which yield mutually conflicting results. It should be stressed that all these schemes are *ad hoc* to some degree; this has to be the case since no one knows what constitutes a good choice for a "system/environment" splitting in quantum cosmology. The standard "unobservability" and coarse-graining arguments are subject to the same criticisms that arise in statistical mechanics.[31]

Apart from this general problem, there is a second, more technical issue. Much of the previous work has been based on density matrices that, in our opinion, are not as intuitively useful as the Wigner function to study the classical limit. As a result, a somewhat intriguing point has been missed; there are times when the Wigner function can be decohered by TOU to the extent that the "linear entropy" $\int dX\, dk\, f^2(X,k)$ increases, yet interference terms remain, in that oscillations of the Wigner function do not entirely vanish.[10] It is not yet clear how this is to be interpreted.

The last feature appears to be unavoidable in the usual TOU approach. This will be motivated by the following simple argument. In Sec. II, it was mentioned that the Husimi distribution is a positive-definite Gaussian-smoothed Wigner function. It is defined to be

$$f_s(X_1,k_1) \equiv \int_{-\infty}^{+\infty} dX\, dk\, f(X,k) \times f_0(X-X_1,k-k_1) , \qquad (52)$$

where

$$f_0(X-X_1,k-k_1) = \frac{1}{\pi\hbar} e^{-(X-X_1)^2/\alpha} e^{-\alpha(k-k_1)^2/\hbar^2} , \qquad (53)$$

α being an arbitrary positive constant. f_0 is the Wigner function for a coherent state.

The density matrix corresponding to $f_s(X_1,k_1)$ is given by the inverse Wigner transform

$$\rho_s(X_1,x_1) = \int_{-\infty}^{+\infty} dX\, dk\, f(X,k) \times \int_{-\infty}^{+\infty} dk_1\, f_0(X-X_1,k-k_1) \times e^{-ik_1 x_1/\hbar} . \qquad (54)$$

Substituting (53) in (54) and evaluating the resulting Gaussian integral, we find immediately

$$\rho_s(X_1,x_1) = e^{-x_1^2/4\alpha} \int_{-\infty}^{+\infty} \frac{dX}{\sqrt{\alpha\pi}} \rho(X,x_1) e^{-(X-X_1)^2/\alpha} . \qquad (55)$$

In some examples,[8] TOU produces

$$\rho_{\rm TOU}(X_1,x_1) = \rho(X_1,x_1) e^{-x_1^2/\sigma} . \qquad (56)$$

Comparison of ρ_s and $\rho_{\rm TOU}$ shows that the latter lacks an extra "position smoothing." This is the reason that $\rho_{\rm TOU}$ is inadequate as far as producing positive-definite Wigner functions is concerned.

A way to avoid these difficulties is to use states that do not possess the dangerous oscillations plaguing $f_{\rm WKB}$. These are the Gaussian or coherent states. In quantum cosmology it is not clear how such states will be present because they are not eigenfunctions of the Hamiltonian. Moreover, it is counterproductive to have a wave packet centered at some value of the radius of the Universe; this implies the lack of any evolution. In any case, these states may also not lead to an acceptable classical limit. This has to do with their dynamics in nonharmonic potentials. For a harmonic potential, a coherent state tracks the classical motion, peaking on the classical trajectory. But this is merely an accident stemming from two reasons: (1) The potential is symmetric and (2) the energy levels are equally spaced. In a series of papers on generalized coherent states (minimum uncertainty states, or MUCS), Nieto and co-workers[32] have shown that for more general potentials it is essentially impossible to find a state that faithfully follows the classical trajectory. The initial "bell shape" of the MUCS degenerates, after some oscillations in a confining potential, into a highly "spiky" form that is not peaked on the classical trajectory, and moreover, the quantum and classical expectation values do not agree. If we were to take the QMIS interpretation literally, it would mean that there is no classical limit for such a system. However, it is precisely such nonlinear systems that are generically encountered.

The problems raised by nonlinear evolution are of extreme importance for the classical limit. When the potential is at most quadratic in X, the quantum Liouville equation reverts to the classical form. Then, with "classical" initial conditions $[f(X,k) \geq 0]$, one obtains "classical" final states. However, when the dynamics is nonlinear, $L_Q \neq 0$, and "classical" initial states evolve into states with Wigner functions that are no longer positive definite. Thus quantum interference may be *dynamically generated;* it has to be continuously damped for a classical limit to exist. More interestingly, there is a possibility that in nonlinear dynamics even decohered states will not follow classical trajectories.[33]

A naive phenomenological approach to incorporate decoherence into the quantum Liouville equation is to couple the system to a "noise" source. This leads to the appearance of diffusion terms and makes the dynamics irreversible. Furthermore, the system is no longer in a pure state, but rather in a mixed state characterized by the noise-averaged Wigner function $\langle f \rangle_N$. The presence of the noise is supposed to represent an interaction with an "environment" and can be shown to lead to destruction of quantum interference in simple examples. This approach can be extended in a straightforward manner to the nonlinear case, including nonlinear interactions with the noise. The last-mentioned situation leads to highly nontrivial effects that could be relevant to quantum cosmology.[33]

Finally, we restate the purpose of this section which was to point out that decoherence is very much an open field with many outstanding unresolved problems. However, the questions involved are of fundamental importance and have important consequences in other areas of physics.

VI. CONCLUSION

The main motivation for this paper is to show that pure quantum states cannot be used to define a classical limit in quantum cosmology. At the very least, it is hoped that some pitfalls in this approach have been uncovered. The follow up to the main conclusion is that suppression of quantum interference is *necessary* for a classical limit to exist. However, it is not clear exactly how this decoherence is to be achieved.

One of the outstanding problems in quantum cosmology is to explain whether the expectation value of the stress tensor $\langle T_{\mu\nu} \rangle$ should be used on the right-hand side of the Einstein equations and, if so, under what conditions. Partial answers have appeared before,[9] but we believe these results to be vitiated by our arguments of Secs. II and III. The analysis needs to be reexamined.

All of our WKB calculations were for one-dimensional systems. Unfortunately, extension to higher dimensions is simple only when the motion is dimensionally separable. In the general case, stationary phase evaluation of f_{WKB} appears to be an intractable problem.

Note added. After the work for this paper was completed, the author received a paper by A. Anderson[34] which also criticizes the methods of Ref. 9.

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