Holomorphic wave function of the Universe

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The quantum behavior of the vacuum Bianchi type-IX universe with the cosmological constant is investigated in terms of the Ashtekar variables. An exact solution to the quantum Hamiltonian constraint in the holomorphic representation is given. This solution reduces to the Hartle-Hawking wave function in the spatially isotropic sector and extends in the triad representation to the classically forbidden region where the determinant of the spatial metric becomes negative. The analysis of the quantum Robertson-Walker universe indicates that if the superspace is extended to such a classically forbidden region, the holomorphic representation picks up some restricted class of solutions in general. This observation leads to a new ansatz on the boundary condition of the Universe. In particular, the behavior of the Lorentzian and Euclidean WKB orbits corresponding to the solution suggests a new picture on the semiclassical behavior of the quantum Universe: that the Universe is created from an ensemble of Euclidean mother spacetimes. Further it is pointed out that the solution is a restriction to the spatially homogeneous sector of an almost exact solution to all the quantum constraints in the holomorphic representation for generic vacuum spacetime with the cosmological constant. The latter generic solution has a WKB structure for which the phase is proportional to the Chern-Simons functional.

I. INTRODUCTION

At present there exist two major approaches to construct a quantum theory of gravity.[1] One is to quantize the Einstein theory or its variants canonically or by path integration. The other is the superstring theories which try to unify gravity with the other fundamental interactions based on two-dimensional field theories. Because there is no reason to believe that the gravitational interaction is described by Einstein's theory on very small scales or in high-energy regions and also because Einstein's theory is unrenormalizable at least perturbatively, it is generally believed that superstring theories are more promising.[2] Recent work has, however, revealed that superstring theories have no predictive power on low-energy physics, in contrast with expectation at the early stage.[3] This implies that one must look for phenomena in which nonperturbative quantum effects of gravity play important roles in order to check the validity of the theories or pick up the true theory from a huge number of candidates. In this respect the present superstring theories are incomplete for they can treat gravitons and their interactions but cannot incorporate the dynamics of spacetime structure.

Canonical or path-integral quantization of Einstein's theory has an advantage at this point since they preserve the fundamental concepts of the classical Einstein theory and directly treat spacetime dynamics unlike the superstring theories. In particular when one tries to discuss quantum gravitational effects on cosmology, the canonical approach or its path-integral expression is the only one that we can rely on at present. From this standpoint, since the interesting proposal on the wave function of the Universe by Hawking and Hartle[4-6] and the fascinating picture on quantum creation of the Universe by Vilenkin,[7,8] a lot of work has been done on the application of canonical quantum gravity to cosmology.[9] Although various interesting results have been obtained, the present status of the research along this line is far from complete. In contrast with its grand motivation only quite simple minisuperspace models or their perturbations have been discussed so far. Even the quantum behavior of spatially homogeneous universes is not known well if we allow for anisotropy. Apart from the problem of interpretation, the largest obstacle there was the complicated nature of the basic equations.

Recently Ashtekar proposed a new canonical formulation of Einstein gravity which may give a breakthrough to this situation.[10-13] In his formulation all of the evolution and constraint equations are written as simple polynomials with the local SO(3,ℂ)-invariant structure. So the structure of the equations are much simplified compared with the conventional Arnowitt-Deser-Misner-(ADM-) Wheeler-DeWitt formulation. In fact it is shown that a large class of solutions for the Hamiltonian constraint can be constructed with the aid of the loop-integral expression.[14] Further it is pointed out that in the loop-space representation of the theory the solutions to the full set of constraint equations are classified in terms of the topological knot and link invariants.[15-17]

Although these developments are important and attractive enough to show the power of the Ashtekar formalism in quantum gravity, the approach based on the loop-space representation is too abstract to find a direct relation to quantum cosmological problems as in the case of the superstring theories (cf. Ref. 18). From this standpoint I applied the Ashtekar formalism directly to spatially homogeneous spacetimes in my previous paper.[19] There I showed that at least for the vacuum Bianchi type-IX universe the quantum constraint equations become quite simple in the holomorphic representation, and gave a small class of exact solutions to them.

In the present paper I extend the analysis to a Bianchi type-IX universe with the cosmological constant to see implications of the Ashtekar formalism, especially a possible new role of the holomorphic representation in quantum cosmology. It is shown that we can find an exact solution which is physically meaningful and can be extended to an almost exact solution to the constraint equations in the holomorphic representation for generic vacuum spacetime with the cosmological constant. On the basis of detailed analysis of this exact solution with general considerations on the wave functions in the canonical quantum gravity, it is proposed that some analyticity and fall-off conditions in the complex connection space may yield a new criterion to select a preferred class of wave functions of the Universe. Further by analyzing the behavior of the WKB orbits corresponding to this solution it is shown that it yields a new picture on the semiclassical behavior of the quantum universe.

The organization of the paper is as follows. In the next section I discuss the relation between the quantum triad ADM theory and the quantum Ashtekar theory with the aid of a generating functional which gives a transformation connecting these two theories. In particular it is shown that the quantum Ashtekar theory can be regarded as a noncanonical representation of the quantized ADM theory if we take the problem of the reality condition seriously. Further the assumption on the operator ordering is explained. On the basis of the framework developed in this section, a possible role of the holomorphic representation in quantum gravity is discused in Sec. III.

Then in Sec. IV the solutions to the Hamiltonian constraint for the closed Robertson-Walker universe is examined in detail in the metric representation and in the holomorphic representation, and the correspondence of the solutions in these two representations are studied. In particular it is shown that the solutions in the holomorphic representation contains a wave function which in the metric representation extends to a classically forbidden region where the determinant of the spatial metric becomes negative. The relation of this solution to the boundary condition of the Universe, that is, the criterion to select out special solutions to the constraint equations representing the wave function of the Universe, is discussed. Further on the basis of the behavior of the WKB orbits, it is shown that the solution yields a new picture on the semiclassical behavior of the quantum universe that the Universe is created from an ensemble of mother Euclidean spacetimes. In Sec. V a nontrivial exact solution to the Hamiltonian constraint in the holomorphic representation for the Bianchi type-IX universe with a nonvanishing cosmological constant is given. The behavior of the Lorentzian and Euclidean WKB orbits corresponding to this solution is analyzed to confirm that the semiclassical picture of the quantum universe given in the spatially isotropic case can be extended to this case. Section VI is devoted to the summary and discussion. There it is pointed out that the solution found in the Bianchi type-IX case is a restriction to the spatially homogeneous sector of an almost exact solution to all the quantum constraint equations for generic vacuum spacetime with the cosmological constant.

Finally in the Appendix I comment on the difficulty associated with the so-called reality condition in quantizing Ashtekar's theory by repeating the proof of the equivalence between the triad ADM theory and Ashtekar's theory at the classical level. This appendix is added because, though this difficulty is well known, it has not been discussed explicitly enough so far as far as the author knows (cf. Refs. 20 and 21). This appendix also gives the detailed definitions of variables and some formulas used in this paper as well as an important identity which leads to the generating functional of the transformation from the triad ADM variables to Ashtekar's variables.

Throughout the present paper units $16\pi G=c=\hbar=1$ are used and the signatures of the fundamental tensors such that $(\eta_{ab})=(-,+,+,+)$, $\varepsilon_{0123}=\varepsilon_{123}=1$ are adopted. As for the index convention, the greek letters $\mu,\nu,\ldots$ denote the spacetime coordinate indices running from 0 to 3, the Latin indices $j,k,\ldots$ their spatial part, the Latin letters $a,b,\ldots$ the internal indices running from 0 to 3, and the capital letters $I,J,\ldots$ their spatial part.

II. QUANTIZATION OF ASHTEKAR'S THEORY

As is well known the Hamiltonian loses the dynamical role in the quantum ADM formalism because it is written as a linear combination of the constraints.[22] In the triad approach this obliges us to extract the dynamics only from the constraint equations on the physical state vector $|\Psi\rangle$,

$$C_R|\Psi\rangle=0\ , \tag{2.1}$$

$$C_{Mj}|\Psi\rangle=0\ , \tag{2.2}$$

$$C_H|\Psi\rangle=0\ , \tag{2.3}$$

and the fundamental commutation relations among the tetrad $\tilde{e}^{\,j}=(\tilde{e}^{\,jI})$ and its conjugate momentum $P_j=(P_{jI})$,

$$[\tilde{e}^{\,jI}(\mathbf{x}),\tilde{e}^{\,kJ}(\mathbf{y})]=[P_{jI}(\mathbf{x}),P_{kJ}(\mathbf{y})]=0\ , \tag{2.4}$$

$$[\tilde{e}^{\,jI}(\mathbf{x}),P_{kJ}(\mathbf{y})]=\tfrac{1}{2}i\delta^j_k\delta^I_J\delta(\mathbf{x}-\mathbf{y})\ . \tag{2.5}$$

Here, for vacuum spacetimes with a cosmological constant Λ, about which the present paper is mainly concerned, C_R, C_{Mj}, and C_H are expressed in terms of $\tilde{e}^{\,j}$ and P_j as Eqs. (A12)–(A14) with Q_j given in Eq. (A31). These yield the basic equations for the triad version of the conventional ADM-Wheeler-DeWitt approach to quantum gravity.

If we assume that quantum gravity theory is completely described by these constraint equations and commutation relations alone, we can construct a consistent quantum version of Ashtekar's theory. Although this has already been shown clearly in the original paper by Ashtekar,[10] I give here a much simpler proof using the generating functional relating the triad ADM theory and

Ashtekar's theory (similar arguments have been already given by some authors[12,23]). It also makes clear the role of the reality condition (as for the background of this condition see the Appendix; there a critical argument on this problem is given). The key point is the fact that from the identity (A32) Q_j given by Eq. (A31) is expressed by the functional derivative of a generating functional F as

$$Q_j = Q_j[\tilde{e}] := \frac{1}{2}\frac{\delta F}{\delta \tilde{e}^j}, \tag{2.6}$$

where

$$F := \int d^3x\, \varepsilon^{ijk} e_{iI} e_{jJ} e_{lI} \partial_k e^{lJ} \tag{2.7}$$

$$= \int d^3x (\tilde{e}_j \times \partial_k \tilde{e}^j)\cdot \tilde{e}^k . \tag{2.8}$$

First let us define the quantum operator corresponding to the complex Ashtekar momentum $\mathcal{A}_j^{\pm} = (\mathcal{A}_{jI}^{\pm})$ in terms of the operators $\tilde{e}^j$ and P_j as

$$\mathcal{A}_j^{\pm} = P_j \pm i Q_j[\tilde{e}] . \tag{2.9}$$

Then it follows from the commutativity of the functional derivative $\delta F/\delta \tilde{e}^j(\mathbf{x})\delta \tilde{e}^k(\mathbf{y}) = \delta F/\delta \tilde{e}^k(\mathbf{y})\delta \tilde{e}^j(\mathbf{x})$ that $\mathcal{A}_j^{\pm}$'s with the same chirality commute with each other. Hence Eq. (2.5) yields the canonical commutation relations among the quantum Ashtekar variables:

$$[\tilde{e}^{jI}(\mathbf{x}), \tilde{e}^{kJ}(\mathbf{y})] = [\mathcal{A}_{jI}^{\pm}(\mathbf{x}), \mathcal{A}_{kJ}^{\pm}(\mathbf{y})] = 0 , \tag{2.10}$$

$$[\tilde{e}^{jI}(\mathbf{x}), \mathcal{A}_{kJ}^{\pm}(\mathbf{y})] = \tfrac{1}{2} i \delta_k^j \delta_J^I \delta(\mathbf{x}-\mathbf{y}) . \tag{2.11}$$

Further, since $C_B = \tilde{e}^k \cdot F_{jk} = D_j(\tilde{e}^j \times \tilde{e}^k) = 0$ for $Q_j[\tilde{e}]$ given by Eq. (2.6) or equivalently by Eq. (A31), we obtain the following relation between the constraints $\mathcal{C}_G$, $\mathcal{C}_{Mj}$, and $\mathcal{C}_H$ in Ashtekar's theory [see Eqs. (A18)–(A20)] and those in the ADM theory:

$$\mathcal{C}_G \equiv C_B \mp i C_R = \mp i C_R , \tag{2.12}$$

$$\mathcal{C}_{Mj} \equiv C_{Mj} \mp 2i\tilde{e}^k \cdot (F_{jk} + P_j \times P_k) = C_{Mj} \pm i C_R \cdot P_j , \tag{2.13}$$

$$\mathcal{C}_H \equiv C_H \pm i D_j(\tilde{e}^j \cdot C_R) \mp D_j(\tilde{e}^j \times \tilde{e}^k)\cdot P_k$$
$$= C_H \pm i D_j(\tilde{e}^j \cdot C_R) . \tag{2.14}$$

Hence, noting that $C_R \cdot P_j = P_j \cdot C_R$ from the commutation relation

$$[C_R^I(\mathbf{x}), P_{jJ}(\mathbf{y})] = i\varepsilon^{IJK} P_{jK}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) , \tag{2.15}$$

we find that the constraints (2.1)–(2.3) are equivalent to the following constraint equations expressed only in terms of the Ashtekar variables:

$$\mathcal{C}_G |\Psi\rangle = 0 , \tag{2.16}$$

$$\mathcal{C}_{Mj} |\Psi\rangle = 0 , \tag{2.17}$$

$$\mathcal{C}_H |\Psi\rangle = 0 . \tag{2.18}$$

Thus the basic equations of the quantum triad ADM theory are formally written only in terms of the Ashtekar variables. However, if we require that $\tilde{e}^j$ and P_j are represented as Hermitian operators, we obtain the additional constraint which is expressed as the operator equations

$$\mathcal{A}_j^{\pm} - (\mathcal{A}_j^{\pm})^\dagger = \pm i \varepsilon_{IJK} \omega_{jJK}(e) . \tag{2.19}$$

We can impose this condition as the operator equation simply because we have neglected the formal time-evolution equations (see the argument in the Appendix). Further we can calculate the commutation relations between $\mathcal{A}_j^{\pm}$ and $(\mathcal{A}_j^{\pm})^\dagger$ with the help of the original definition of $\mathcal{A}_j^{\pm}$, though the results are quite complicated.

Since this constraint equation contains the Hermitian conjugation, it is a requirement on the inner product in the state space. In general it is difficult to find an inner product which satisfies such intricate conditions. In the present case, however, we can reduce the problem to a much simpler one in the mathematical sense because the definition (2.9) is written as

$$\mathcal{A}_j^{\pm} = e^{\pm F} P_j e^{\mp F} . \tag{2.20}$$

For example, in the usual representation of the ADM theory in which $\tilde{e}^j$ becomes diagonal,

$$|\Psi\rangle \rightarrow \Psi[\tilde{e}] , \tag{2.21}$$

$$P_j \rightarrow \frac{1}{2}\frac{\delta}{i\delta \tilde{e}^j} , \tag{2.22}$$

which is referred to as the $\tilde{e}$ representation for simplicity from now on, the Hermiticity of P_j fixes the inner product to be the natural one given by

$$\langle \Psi|\Psi\rangle := \int [d\tilde{e}] \overline{\Psi[\tilde{e}]} \Psi[\tilde{e}] . \tag{2.23}$$

If we introduce the new representation

$$|\Psi\rangle \rightarrow \Psi^{\pm}[\tilde{e}] := e^{\pm F}\Psi[\tilde{e}] , \tag{2.24}$$

with the inner product

$$\langle \Psi|\Psi\rangle = \int [d\tilde{e}] e^{-2F} \overline{\Psi^{\pm}[\tilde{e}]} \Psi^{\pm}[\tilde{e}] , \tag{2.25}$$

which is referred to the chiral $\tilde{e}$ representation from now on, the action of the operator $\mathcal{A}_j^{\pm}$ on $|\Psi\rangle$ is represented as

$$\mathcal{A}_j^{\pm} \rightarrow \frac{1}{2}\frac{\delta}{i\delta \tilde{e}^j} . \tag{2.26}$$

It is obvious that these two representations are equivalent and $\mathcal{A}_j^{\pm}$ satisfies the reality condition (2.19). Further in the chiral $\tilde{e}$ representation, which is a kind of noncanonical representation of the ADM canonical variables, the constraint equations become simple polynomial differential equations. From this standpoint the important achievement of Ashtekar formalism is that it explicitly gives us the generating functional F which transforms the canonical representation to a noncanonical representation in which the constraint equations become simple.

Thus, as far as we require that $\tilde{e}^j$ and P_j are represented as Hermitian operators, the quantized Ashtekar theory is just a convenient exotic representation of the quantized triad ADM theory. However, if we look into the content of the theory in detail, we find that the Hermiticity requirement on $\tilde{e}^j$ and P_j has no firm ground. In fact, solutions to the quantum constraint equations are not normalizable with respect to the inner product (2.23) even in the minisuperspace models. Thus we cannot apply the usual probability interpretation to the inner product. Since the Hermiticity requirement on observables in quantum mechanics is closely connected with the probability interpretation of the inner product, this implies that there exists no direct connection between the Hermiticity of operators with respect to the formal inner product and the classical reality of them in the ADM-Wheeler-DeWitt approach to quantum gravity. Of course this does not imply that the classical reality of physical quantities loses meaning, but indicates that it should be formulated in a way different from conventional quantum mechanics. For example, in the WKB-type interpretation to the wave functions, the reality condition on the physical quantities is used to pick up a region of superspace where the Universe behaves semiclassically. In such an approach the reality condition does not restrict the acceptable wave functions. Hence the quantum triad ADM theory and the quantum Ashtekar theory become completely equivalent. The complex nature of Ashtekar momentum or its relation to the ADM variables becomes important only when we find the semiclassical regions. This problem as well as problems connected with the inner product will be discussed in the following sections in more detail.

In order to make the quantum theory well defined, we must eliminate the ambiguity associated with the ordering of operators in the constraint equations. If we do not introduce the inner product of the state vectors, the only requirement on the operator ordering is that the algebra of the constraint operators close and that the constraint equations (2.16), (2.17), and (2.18) are consistent. This problem was extensively discussed by Jacobson and Smolin.[14] As pointed out there, there exist two natural orderings. The first one is the ordering suggested by Ashtekar[10] in which all $\tilde{e}^j$'s are gathered left to $\mathcal{A}_j^{\pm}$. This ordering formally guarantees the consistency of the constraint equations even when the cosmological constant exists. Actually for this ordering, from Eq. (2.11), the commutation relations among the constraint operators are given by

$$\left[\int g_1\cdot\mathcal{C}_G, \int g_2\cdot\mathcal{C}_G\right] = \mp\int (g_1\times g_2)\cdot\mathcal{C}_G , \qquad (2.27)$$

$$\left[\int g\cdot\mathcal{C}_G, \int f^j\mathcal{C}_{Mj}\right] = 0 , \qquad (2.28)$$

$$\left[\int g\cdot\mathcal{C}_G, \int h\,\mathcal{C}_H\right] = 0 , \qquad (2.29)$$

$$\left[\int f_1^j\mathcal{C}_{Mj}, \int f_2^k\mathcal{C}_{Mk}\right] = \pm\int f_1^j f_2^k \mathcal{C}_G\cdot\mathcal{F}_{jk} + i\int [f_1,f_2]^k\mathcal{C}_{Mk} , \qquad (2.30)$$

$$\left[\int f^j\mathcal{C}_{Mj}, \int h\,\mathcal{C}_H\right] = i\int\left\{(f^j\partial_j h - h\partial_j f^j)\mathcal{C}_H + f^j h\,\mathcal{C}_G\cdot\left[\tilde{e}^k\times\left[\mathcal{F}_{jk} - \frac{\Lambda}{2}\varepsilon_{jkl}\tilde{e}^l\right]\right]\right\} , \qquad (2.31)$$

$$\left[\int h_1\mathcal{C}_H, \int h_2\mathcal{C}_H\right] = i\int (h_2\partial_j h_1 - h_1\partial_j h_2)(\tilde{e}^k\cdot\tilde{e}^j)\mathcal{C}_{Mk} , \qquad (2.32)$$

where $g^I(\mathbf{x}), f^j(\mathbf{x}), h(\mathbf{x}), \ldots$ are smooth functions of the space coordinates with compact supports. Thus the constraint algebra closes weakly. Further, from the commutation relation between $\mathcal{C}_G$ and an arbitrary quantity $V^I(\tilde{e}(\mathbf{x}), \mathcal{F}(\mathbf{x}))$ which transforms as an internal vector,

$$\left[\int g_1\cdot\mathcal{C}_G, \int g_2\cdot V\right] = \mp\int (g_1\times g_2)\cdot V , \qquad (2.33)$$

it follows that $C_G(\mathbf{x})\cdot V(\mathbf{x}) = V(\mathbf{x})\cdot C_G(\mathbf{x})$. Thus the constraint operators are located at the right ends in all of the terms on the right-hand sides of the commutation relation, which guarantees the consistency of the constraints. As pointed out by Jacobson and Smolin, however, this ordering has the difficulty that $\mathcal{C}_{Mj}$ differs from the generator of the spatial coordinate transformation by a divergent constant.

The second ordering which is proposed by Jacobson and Smolin is the reversed one of the first ordering, which gives the commutation relations in which the orders of the constraint operators and the coefficient operators are reversed on the right-hand sides of the above equations. For this ordering $\mathcal{C}_{Mj}$ exactly generates the spatial coordinate transformation. However, it does not yield a consistent system of quantum constraints at least formally. Although the inconsistency is shown to disappear for a family of solutions to the Hamiltonian constraint represented as a loop integral due to the degeneracy of the metric,[14,17] it seems to be not the case for general solutions representing regular spacetimes. Furthermore it cannot be denied that the anomaly associated with $\mathcal{C}_{Mj}$ in the first ordering may be removed by some regularization of the operator products. For these reasons I adopt the first ordering in the present paper.

Finally note that the relation (2.9) is a generalization of the relation between the Ashtekar and the ADM momentum given in my previous paper.[19] In fact for the Bianchi spacetime with the metric

$$ds^2 = -N^2dt^2 + e^{2\alpha}(e^{2\beta})_{IJ}\chi^I\otimes\chi^J , \qquad (2.34)$$

the fundamental variables are parametrized by functions only of time, σ_{IJ}, P_{IJ}, and $A_{IJ}^{\pm}$ as

$$\tilde{e}^{jI} = \frac{1}{2\Omega}\sigma_{IJ}(t)|\chi|X_J^j, \quad \sigma_{IJ} = 2\Omega e^{2\alpha-\beta_{IJ}} , \qquad (2.35)$$

$$P_{jI} = P_{IJ}(t)\chi_j^J , \qquad (2.36)$$

$$\mathcal{A}_{jI}^{\pm} = A_{IJ}^{\pm}(t)\chi_j^J , \qquad (2.37)$$

where $\chi^I = \chi^I{}_j dx^j$ is a basis of invariant forms normalized by

$$d\chi^I = \tfrac{1}{2}C^I{}_{JK}\chi^J\wedge\chi^K , \qquad (2.38)$$

$|\chi|$ is the determinant of χ_j^I, $\Omega := \int d^3x|\chi|$ is the coordinate volume of space, and X_J^j are the invariant fields dual to χ^I_j. In terms of these variables F is written as

$$F=\frac{1}{4}|\sigma|(\tilde{\sigma}\sigma)^{-1}_{KL}\varepsilon_{KIJ}C^L{}_{IJ} , \tag{2.39}$$

where $\tilde{\sigma}$ is the transposed matrix of σ and Eq. (2.9) reduces to

$$A^{\pm}_{IJ}=P_{IJ}\pm i\frac{\partial F}{\partial \sigma_{IJ}} . \tag{2.40}$$

III. HOLOMORPHIC REPRESENTATION

As explained in the previous section the quantum theory of Ashtekar's formalism can be formulated as a noncanonical representation of the quantized triad ADM formalism as far as the constraint equations are concerned. As is well known, however, the quantum constraints alone do not yield any sensible quantum theory of gravity even formally. First the solutions to the quantum constraint equations in the e representation of the triad ADM theory are not normalizable with respect to the natural inner product which makes the momentum operator Hermitian. Thus the wave function does not represent a state in the usual quantum-mechanical sense. Second the wave functions do not depend on time and, hence, we have no concept of evolution.

Although, generally speaking, all of these difficulties arise from the general covariance, i.e., invariance under the general coordinate transformations of Einstein's theory, it is more appropriate to classify them into two types. Those belonging to the first class are the spatial coordinate transformations. In the triad formulation we must also include local triad rotations into this type. Invariance of this type is quite similar to the local gauge invariance in the gauge field theories. If theories with such invariance are quantized by the Dirac procedure without gauge fixing, it makes the inner product divergent because, naively speaking, wave functions become constant along the gauge orbits. For example, in the Abelian gauge theory the gauge invariance leads to the constraint on the state vector $|\Psi\rangle$, $\partial_j E^j|\Psi\rangle=0$. In the presentation in which the potential A_j is diagonal, this quantum constraint implies that the wave function $\Psi[A]$ does not depend on the longitudinal part $-\partial_j\phi$ of the potential. Thus the inner product $\int DA|\Psi[A]|^2$ diverges along the ϕ integration. This type of divergence can be, however, eliminated by introducing δ-function-type terms to the integration measure in the definition of the inner product.

The second type of invariance is the one associated with the time coordinate transformation and can be regarded as the time reparametrization invariance. This type of invariance does not exist in the usual local gauge theories and have properties quite different from the first one. In particular it is due to this invariance that time dependence of the wave functions disappears. Actually in a theory with the time reparametrization invariance, the same value of the parameter time can be assigned to any physical time. Thus any state which is invariant under the time reparametrization should be a superposition of states at arbitrary physically different times.

For example, if we rewrite the dynamics of a system with the Hamiltonian $H(q,p)$ into a time-reparametrization-invariant form by introducing a parameter time τ and treating the time t as a canonical variable with momentum π, it becomes a canonical system with the Hamiltonian $\tilde{H}=N(\pi+H)$ and a constraint $\phi:=\pi+H=0$ (Ref. 24). Dirac quantization of this system yields the quantum constraint $\phi|\Psi\rangle=0$. Solutions lose time dependence like the quantum gravity case. However, if we expand $|\Psi\rangle$ by the eigenstates of the time operator t as

$$|\Psi\rangle=\int dt\,|\Psi(t)\rangle\otimes|t\rangle , \tag{3.1}$$

the quantum constraint is translated to the ordinary Schrödinger equation for the decomposed states $|\Psi(t)\rangle$:

$$i\partial_t|\Psi(t)\rangle=H|\Psi(t)\rangle . \tag{3.2}$$

Although the inner product of $|\Psi\rangle$ diverges, the decomposed states $|\Psi(t)\rangle$ have a constant and finite inner product.

It is expected that the similar treatment can be applied to the quantum gravity case. Unfortunately, however, the quantum gravity system differs from the simple reparametrized quantum mechanical system in two respects. First there exists no natural time operator or a momentum conjugate to it. Actually the Hamiltonian constraint is second order in momentum. Hence no fundamental variable can be adopted as the time operator in the exact sense.[25] Second the time reparametrization invariance in the general relativity has an infinite degrees of freedom. Hence a single time variable does not eliminate the degeneracy of the state. In other words a field-type time variable is required. In general it is difficult to find such a field variable which can play the role of time over the whole space. These two difficulties are intimately connected and seem to be deeply rooted in the local nature of the general relativity. Actually it is generally impossible to find a global coordinate condition which can be applied to any spacetime even in the classical level. Thus they are not intrinsic to the Wheeler-DeWitt-type approach and cannot be avoided also in the gauge-fixing approaches. Anyway without a field-type time variable which completely eliminates the degeneracy of the state, we cannot introduce the inner product or the concept of probability into the theory.

One possible approach to resolve this difficulty is the one based on the WKB-type approximation to the wave functions or the Wigner function.[26–30] In this approach the variables are classified into semiclassical ones and quantum ones, and the Hamiltonian constraint yields semiclassical evolution equations for the former in some region of the superspace or of the corresponding phase space, and an approximate Schrödinger equation for the latter there. The semiclassical variables play the role of the field-type time variable and the Schrödinger equation is derived for a reason quite similar to the simple example of the time-reparametrization-invariant quantum-mechanical system explained above. One large difference is that the inner product and the probability interpretation cannot be defined exactly in this approach. However, it may not be a defect of this approach. In fact, if we take seriously the wave-packet reduction problem in the standard quantum mechanics, the final theory describing the whole Universe should take a form different from the conventional quantum mechanics. It is quite possible that the probability concept has a limited applicability and depends on the semiclassical history of the Universe (cf. Ref. 31). Although these arguments do not fully justify the WKB-type approach, they at least suggest that it is meaningful to investigate the behavior of the solutions to the quantum constraint equations.

Provided that the Universe is described by a solution to the constraint equations, one must find a rule to pick up solutions which are physically meaningful. In the standard quantum mechanics, this rule is given by the normalizability of state vectors on the basis of the probability interpretation. However, we have no inner product in the quantum gravity case, at least in the usual sense. Furthermore, in the final theory, we must select one solution. In the framework of the Wheeler-DeWitt theory, various *Ansätze* have been proposed to give such criterion, so called *the boundary condition of the Universe.*[5,6,7,8,32,33,34] The most famous one is the Hartle-Hawking proposal based on the Euclidean path integral. Although it is quite elegant and attractive, recent work shows that it does not work well due to the ill definedness of the Euclidean path integral.[35-37]

New aspects brought about by the polynomial formulation is that we can consider the so-called *holomorphic representation* which is a kind of momentum representation.[14] Here we discuss the possibility that we may utilize this holomorphic representation to give a new criteria to select out the wave function of the Universe.

Before going to its application to quantum gravity, we first review its definition and fundamental properties for the case of quantum mechanics. For a quantum system with the fundamental canonical variables $(\hat{q},\hat{p})$ let us consider a non-Hermitian operator $\hat{\alpha}$ defined as

$$\hat{\alpha}:=\hat{p}+if(\hat{q})=e^{F}\hat{p}e^{-F}, \quad f(q)=\partial_q F(q) , \tag{3.3}$$

and its eigenbras

$$\langle \alpha|\hat{\alpha}=\langle \alpha|\alpha . \tag{3.4}$$

In the q representation these eigenbras are expressed apart from the normalization as

$$\langle \alpha|q\rangle=e^{-i\alpha q-F} . \tag{3.5}$$

In terms of these eigenbras we can introduce the α representation of the state vector $|\Psi\rangle$ defined by

$$\Phi(\alpha):=\langle\alpha|\Psi\rangle=\int dq\, e^{-i\alpha q-F}\Psi(q) , \tag{3.6}$$

where $\Psi(q)$ is the q representation of the state vector. Since $\hat{\alpha}$ acts as $-i\partial_q$ on the wave function $\tilde{\Psi}(q)=e^{-F}\Psi(q)$, $\Phi(\alpha)$ is the wave function in a kind of momentum representation to the noncanonical representation of $\hat{p}$. In particular the operators $\hat{q}$ and $\hat{\alpha}$ are expressed as

$$\hat{q}|\Psi\rangle\mapsto i\frac{d}{d\alpha}\Phi(\alpha) , \tag{3.7}$$

$$\hat{\alpha}|\Psi\rangle\mapsto\alpha\Phi(\alpha) . \tag{3.8}$$

What is different from the usual momentum representation is that α takes complex values and the corresponding eigenbras form a nonorthogonal overcomplete set in general due to the non-Hermiticity of $\hat{\alpha}$. As a result $\Phi(\alpha)$ is a function defined in some domain of a complex plain.

An important property of this α representation is that for the normalizable state vector $|\Psi\rangle$ for which $\Psi(q)$ belongs to $L^2(q)$, i.e., square integrable, $\Phi(\alpha)$ becomes a holomorphic function in the whole complex plane of α if $e^{-i\alpha q-F}\in L^2(q)$ for any complex value of α. This is the reason why this representation is called the holomorphic representation. In this case we can find the inversion formula to Eq. (3.6):

$$\Psi(q)=\frac{1}{(2\pi)^n}e^{F}\int_C d\alpha\, e^{i\alpha q}\Phi(\alpha) , \tag{3.9}$$

where the integration path is along the real axis.

An interesting point of this representation is that the requirement of the holomorphicity of $\Phi(\alpha)$ yields an criterion to select out the normalizable wave functions for some cases. For example, let us consider a one-dimensional harmonic oscillator with the Hamiltonian

$$H=\frac{1}{2m}p^2+\frac{1}{2}m\omega^2q^2 . \tag{3.10}$$

The eigenvalue equation of this Hamiltonian in the q representation,

$$\hat{H}\Psi(q)\equiv\left[-\frac{1}{2m}\frac{d^2}{dq^2}+\frac{1}{2}m\omega^2q^2\right]\Psi(q)=E\Psi(q) , \tag{3.11}$$

has two parameter families of solutions:

$$\Psi(q)=AD_\nu(\sqrt{2m\omega}\,q)+BD_{-\nu-1}(i\sqrt{2m\omega}\,q) , \tag{3.12}$$

where A and B are arbitrary constants, $E=(\nu+\frac{1}{2})\omega$, and $D_\nu(z)$ is the Weber function. If we require that $\Psi(q)$ is square integrable, only the first term with non-negative integer ν is allowed and we get the correct energy spectrum.

By introducing the operator

$$\hat{\alpha}=\hat{p}+im\omega\hat{q}, \quad F(q)=\tfrac{1}{2}m\omega q^2 , \tag{3.13}$$

we can solve the same problem in the α representation, which corresponds to the so-called coherent-state representation. In this representation the energy-eigenvalue equation

$$\hat{H}\Phi(\alpha)\equiv\left[\frac{1}{2m}\alpha\left[\alpha+2m\omega\frac{d}{d\alpha}\right]+\frac{\omega}{2}\right]\Phi(\alpha) = (\nu+\tfrac{1}{2})\omega\Phi(\alpha) \tag{3.14}$$

has a single-parameter family of solutions

$$\Phi_\nu(\alpha)=C\left[\frac{\alpha}{\sqrt{2m\omega}}\right]^\nu\exp\left[-\frac{\alpha^2}{4m\omega}\right] . \tag{3.15}$$

Here if we require that Φ_ν is holomorphic in α, ν is restricted to non-negative integers. Thus we obtain the correct energy spectrum.

What is happening here is that the transformation from the q wave function to α wave function,

$$\Phi(\alpha)=\int_{-\infty}^{\infty} dq \exp(i\alpha q-\tfrac{1}{2}m\omega q^2)\Psi(q) , \tag{3.16}$$

gives holomorphic functions in the whole complex plane only for the case when ν is a non-negative integer. In fact the transform of $D_\nu(\sqrt{2m\omega}\,q)$ exists only in the region $\mathrm{Im}\alpha>0$ for $\nu\leq-1$ and $\mathrm{Im}\alpha\geq0$ for $\nu>-1$ and $\nu\neq0,1,2,\ldots$, though the corresponding $\Phi(\alpha)$ is always given $\Phi_\nu(\alpha)$. The solution $D_{-\nu-1}(i\sqrt{2m\omega}\,q)$ has the transform only for $\nu>-1$ and on the real axis of α. Further it does not even have an analytic extension because the corresponding $\Phi(\alpha)$ is proportional Φ_ν for $\alpha>0$ but vanishes for $\alpha<0$.

Thus in the case of the harmonic oscillator the requirement of holomorphicity on the wave function in the coherent-state representation yields a necessary and sufficient criterion to select out the normalizable wave function. Of course this result cannot be extended to the general case because the requirement of holomorphicity is weaker than the condition of square integrability of the wave functions in the q representation in general.

In the case of quantum gravity $\mathcal{A}^{\pm}$ representation in which $\mathcal{A}_j^{\pm}$ becomes diagonal corresponds to the holomorphic representation. From the relation (2.20) the wave function $\Phi[\mathcal{A}^{\pm}]$ in this representation is connected with the wave function $\Psi[\tilde{e}]$ in the canonical $\tilde{e}$ representation by

$$\Phi[\mathcal{A}^{\pm}]=\int d\tilde{e} \exp\left[-i\int d^3x\, \mathcal{A}_j^{\pm}\cdot\tilde{e}^{\,j}\mp F\right]\Psi[\tilde{e}] . \tag{3.17}$$

Although we cannot apply the reasoning leading to the holomorphicity of $\Phi(\mathcal{A}^{\pm})$ now, it seems natural to require some analyticity condition on it for the following reasons. First for the physical degrees of freedom which correspond to the gravitational waves in the weak-field limit the Hamiltonian is positive definite and wave functions are expected to damp rapidly enough for large excitations like the harmonic oscillator. Hence the wave function in the $\mathcal{A}^{\pm}$ representation will be holomorphic with respect to the $\mathcal{A}^{\pm}$'s conjugate to this freedom. Actually it is shown by Ashtekar[38] that the holomorphic requirement is appropriate in the weak-field limit. Second the gauge freedom associated with the local triad rotation and the spatial coordinate transformation can be eliminated by introducing the gauge-fixing δ functions in the definition of the transform. Finally to see the behavior in the sector associated with the freedom of the local time evolution, let us recall the case of the quantum-mechanical system written in the reparametrization-invariant form. As noted above, the state vector $|\Psi\rangle$ is decomposed as in Eq. (3.1). Since $|\Psi(t)\rangle$ obeys the Schrödinger equation with the Hamiltonian H, it is written in terms of the eigenstates of H as

$$|\Psi(t)\rangle=\sum_n e^{-iE_n t}|E_n\rangle . \tag{3.18}$$

Hence the Laplace transform of $|\Psi(t)\rangle$,

$$\int_{t_0}^{\infty} dt\, e^{-it\alpha}|\Psi(t)\rangle , \tag{3.19}$$

becomes analytic in the lower half complex plane of α. In quantum gravity solutions to the quantum constraints can be also expanded in the form like Eq. (3.1). The difference is that the variable t can be regarded as a time variable only in a limited range of values and the Schrödinger-type equation holds only in this range. If the solutions represent quantum states of the Universe with a beginning, this range will have a lower bound and below this range $|\Psi(t)\rangle$ will damp or vanish. Thus the wave function in the $\mathcal{A}^{\pm}$ representation is expected to be analytic in the lower half plane with respect to the $\mathcal{A}^{\pm}$'s conjugate to the time variables.

Thus these observations suggest that we should require that the wave function in the $\mathcal{A}^{\pm}$ representation should be holomorphic in the half complex planes with respect to the components of $\mathcal{A}^{\pm}$ which correspond to time, and, in the entire complex planes with respect to the remaining components. In the next section we will examine whether or not this is the case for a simple system.

IV. CLOSED ROBERTSON-WALKER UNIVERSE

The spacetime metric of the closed Roberston-Walker universe is given by

$$ds^2=-N^2dt^2+e^{2\alpha}d\Omega_3^2 , \tag{4.1}$$

where $d\Omega_3^2$ is the metric of a Euclidean three-sphere and written in terms of an appropriate one-form basis $\chi^I=\chi_j^I dx^j$ which is invariant under the SO(3) isometry group as

$$d\Omega_3^2=\chi^I\otimes\chi^I . \tag{4.2}$$

We normalize the one-form basis by

$$d\chi^I=\tfrac{1}{2}\varepsilon_{IJK}\chi^J\wedge\chi^K , \tag{4.3}$$

which determines the sectional curvature to be $K=\frac{1}{4}$ and the volume $\Omega=16\pi^2$. For this metric the triad variables $\tilde{e}^{\,j}$ are parametrized by a single variable σ as

$$\tilde{e}^{\,jI}=\frac{1}{6\Omega}\sigma|\chi|X_I^j, \quad \sigma=6\Omega e^{2\alpha} , \tag{4.4}$$

where $|\chi|$ is the determinant of χ_j^I and $X_I=(X_I^j)$ is the vector field basis dual to χ^I. Hence normalizing N and Λ as

$$\underset{\sim}{n}=\frac{1}{6\Omega}e^{-3\alpha}N , \tag{4.5}$$

$$\lambda=\frac{1}{18\Omega}\Lambda , \tag{4.6}$$

the ADM Lagrangian is written as

$$L=-\sigma\dot{P}+\underset{\sim}{n}C_H , \tag{4.7}$$

where P is the ADM momentum conjugate to σ and C_H is given by

$$C_H=\sigma^2(-P^2+\lambda\sigma-\tfrac{1}{4}) . \tag{4.8}$$

Since we have only the Hamiltonian constraint in the present case, the quantum constraint in the ADM formalism gives the following Wheeler-DeWitt equation for the wave function $\Psi(\sigma)$:

$$\sigma^2\left[\frac{d^2}{d\sigma^2}+\lambda\sigma-\frac{1}{4}\right]\Psi(\sigma)=0\ . \tag{4.9}$$

Here we have adopted the operator ordering such that σ sits left to P on the basis of the general argument on the operator ordering in Sec. II.

Because of the spatial homogeneity and isotropy the Ashtekar momentum $\mathcal{A}_j^{\pm}$ is parametrized by a single function of time, $A^{\pm}$ as

$$\mathcal{A}_{jI}^{\pm}=A^{\pm}\chi_j^I\ . \tag{4.10}$$

Since the generating functional F in the present case is given by

$$F=\frac{\sigma}{2}\ , \tag{4.11}$$

the Ashtekar momentum $A^{\pm}$ is related to the ADM momentum P simply as

$$A^{\pm}=P\pm\frac{i}{2}\ . \tag{4.12}$$

For simplicity we shall omit the index $\pm$ on A in the rest of this section as far as no confusion occurs. Then C_H is expressed in the A representation as

$$C_H=\sigma^2[-A(A\mp i)+\lambda\sigma]\ . \tag{4.13}$$

Thus the wave function $\Phi(A)$ in the A representation should satisfy

$$\frac{d^2}{dA^2}\left[-A(A\mp i)+i\lambda\frac{d}{dA}\right]\Phi(A)=0\ . \tag{4.14}$$

Now we examine the solutions to the Wheeler-DeWitt equations in the σ and the A representations and their relation in order to see the role of the holomorphic condition on the wave functions. Since the structures of the equations change depending on whether or not $\lambda=0$, we discuss these two cases separately.

First for $\lambda=0$ the Wheeler-DeWitt equation (4.9) has a two-parameter family of solutions:

$$\Psi(\sigma)=C_1e^{-\sigma/2}+C_2e^{\sigma/2}\ . \tag{4.15}$$

The corresponding equation in the A representation (4.14) has also a two-parameter family of solutions:

$$\Phi(A)=\frac{D_1}{A\mp i}+\frac{D_2}{A}\ . \tag{4.16}$$

This general solution is analytic in the complex A plane for a sufficiently small value of $\operatorname{Im}A$. Hence the analyticity requirement is satisfied for all the solutions. Further the transformation

$$\Phi(A)=\int_0^{\infty}d\sigma\, e^{-i\sigma A\mp F}\Psi(\sigma) \tag{4.17}$$

exists in $\operatorname{Im}A<-\frac{1}{2}\mp\frac{1}{2}$ for all the solutions given by Eq. (4.15) and the solutions in the two representations are in the one-to-one correspondence through this transformation, as is expected.

In contrast the situation changes for $\lambda\neq0$. The Wheeler-DeWitt equation (4.9) again gives a two-parameter family of solutions:

$$\Psi(\sigma)=C_1\mathrm{Ai}(x)+C_2\mathrm{Bi}(x)\ , \tag{4.18}$$

where $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are Airy functions and x is given by

$$x:=\frac{1-4\lambda\sigma}{4\lambda^{2/3}}\ . \tag{4.19}$$

However, the corresponding equation in the A representation (4.14) is now third order and has a three-parameter family of solutions:

$$\Phi(A)=e^{-iS}\left[D_3+\int_{-i\lambda\infty}dA(D_1+D_2A)e^{iS}\right]\ , \tag{4.20}$$

$$S=\frac{1}{\lambda}\left[\frac{A^3}{3}\mp\frac{i}{2}A^2\right]\ . \tag{4.21}$$

Thus an extra solution appears in the A representation.

The origin of this difference in the degrees of freedom of the solutions is clarified by looking at the correspondence of the solutions. The key point here is the range of σ. Since the formalism itself does not constrain the range of σ explicitly in the quantum theory, we must specify it by additional physical arguments or the consistency of the theory. Mathematically the general transform of the wave function $\Psi(\sigma)$ defined by

$$\Phi(A):=\int_{\sigma_-}^{\sigma_+}d\sigma\, e^{-i\sigma A\mp F}\Psi(\sigma) \tag{4.22}$$

may be a solution to the Wheeler-DeWitt equation in the A representation. However, if we require that this transform satisfies Eq. (4.14) the range of σ is restricted. Actually this requirement is satisfied only if

$$\left[\left[\sigma^2\frac{d\Psi}{d\sigma}+(\mp iA\sigma^2\pm\tfrac{1}{2}\sigma-2\sigma^2)\Psi\right]e^{-i\sigma A\mp F}\right]_{\sigma_-}^{\sigma_+}=0 \tag{4.23}$$

for an arbitrary value of A. Thus the range of σ is restricted to either $[0,+\infty]$ or $[-\infty,0]$ or $[-\infty,\infty]$.

Now let us look at the correspondence between $\Psi(\sigma)$ and $\Phi(A)$ given by Eq. (4.22) for each of these three ranges. In principle we must discuss the cases $\lambda>0$ and $\lambda<0$ separately because the behavior of $\Phi(A)$'s given above largely depends on the sign of λ. However, since the equations for the case $\lambda<0$ is obtained from those for the case $\lambda>0$ by the transformation $\sigma\to-\sigma$ and $A^{\pm}\to-A^{\mp}$ due to the structure of the constraint equations, we only give the equations for the case $\lambda>0$. Some implications of this correspondence will be discussed later.

From the asymptotic behavior of the Airy functions

$$\mathrm{Ai}(x)\sim\begin{cases}\frac{1}{2}\pi^{1/2}x^{-1/4}e^{-2x^{3/2}/3}, & x>0,\\ \pi^{1/2}|x|^{-1/4}\sin\left[\frac{2}{3}|x|^{3/2}+\frac{\pi}{4}\right], & x<0\ ,\end{cases} \tag{4.24}$$

$$\mathrm{Bi}(x)\sim\begin{cases}\pi^{1/2}x^{-1/2}e^{2x^{3/2}/3} & x>0,\\ \pi^{1/2}|x|^{-1/4}\cos\left[\frac{2}{3}|x|^{3/2}+\frac{\pi}{4}\right], & x<0\ ,\end{cases} \tag{4.25}$$

both $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ have the transform (4.22) for the range of σ in $[0,\infty]$, but for the range $[-\infty,0]$ or $[-\infty,\infty]$ only $\mathrm{Ai}(x)$ has the transform.

First for the range $[0,\infty]$, from the integral representations for the Airy functions

$$\mathrm{Ai}(x),\mathrm{Bi}(x)=\frac{1}{2\pi}\int_C dt\,\exp\left[i\left[\frac{t^3}{3}+xt\right]\right], \quad (4.26)$$

where the integration path C for $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ is given by C_A and C_B shown in Fig. 1, respectively, the transformations of $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ exist for $\mathrm{Im}\,A<\frac{1}{2}$ and are given by

$$\Phi(A)=\frac{2i}{\pi\lambda^{1/3}}\int_C \frac{dt}{t+A\mp i/2}\exp\left[\frac{i}{\lambda}\left[\frac{t^3}{3}+\frac{t}{4}\right]\right]. \quad (4.27)$$

From this equation it follows that

$$\left[-A(A-i)+i\lambda\frac{d}{dA}\right]\Phi(A) = -4[\Psi'(0)+(iA\pm\tfrac{1}{2})\Psi(0)]\,. \quad (4.28)$$

Thus the transform of the solution (4.18) is given by a holomorphic function

$$\Phi(A)=\frac{4i}{\lambda}e^{-iS}\int_{-i\infty} dA[\Psi'(0)+(iA\pm\tfrac{1}{2})\Psi(0)]e^{iS}\,, \quad (4.29)$$

which is a two-parameter subset of the solutions given in Eq. (4.20).

On the other hand, for the range $[-\infty,\infty]$ the transformation of $\mathrm{Ai}(x)$ is well defined only in $\mathrm{Im}\,A<\frac{1}{2}$, but the corresponding image $\Phi(A)$ turns out to be holomorphic in the whole complex plane and given by

$$\Phi_0(A)=-4\lambda^{-1/3}e^{\mp 1/12\lambda}e^{-iS}\,. \quad (4.30)$$

This corresponds to the remaining one-parameter family of solutions in Eq. (4.20). Finally for the range $[-\infty,0]$ the transformation of $\mathrm{Ai}(x)$ is well defined in the whole complex plane, and the image is given by the difference of $\Phi(A)$'s given in Eqs. (4.30) and (4.29).

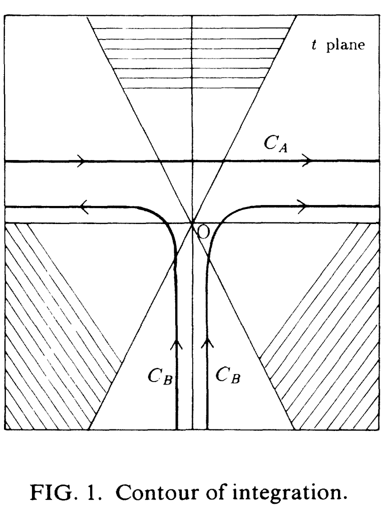

FIG. 1. Contour of integration.

Conversely, if we take the inverse transform of (4.22) defined by

$$e^{\mp F}\Psi(\sigma)=\frac{1}{2\pi}\int_{\tilde{C}} dA\, e^{iA\sigma}\Phi(A)\,, \quad (4.31)$$

where the integration path $\tilde{C}$ is a line of $\mathrm{Im}\,A=\beta=\mathrm{const}$ with a sufficiently small negative β, $\Phi(A)$ given by Eq. (4.29) yields $\Psi(\sigma)$ which vanishes for $\sigma<0$, while $\Phi_0(A)$ yields $\mathrm{Ai}(x)$ in the full range of $\sigma\in[-\infty,\infty]$.

We can learn several things from this study of the simple system. First the wave functions which are defined in different ranges of σ in the σ representation are represented by different analytic functions in the A representation even if they coincide with each other in the common domain of σ. In particular for nonvanishing λ the Hamiltonian constraint in the A representation allows for a solution which extends across the classical singularity $q=\det(q_{jk})=0$ (note that this corresponds to $\sigma=0$ in the above system). This conclusion holds also for general systems because the Hamiltonian constraint becomes a third-order differential equation in the $\mathcal{A}^{\pm}$ representation for systems with nonvanishing potential of material fields. Second, though the analyticity requirement on the wave functions in the A representation does not restrict the class of solutions strongly, the A representation picks up some restricted class of solutions if we require that the wave functions in the σ representation do not have an artificial cutoff at $q=0$. In the present case this requirement selects out a single solution Φ_0 given by Eq. (4.30) for $\lambda\neq 0$, which coincides with the Hartle-Hawking wave function.[36]

This last point is interesting in connection with the boundary condition on the wave function of the Universe. In the conventional ADM approach in which the spatial metric q_{jk} is taken as the fundamental variable the Wheeler-DeWitt equation becomes singular at $q=0$. Hence the range of q_{jk} is restricted to $q>0$ and the $q=0$ portion is taken as a boundary of the superspace. The intuitive picture of the quantum universe so-called "creation of the Universe from nothing" is based on this standpoint. In contrast, in Ashtekar's formalism in which $\tilde{e}^{\,j}$ is taken as the fundamental variable the equations are regular at $q=0$ due to the polynomiality. Hence it is natural to extend the domain of the wave function to the region where $q<0$ (Ref. 17). Since the inverse of the spatial metric

$$q^{jk}=\tilde{e}^{\,j}\cdot\tilde{e}^{\,k}/q \quad (4.32)$$

changes its signature in this region, it corresponds to the Euclidean spacetime in general. Thus allowing for the wave function to extend across the classical singularity $q=0$ may yield an intuitive picture of the quantum universe different from that based on the metric theory.

In order to see this, let us examine the behavior of the wave function $\Phi_0(A)$ more closely. Since the solution $\Phi_0(A)$ has the WKB-type structure, it can be interpreted by the corresponding classical solutions. Actually the phase factor S satisfies the classical Hamilton Jacobi equation

$$h\left[\sigma, \frac{\partial S}{\partial \sigma}\right]=0 . \tag{4.33}$$

Thus the WKB orbits determined by the equations

$$\sigma=\frac{\partial S}{\partial A}=\frac{1}{\lambda}(A^2 \mp iA) , \tag{4.34}$$

$$\frac{dA}{d\tau}=\{h, A\}=-\lambda\sigma^2 , \tag{4.35}$$

where

$$d\tau := \underset{\sim}{n} dt=\frac{\sqrt{6\Omega}\, dt}{\sigma^{3/2}} , \tag{4.36}$$

satisfy the classical equation of motion. The solution to this equation is given by

$$A=-\tfrac{1}{2}\sinh\xi \pm \frac{i}{2} , \tag{4.37}$$

$$\lambda\sigma=\tfrac{1}{4}\cosh^2\xi , \tag{4.38}$$

$$\sqrt{6\Omega\lambda}\, t=\xi+\text{const} , \tag{4.39}$$

where we have taken $N=1$ for simplicity. If we assume that t is real and $\sigma>0$, we obtain a classical solution only for $\lambda>0$, which represents the de Sitter spacetimes (dS^4)

$$2\Omega\lambda\, ds^2=-d\xi^2+\tfrac{1}{4}\cosh^2\xi\, d\Omega_3^2 , \tag{4.40}$$

as expected. Here the unusual factor $\frac{1}{4}$ appears because we have chosen the metric of the three-sphere so that it has the sectional curvature $K=\frac{1}{4}$. This solution covers the region of σ where $\lambda\sigma>\frac{1}{4}$.

For $\lambda>0$ the remaining region is covered by Euclidean solutions. First for $\sigma>0$ ξ hence t should be imaginary. By replacement $\xi \to \pm i(\xi+\pi/4)$, Eqs. (4.38) and (4.39) yield a Euclidean solution

$$\sqrt{6\Omega\lambda}\, t=\pm i\xi+\text{const}, \quad \lambda\sigma=\tfrac{1}{4}\sin^2\xi , \tag{4.41}$$

which covers the region $0 \le \lambda\sigma \le \frac{1}{4}$ and represents the Euclidean four-sphere (S^4)

$$2\Omega\lambda\, ds^2=d\xi^2+\tfrac{1}{4}\sin^2\xi\, d\Omega_3^2 . \tag{4.42}$$

Next for $\sigma<0$ the replacement $\xi \to \xi+(\pi/2)i$ yields a solution

$$\sqrt{6\Omega\lambda}\, t=\xi+\text{const}, \quad \lambda\sigma=-\tfrac{1}{4}\sinh^2\xi . \tag{4.43}$$

Now the time coordinate t is apparently real but the solution represents the Euclidean four-dimensional hyperbolic space (H^4)

$$2\Omega\lambda\, ds^2=-(d\xi^2+\tfrac{1}{4}\sinh^2\xi\, d\Omega_3^2) . \tag{4.44}$$

Of course the last solution is not new. It can be obtained also in the metric theory if we allow for q to become negative. What is important here is that this solution is obtained with other solutions from the single holomorphic wave function in Ashtekar's theory. In the Wheeler-DeWitt approach based on the metric formulation there exists no natural connection between the solutions in the regions $q<0$ and $q>0$ in general. Of course in the present simple system the solution in $q>0$ is uniquely extended into the region $q<0$ also in the Wheeler-DeWitt theory since the superspace is one dimensional. However, it is not the case when the dimension of the superspace is greater than one.

Anyway the wave function $\Phi_0(A)$ gives under the WKB approximation a sequence of three classical spacetimes: $H^4 \to S^4 \to dS^4$. Hence, instead of creation from nothing, we get a picture that the Universe is created from a Euclidean space. Here the Euclidean four sphere plays the role of a temporary bridge connecting the mother Euclidean space and the Lorentzian universe as illustrated in Fig. 2. As we will see in the next section this picture can be extended to anisotropy spacetimes.

Now let us look at the case $\lambda<0$. As noted before, the solutions in this case are obtained from the case $\lambda>0$ by the replacement $\sigma \to -\sigma$ and $A^{\pm} \to -A^{\mp}$. Hence the classical WKB orbits are given by the replacement $\sigma \to -\sigma$ from the above solutions. This implies that the wave function now gives the sequence of spacetimes; $dS^4 \to S^4 \to H^4$. A curious point of this result is that dS^4 which is a classical Lorentzian solution for $\lambda>0$ appears. Although it appears to be contradictory, it is not the case for the following reason. From Eq. (4.38) ξ is real for $\lambda<0$ and $\sigma<0$. However, from Eq. (4.39), it implies that t is imaginary since $\sigma<0$. Thus the Lorentzian signature is obtained by the simultaneous changes of the signature of the temporal metric and the spatial metric. If we assume that the original coordinate time should be real, such solution is not allowed classically. However, in the framework where the constraint equations alone give the full theory, the coordinate time has no physical meaning. The physical time should be derived from the wave function. From this standpoint what should be real is not t but τ which is real for real ξ. Hence the appearance of the Lorentzian de Sitter solution may be regarded as physically meaningful. In this standpoint the signature of the cosmological constant has no meaning if we allow for the wave function to extend to the region $q<0$. Furthermore, since the wave function in the σ representation falls off exponentially in the region $\lambda\sigma<0$, the

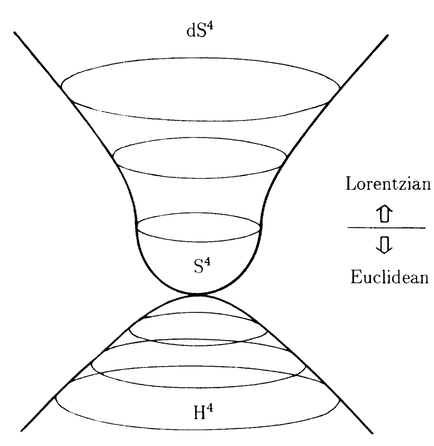

FIG. 2. Creation of the Universe from Euclidean spacetime.

cosmological constant becomes effectively positive in the classical universe irrespective of the input sign of λ. Although this result is quite attractive, its generality is not clear. For example, for the anisotropic universe the $\mathcal{A}^+$ and the $\mathcal{A}^-$ representations are not equivalent as we will see in the next section. For such a case the argument of this section does not apply since the above transformation connecting the case $\lambda>0$ and the case $\lambda<0$ changes the representation.

These analyses show that the wave function $\Phi_0(A)$ is a quite attractive candidate of the wave function of the Universe. Thus it is interesting to see whether there is a criterion to select out this type of wave functions in the general case. In the σ representation the wave function Φ_0 is characterized by two features. First it has no cutoff at $q=0$. Second it does not grow in the classically forbidden region $\lambda\sigma<0$. The second property is important since if the wave function grows there it will become quite improbable for the classical universe to emerge, intuitively speaking. One attractive point of the $\mathcal{A}^\pm$ representation is that this second requirement is automatically included. In fact if the wave function grows in the forbidden region, it will not have a holomorphic image because the integration defining the transformation diverges. However, there occurs another difficulty in the $\mathcal{A}^\pm$ representation. As the above example shows, the solutions to Hamiltonian constraint in the $\mathcal{A}^\pm$ representation may contain the wave functions which have a cutoff at $q=0$ in the chiral $\tilde{e}$ representation. Thus we must give a criterion to rule out such solutions. Fortunately we can find such criterion. In the above example, the inverse transform from $\Phi(A)$ to $\Psi^\pm(\sigma)=e^{\mp F}\Psi(\sigma)$ is given by Eq. (4.31). If $\Psi^\pm(\sigma)$ has a cutoff at $\sigma=0$, its second derivative with respect to σ will diverge there. Hence if the integral

$$\left.\frac{d^2}{d\sigma^2}\Psi^\pm(\sigma)\right|_{\sigma=0}=-\frac{1}{2\pi}\int_C dA\, A^2e^{iA\sigma}\Phi(A) \qquad (4.45)$$

converges, it will correspond to the solutions with no cutoff. This condition is equivalent to the one that $\Phi(A)$ falls off more rapidly than $O(A^{-2})$ for $A\to\pm\infty$. In fact it is easily shown that this condition selects out the solution Φ_0 uniquely. Thus we can formulate the above condition on the wave function in the $\mathcal{A}^\pm$ representation that it is holomorphic and falls off faster than $O(1/\mathcal{A}^{\pm 2})$ at infinity along some path without endpoint.

V. BIANCHI TYPE-IX UNIVERSE WITH Λ

As noted in Sec. IV the Ashtekar variables $\tilde{e}^J$ and $\mathcal{A}_j^\pm$ are parametrized by time-dependent matrices σ^{IJ} and A_{IJ} for general spatially homogeneous spacetimes. Hence the system has 9 degrees of freedom. However, as was shown in Ref. 19, 6 of them which corresponds to the freedom of the local Lorentz transformation decouples from the dynamics and can be eliminated preserving the polynomiality of the constraint equation for vacuum Bianchi type-IX spacetimes. We take this reduced theory as a starting point. In this reduced theory the spacetime metric is given by the diagonal form

$$ds^2=-N^2dt^2+e^{2\alpha+2\beta_I}\chi^I\otimes\chi^I\ , \qquad (5.1)$$

where χ^I is the same as in the previous section. Correspondingly σ^{IJ}, P_{IJ}, and A_{IJ} given in Eqs. (2.35)–(2.37) become diagonal:

$$\sigma^{IJ}=\sigma^I\delta_{IJ},\quad \sigma^I=2\Omega e^{2\alpha-\beta_I}\ , \qquad (5.2)$$

$$P_{IJ}=P_I\delta_{IJ}\ , \qquad (5.3)$$

$$\mathcal{A}_{IJ}^\pm=A_I^\pm\delta_{IJ}\ . \qquad (5.4)$$

By introducing variables

$$\underset{\sim}{n}:=\frac{e^{-3\alpha}}{2\Omega}N\ , \qquad (5.5)$$

$$\lambda:=\frac{\Lambda}{6\Omega}\ , \qquad (5.6)$$

the chiral Lagrangian $L^\pm$ is written as

$$L^\pm=-\sigma_I\dot{A}_I^\pm-\underset{\sim}{n}h\ , \qquad (5.7)$$

where h is given by

$$h=-\sigma_1\sigma_2(A_1^\pm A_2^\pm\mp iA_3^\pm)-\sigma_2\sigma_3(A_2^\pm A_3^\pm\mp iA_1^\pm)$$
$$-\sigma_3\sigma_1(A_3^\pm A_1^\pm\mp iA_2^\pm)+3\lambda\sigma_1\sigma_2\sigma_3\ . \qquad (5.8)$$

The Ashtekar momentum $A_I^\pm$ is related to the corresponding ADM momentum P_I by

$$A_I^\pm=P_I\pm i\frac{\partial F}{\partial\sigma_I}\ , \qquad (5.9)$$

where the generating functional F is now given by

$$F=\frac{1}{2}\left[\frac{\sigma_2\sigma_3}{\sigma_1}+\frac{\sigma_3\sigma_1}{\sigma_2}+\frac{\sigma_1\sigma_2}{\sigma_3}\right]\ . \qquad (5.10)$$

From now on we omit $\pm$ in A_I for simplicity as far as no confusion occurs.

By the replacement $\sigma_I\to i\partial/\partial A_I$, h gives the Wheeler-DeWitt equation in the A representation:

$$\left[\frac{\partial^2}{\partial A_1\partial A_2}(A_1A_2\mp iA_3)+\frac{\partial^2}{\partial A_2\partial A_3}(A_2A_3\mp iA_1)+\frac{\partial^2}{\partial A_3\partial A_1}(A_3A_1\mp iA_2)-3i\lambda\frac{\partial^3}{\partial A_1\partial A_2\partial A_3}\right]\Phi(A)=0\ , \qquad (5.11)$$

where we adopted the operator ordering suggested by the general argument in Sec. II.

Unlike the equation in the Roberston-Walker universe discussed in the previous section we cannot find the general solution to this equation. However, we can find a nontrivial special solution which reduces to the solution Φ_0 for the Robertson-Walker universe when restricted to the isotropic sector. It is given by

$$\Phi(A)=e^{-iS}, \tag{5.12}$$

$$S=\frac{1}{\lambda}\left[A_1A_2A_3\mp\frac{i}{2}(A_1^2+A_2^2+A_3^2\right]. \tag{5.13}$$

The wave function in the σ representation corresponding to this solution is in general given by the transformation

$$e^{\mp F}\Psi(\sigma)=\left[\frac{\Omega}{\pi}\right]^3\int_{C_1}A_1\int_{C_2}dA_2\int_{C_3}dA_3e^{i\sigma\cdot A}\Phi(A). \tag{5.14}$$

Here there arises a new problem associated with the contour of integration. In the isotropic case the requirement that this transformation becomes the inverse of the transformation which maps $\Psi(\sigma)$ to $\Phi(A)$ determined the contour to be a path which tends to $\pm\infty-i\beta$ with sufficiently large β. However, in the present case, the integration in the above transformation does not converge along such contour in general. In fact for the A^+ representation with positive λ or for the A representation with negative λ, the integration converges only when $|\beta|$ is sufficiently small for all C_I. Further for the A^- representation with positive λ or for the A^+ representation with negative λ the integration does not converge along such contours for any value of β.

At present we have no idea how to treat this problem. So we only consider the case $\lambda>0$ in the A^+ representation (which is equivalent to the case $\lambda<0$ in the A^- representation). In this case the above transformation yields the wave function in the σ representation

$$e^{-F}\Psi(\sigma)=\frac{2\lambda\Omega^3}{\pi^2}\int_{-\infty}^{\infty}\frac{dz}{\sqrt{1+z^2}}\exp\left[\frac{\lambda}{2}\frac{-(\sigma_1^2+\sigma_2^2)+2i\sigma_1\sigma_2 z}{1+z^2}+i\sigma_3 z-\frac{z^2}{2\lambda}\right]. \tag{5.15}$$

Although this expression is not symmetric in σ_1, σ_2, and σ_3, it is actually a symmetric function of them since $\Phi(A)$ is symmetric in A_1, A_2, and A_3.

Although we cannot express $\Psi(\sigma)$ in terms of well-known functions explicitly we can infer its asymptotic behavior with help of the WKB approximation. In the region where $|\sigma_I|$ is large, $\Psi(\sigma)$ is approximately given by

$$\Psi(\sigma)\sim e^{iS'(\sigma)},\quad S'=S(\sigma,A)-\sigma\cdot A, \tag{5.16}$$

with A given by the solution of

$$\sigma_I=\frac{\partial S}{\partial A_I}=\frac{1}{\lambda}[A_JA_K\mp iA_I], \tag{5.17}$$

where (I,J,K) is (1,2,3) or its cyclic permutation. In the lowest-order approximation it is given by

$$\Psi(\sigma)\sim C\exp(-2i\sqrt{\lambda\sigma_1\sigma_2\sigma_3}) + C^*\exp(2i\sqrt{\lambda\sigma_1\sigma_2\sigma_3}), \tag{5.18}$$

where C is some constant [note that $\Psi(\sigma)$ is real from the integral expression (5.15)].

Next we examine the semiclassical picture for the quantum universe the solution gives by examining the WKB orbits corresponding to the wave function as in the previous section. Since the phase S of Φ given in Eq. (5.13) is an exact solution to the classical Hamiltonian-Jacobi equation

$$h\left[\sigma=\frac{\partial\mathcal{S}}{\partial A},A\right]=0, \tag{5.19}$$

the WKB orbits are given by the solutions to Eq. (5.17) and

$$\frac{dA_I}{d\tau}=\{A_I,h\}=-\lambda\sigma_J\sigma_K, \tag{5.20}$$

where τ is defined by

$$d\tau=\underset{\sim}{n}\,dt=\left[\frac{2\Omega}{\sigma_1\sigma_2\sigma_3}\right]^{1/2}dt. \tag{5.21}$$

First we consider the solutions which correspond to Lorentzian spacetimes. For such solutions σ_I and τ should be real. As is expected from the general argument in the Appendix, the consistency of this condition with time evolution requires that A_I is written as

$$A_I=P_I\pm iF_I, \tag{5.22}$$

where P_I is real and F_I is the gradient of F

$$F_I:=\frac{\partial F}{\partial\sigma_I}. \tag{5.23}$$

Here note that F_1–F_3 are not independent and satisfy the identity

$$F_1^2+F_2^2+F_3^2+2F_1F_2F_3-1\equiv 0. \tag{5.24}$$

If we decompose Eq. (5.17) into the real and imaginary parts according to Eq. (5.22) we obtain

$$P_JP_K-F_JF_K+F_I=\lambda\sigma_I, \tag{5.25}$$

$$P_JF_K+P_KF_J=P_I. \tag{5.26}$$

From the identity (5.24), the second equation turns out to be equivalent to that P_I is parametrized by a single function y as

$$P_I=y/(F_JF_K+F_I). \tag{5.27}$$

Similarly the evolution equation (5.20) yields

$$\frac{dP_I}{d\tau}=-\lambda\sigma_J\sigma_K, \tag{5.28}$$

$$\frac{dF_I}{d\tau}=0. \tag{5.29}$$

The second equation implies that F_I is constant in time. Hence, from the identity (5.24), σ_I is parametrized by a single function x as

$$\sigma_I = C_I x \ , \tag{5.30}$$

where C_I is a constant defined by

$$C_I := F_I + F_J F_K \ . \tag{5.31}$$

Inserting these expressions for P_I and σ_I into Eq. (5.25), we find that y should satisfy

$$y^2 = C_J C_K (F_J F_K - F_I) + \lambda C_1 C_2 C_3 x \tag{5.32}$$

for $(I,J,K)=(1,2,3)$, (2,3,1), and (3,1,2). If we require that $C_1 C_2 C_3 \neq 0$, these equations are consistent only when

$$F_I^2 = F_2^2 = F_3^2 \ . \tag{5.33}$$

This condition together with the identity (5.24) determines the values of F_I to be $F_1 = F_2 = F_3 = \frac{1}{2}$ or $F_1 = F_2 = -F_3 = \frac{1}{2}$ or its permutation. It is obvious that the first choice yields the isotropic solution discussed in the previous section, i.e., de Sitter spacetime. Since the simultaneous change of sign in two of σ_I's does not change the spatial metric q_{jk}, the latter choices yield the same spacetime. Thus if we restrict to the Lorentzian solutions, the isotropic de Sitter solution is the only exact solution to Eqs. (5.17) and (5.20). This implies that WKB orbits filling the region with large positive σ_I are not exactly Lorentzian. However, for large x, the first terms on the right-hand sides of Eqs. (5.32) become much smaller than the second terms. If we neglect these first terms, F_I's are no longer restricted by the consistency and we obtain the approximate solutions

$$x \simeq \frac{1}{\lambda C_1 C_2 C_3} \left[\frac{\lambda C_1 C_2 C_3}{3} \frac{1}{\tau_* - \tau} \right]^{2/3} , \tag{5.34}$$

$$y \simeq \left[\frac{\lambda C_1 C_2 C_3}{3} \frac{1}{\tau_* - \tau} \right]^{1/3} , \tag{5.35}$$

where τ_* is an integration constant. Since τ is related to t as

$$\tau_* - \tau \simeq \exp(-\sqrt{2\Omega\lambda}\, t) \ , \tag{5.36}$$

from the relation

$$\sqrt{2\Omega}\, dt = \sqrt{\sigma_1 \sigma_2 \sigma_3}\, d\tau = \sqrt{C_1 C_2 C_3}\, x^{3/2} d\tau \ , \tag{5.37}$$

these solutions represent spacetimes expanding exponentially with constant anisotropy ratio $\beta_1 : \beta_2 : \beta_3$. These solutions fill out the region with large positive $\sigma_1 \sigma_2 \sigma_3$.

Similarly to the case of the isotropic universe, the region with $\sigma_1 \sigma_2 \sigma_3 \lesssim 0$ is covered by Euclidean solutions. By the replacement

$$d\tau \to \pm i\, d\tau \ , \tag{5.38}$$

$$A_I = \pm i B_i \ , \tag{5.39}$$

Eqs. (5.17) and (5.20) yield the following equations for the Euclidean orbits:

$$\sigma_I = \frac{1}{\lambda} (B_J B_K - B_I) \ , \tag{5.40}$$

$$\frac{dB_I}{d\tau} = \lambda \sigma_J \sigma_K \ , \tag{5.41}$$

Now the reality of σ gives rise to no constraint. After a long calculation we find that these equations are reduced to the second-order nonlinear equation for $Z := B_1 B_2 B_3$ given by

$$x(2x+1) Z \frac{d^2 Z}{dx^2} + \left[\frac{2x+1}{1-x} \left[\frac{dZ}{dx} \right]^2 + \frac{x(4Z + 10x^3 - 15x^2 + 1)}{(1-x)^2} \frac{dZ}{dx} + \frac{18x^2+1}{1-x} Z + 8x^4 \frac{2x-3}{1-x} \right] \frac{dZ}{dx} + 2x^3 (5Z + 4x^3 - 6x^2) = 0 \ , \tag{5.42}$$

where x is related to τ by

$$\frac{d\tau}{\lambda} = \frac{1}{Z} \frac{\dfrac{dZ}{dx} + x(1-x)}{(1-x)^2 (2x+1)} dx \ . \tag{5.43}$$

For each solution to this equation B_I's are given by the solution to the algebraic equations

$$B_1^2 + B_2^2 + B_3^2 = 2Z + 3x^2 - 2x^3 \ , \tag{5.44}$$

$$B_1^2 B_2^2 + B_2^2 B_3^2 + B_3^2 B_1^2 = Z \left[Z + 1 + 3x^2 - 2x^3 - \frac{x(1-x)^3 (2x+1)}{\dfrac{dZ}{dx} + x(1-x)} \right] , \tag{5.45}$$

$$B_1^2 B_2^2 B_3^2 = Z^2 \ . \tag{5.46}$$

Thus we obtain two-parameter family of Euclidean solutions which covers the three-dimensional space of σ.

Unfortunately it is difficult to solve Eq. (5.42) generally. We could find only two exact solutions to this equation. One is $Z = x^3$ which corresponds to the Euclidean isotropic solutions discussed in the previous section. The other is $Z = (x-2)(x^2-2)$. This solution yields

$$B_1 = B_2 = 2 - x^2, \quad B_3 = (x-2)^2 \tag{5.47}$$

and its permutations. The corresponding expressions for σ_I are given by

$$\lambda\sigma_1 = \varepsilon_1\sqrt{2-x^2}(x-1)\ , \tag{5.48}$$

$$\lambda\sigma_2 = \varepsilon_2\sqrt{2-x^2}(x-1)\ , \tag{5.49}$$

$$\lambda\sigma_3 = \varepsilon_1\varepsilon_2(x^2-x)\ , \tag{5.50}$$

and its permutations where ε_1 and ε_2 are $\pm$, and the Euclidean time $\chi := \pm i\sqrt{2\lambda\Omega}\, t$ is expressed in terms of x as

$$d\chi = (\lambda\sigma_1\sigma_2\sigma_3)^{1/2}d\tau = \left[\frac{x}{(2-x^2)(x-1)}\right]^{1/2} dx\ . \tag{5.51}$$

If necessary, σ_I can be expressed in terms of χ by the elliptic functions.

The behavior of the latter solution in σ space and in q space is illustrated in Fig. 3. It starts from a degenerate Euclidean space (two-dimensional disk), expands and deforms to another degenerate Euclidean space (one-dimensional line), and finally isotropizes and shrinks to a point. After passing the point, it expands to a deformed three-sphere and finally collapses to a two-dimensional disk. This latter part represents a deformed Euclidean four-sphere with a degenerate hypersurface. If the behavior of a generic solution is similar to this special orbit, the above wave function gives us a semiclassical picture of the quantum universe similar to the one given in the previous section. The Universe is created from a small region of an ensemble of fluctuating four-dimensional Euclidean spaces. In the present case an ensemble of anisotropic Euclidean spaces corresponds to the mother Euclidean spacetimes, and the deformed Euclidean four-spheres represent local fluctuations of the mother spacetimes. The reason why such local fluctuations occur only at a single point (or at its neighborhood) in the present case is the spatial homogeneity assumption on spacetimes. Thus, following this picture, it is expected that such a creation of universes from fluctuations occurs at many regions of the mother Euclidean space in the generic situation. In order to check whether or not this picture is valid we must study the behavior of the above exact solution in detail. We should also examine whether or not the holomorphic condition and the asymptotic condition on the wave functions select this solution uniquely. These problems are now under investigation.

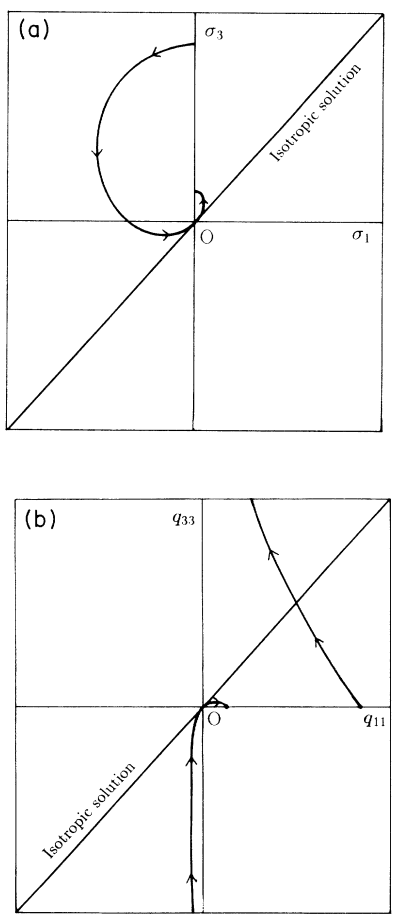

FIG. 3. WKB paths for the special solution.

VI. DISCUSSIONS

In this paper I studied the quantization of the complex canonical theory of gravity proposed by Ashtekar and its application to Bianchi type-IX cosmology in order to see the consequences of the polynomiality of Ashtekar's formalism in quantum gravity. In particular I examined in detail the structure of the solutions to the Hamiltonian constraint in the holomorphic and the triad representation, and the relation between these representations for the Robertson-Walker universe with the cosmological constant. On the basis of the analysis I argued that in the quantized Ashtekar formalism it is natural to extend the superspace on which the wave functions are defined to the region where the determinant q of the spatial metric is negative. Further it was found that if we extend the superspace in such a way, the holomorphic representation selects out a solution which is an analytic extension of the Hartle-Hawking wave function in the conventional ADM-Wheeler-DeWitt approach. By studying the behavior of the semiclassical orbits corresponding to the solution, it was shown that the extended wave function represents a sequence of the spacetime: a Euclidean hyperbolic space $\to$ a Euclidean four-sphere $\to$ a Lorentzian de Sitter spacetime.

On the basis of these results I proposed a new *Ansatz* on the criterion to select out the wave function of the Universe that the wave function should be represented by a holomorphic function with an appropriate asymptotic behavior in the holomorphic representation. I further conjectured that under this *Ansatz* the quantum behavior of the Universe is described by a semiclassical picture such that the Lorentzian classical universe is created from an ensemble of Euclidean mother spacetimes through their local fluctuations, which is different from the conventional picture "creation of the Universe from nothing." I also pointed out that the sign of the cosmological constant may become meaningless by the extension of the superspace and an effectively positive value are automatically realized in the classical universe by dynamics.

I further extended the analysis to the vacuum Bianchi type-IX universe with the cosmological constant, and examined the behavior of a special wave function which coincides in the isotropic sector with the special wave function for the Robertson-Walker universe discussed above. The semiclassical analysis of this solution confirmed the above picture on the quantum behavior of the Universe. I also pointed out that in the holomorphic representation the quantum theory may yield different results depending on the choice of the complex momentum, the left chiral one or the right chiral one.

Although the analysis in the present paper is far from complete even if we limit the scope to the Bianchi cosmology, the obtained results are attractive enough to show that Ashtekar's formalism is more than useful in the study of quantum cosmology. In particular it is interesting to see whether the *Ansatz* and the proposed picture on the quantum universe can be extended to the generic case.

In this connection I here point out that the special holomorphic solution found for the Bianchi type-IX spacetime can be extended to an almost exact solution to all the quantum constraint equations for the general vacuum spacetimes with the cosmological constant. It is given by

$$\Phi(\mathcal{A}^{\pm})=e^{-i\mathcal{S}}, \tag{6.1}$$

$$\mathcal{S}=\frac{1}{\Lambda}\int d^3x\, \varepsilon^{jkl}[\mp 3i\mathcal{A}_j^{\pm}\cdot\partial_k\mathcal{A}_l^{\pm}+\mathcal{A}_j^{\pm}\cdot(\mathcal{A}_k^{\pm}\times\mathcal{A}_l^{\pm})] . \tag{6.2}$$

To prove that this is a solution to the constraint equations, first note the relation

$$\frac{\delta\mathcal{S}}{\delta\mathcal{A}_j^{\pm}}=\frac{6}{\Lambda}\mathcal{F}^j , \tag{6.3}$$

where $\mathcal{F}^j$ is defined by

$$\mathcal{F}^j:=\tfrac{1}{2}\varepsilon^{jkl}\mathcal{F}_{kl}=e^{jkl}(\mp i\partial_k\mathcal{A}_l^{\pm}+\tfrac{1}{2}\mathcal{A}_k^{\pm}\times\mathcal{A}_l^{\pm}) . \tag{6.4}$$

From this relation the operation of $\tilde{e}^{\,j}$ on the wave function is given by

$$\tilde{e}^{\,j}\Phi=\frac{i}{2}\frac{\delta\Phi}{\delta\mathcal{A}_j^{\pm}}=\frac{3}{\Lambda}\mathcal{F}^j\Phi . \tag{6.5}$$

Hence, noting that $\tilde{e}^{\,j}\times\mathcal{A}_j^{\pm}=-\mathcal{A}_j^{\pm}\times\tilde{e}^{\,j}$ holds locally even in the operator sense, the gauge constraint is satisfied owing to the Bianchi identity for the complex connection $\mathcal{A}_j^{\pm}$:

$$\mathcal{C}_G\Phi=\frac{6}{\Lambda}\mathcal{D}_j\mathcal{F}^j\Phi=\frac{3}{\Lambda}\varepsilon^{jkl}\mathcal{D}_j\mathcal{F}_{kl}\Phi\equiv 0 . \tag{6.6}$$

For the operator ordering adopted in this paper the Hamiltonian constraint is also satisfied trivially:

$$\mathcal{C}_H\Phi=\frac{1}{4}\varepsilon_{jkl}\frac{\delta}{\delta\mathcal{A}_k^{\pm}}\times\frac{\delta}{\delta\mathcal{A}_l^{\pm}}\cdot\left[\left(\mathcal{F}^j-\frac{\Lambda}{3}\tilde{e}^{\,j}\right)\Phi\right]=0 . \tag{6.7}$$

In contrast, since the operation of $\mathcal{C}_{Mj}$ on Φ for the current ordering is given by

$$\mathcal{C}_{Mj}\Phi=\mp 2i\varepsilon_{jkl}(\tilde{e}^{\,k}\cdot\mathcal{F}^l-\mathcal{F}^l\cdot\tilde{e}^{\,l})\Phi , \tag{6.8}$$

and the operators $\tilde{e}^{\,j}$ and $\mathcal{F}^j$ do not commute,

$$[\tilde{e}^{\,kI}(\mathbf{x}),\mathcal{F}_{jkl}(\mathbf{y})]=\pm\partial_j\delta(\mathbf{y}-\mathbf{x}) , \tag{6.9}$$

the momentum constraint is satisfied only in the semiclassical approximation. This difficulty is a generic feature of the ordering adopted in the present paper as was discussed in Sec. II. Actually the solution is invariant under the spatial diffeomorphism and satisfies the momentum constraint for the reversed ordering. As noted in Sec. II it is not clear whether or not this apparent defect of the solution is serious. It may be removed under some regularization of the products of local operators at the same point, or by some gauge fixing.

Apart from this difficulty, this solution has a quite interesting feature: the phase $\mathcal{S}$ is proportional to the Chern-Simons functional discussed in Ref. 21 in connection with the problem of topological *CP* violation in quantum gravity. Further $\mathcal{S}$ is an exact solution to the classical constraint equations in the Hamilton-Jacobi sense. It just corresponds to the classical solutions to the Einstein equations studied by Samuel.[39] These facts as well as the various fascinating features in the spatially homogeneous sector pointed out in the present paper suggest that it is important to study the structure and behavior of the above holomorphic wave function in detail. This problem is now under investigation.

APPENDIX: RELATION BETWEEN ASHTEKAR'S VARIABLES AND THE ADM VARIABLES

In the first-order formalism of Einstein's theory, the Lagrangian density for vacuum spacetimes with cosmological constant Λ is expressed in terms of the tetrad $e^{\mu}{}_a$ and the connection form $A_{\mu ab}$ as

$$\mathcal{L}=|\omega|(e^{\mu a}e^{\nu b}F_{\mu\nu ab}-2\Lambda) , \tag{A1}$$

where $|\omega|=1/\det(e^{\mu}{}_a)$ and $F_{\mu\nu ab}$ is the curvature tensor

$$F_{\mu\nu ab}:=\partial_\mu A_{\nu ab}-\partial_\nu A_{\mu ab}+A_{\mu a}{}^{c}A_{\nu cb}-A_{\nu a}{}^{c}A_{\mu cb} . \tag{A2}$$

As was shown by Jacobson and Smolin,[40] this Lagrangian is equivalent to the chiral ones obtained by replacing $A_{\mu ab}$ in Eq. (A2) by its complex chiral combination

$$\mathcal{A}_{\mu ab}^{\pm}:=\tfrac{1}{2}(A_{\mu ab}\pm i\,{}^*A_{\mu ab}) , \tag{A3}$$

where ${}^*A_{\mu ab}$ is the dual tensor of $A_{\mu ab}$ defined by

$${}^*A_{\mu ab}:=\tfrac{1}{2}\varepsilon_{ab}{}^{cd}A_{\mu cd} . \tag{A4}$$

If one puts this chiral theory into a canonical form under some appropriate gauge condition, it gives an apparently polynomial canonical theory which coincides with Ashtekar's theory. This yields an elegant proof of the equivalence of the triad ADM theory and Ashtekar's theory, and clarifies why Einstein's theory can be rewritten as a polynomial theory. As is pointed out by some people, however, there is a hidden difficulty associated with the reality condition on the triad in this equivalence proof, which becomes serious when one quantizes the theory. In this appendix I make this point clear by repeating the equivalence proof in the (3+1)-decomposition framework under the spatial gauge.

In the spatial gauge in which the vector $e^{\mu}{}_0$ is orthogonal to the $t=$const hypersurfaces, e^{0I} vanishes. Hence by defining the lapse function N and the shift vector N^j as

$$(e^{\mu}_0)=\left[\frac{1}{N},-\frac{N^j}{N}\right], \tag{A5}$$

the spacetime metric is expressed only in terms of N, N^j, and the triad e^{jI} as

$$ds^2=-N^2dt^2+q_{jk}(dx^j+N^jdt)(dx^k+N^kdt), \tag{A6}$$

where q_{jk} is the spatial metric tensor defined as the inverse of $q^{ik}:=e^{jI}e^{kI}$. Under this 3+1 decomposition the Lagrangian density (A1) is written in terms of N^j, $\underset{\sim}{N}:=N/q^{1/2}[q:=\det(q_{jk})]$ and

$$\tilde{e}^{\,j}:=(\tilde{e}^{\,jI}),\quad \tilde{e}^{\,jI}:=q^{1/2}e^{jI}, \tag{A7}$$

$$P_{\mu}:=(P_{\mu I}),\quad P_{\mu I}:=A_{\mu 0I}, \tag{A8}$$

$$Q_{\mu}:=(Q_{\mu I}),\quad Q_{\mu I}:=\tfrac{1}{2}\varepsilon_{IJK}A_{\mu JK}, \tag{A9}$$

as

$$\mathcal{L}=-2\tilde{e}^{\,j}\cdot\dot{P}_j-P_0\cdot C_B-Q_0\cdot C_R-N^jC_{Mj}-\underset{\sim}{N}C_H, \tag{A10}$$

where

$$C_B:=2D_j\tilde{e}^{\,j}, \tag{A11}$$

$$C_R:=2\tilde{e}^{\,j}\times P_j, \tag{A12}$$

$$C_{Mj}:=-2\tilde{e}^{\,k}\cdot(D_jP_k-D_kP_j), \tag{A13}$$

$$C_H:=-(\tilde{e}^{\,j}\times\tilde{e}^{\,k})\cdot\left[F_{jk}+P_j\times P_k-\frac{\Lambda}{3}\varepsilon_{jkl}\tilde{e}^{\,l}\right]. \tag{A14}$$

Here $\cdot$ and $\times$ denote the inner and exterior products with respect to the internal index, respectively, D_j is the spatial covariant derivative

$$D_j:=\partial_j-Q_j\times, \tag{A15}$$

and F_{jk} is the corresponding curvature tensor

$$F_{jk}:=\partial_jQ_k-\partial_kQ_j-Q_j\times Q_k. \tag{A16}$$

If we replace $A_{\mu ab}$ by its complex chiral combination $\mathcal{A}^{\pm}_{\mu ab}$ in all the equations above, we obtain the corresponding expression for the complex chiral Lagrangian density $\mathcal{L}^{\pm}$:

$$2\mathcal{L}^{\pm}=-2\tilde{e}^{\,j}\cdot\dot{\mathcal{A}}^{\pm}_j-\mathcal{A}^{\pm}_0\cdot\mathcal{C}_G-N^j\mathcal{C}_{Mj}-\underset{\sim}{N}\mathcal{C}_H, \tag{A17}$$

where

$$\mathcal{C}_G:=2\mathcal{D}_j\tilde{e}^{\,j}, \tag{A18}$$

$$\mathcal{C}_{Mj}:=\mp 2i\tilde{e}^{\,k}\cdot\mathcal{F}_{jk}, \tag{A19}$$

$$\mathcal{C}_H:=-(\tilde{e}^{\,j}\times\tilde{e}^{\,k})\cdot\left[\mathcal{F}_{jk}-\frac{\Lambda}{3}\varepsilon_{jkl}\tilde{e}^{\,l}\right]. \tag{A20}$$

Here $\mathcal{D}_j$ and $\mathcal{F}_{jk}$ are quantities obtained by replacing P_j and Q_j in D_j and F_{jk} with $\mathcal{A}^{\pm}_j$ and $\mathcal{B}^{\pm}_j$ defined by

$$\mathcal{A}^{\pm}_{\mu I}:=2\mathcal{A}^{\pm}_{\mu 0I}, \tag{A21}$$

$$\mathcal{B}^{\pm}_{\mu I}:=\varepsilon_{IJK}\mathcal{A}^{\pm}_{\mu JK}, \tag{A22}$$

which are expressed in terms of P_{μ} and Q_{μ} as

$$\mathcal{A}^{\pm}_{\mu}=P_{\mu}\pm iQ_{\mu}, \tag{A23}$$

$$\mathcal{B}^{\pm}_{\mu}=Q_{\mu}\mp iP_{\mu}. \tag{A24}$$

In particular as a result of the (anti-)self-duality of $\mathcal{A}^{\pm}_{\mu ab}$, they are related as $\mathcal{B}^{\pm}_{\mu}=\mp i\mathcal{A}^{\pm}_{\mu}$.

Inserting Eqs. (A23) and (A24) into Eq. (A17) we obtain

$$2\mathcal{L}^{\pm}=\mathcal{L}\pm i\mathcal{L}', \tag{A25}$$

$$\mathcal{L}'=-2\tilde{e}^{\,j}\cdot\dot{Q}_j+N^j\varepsilon_{IJK}F_{jIJK}-Q_0\cdot C_B+P_j\cdot\psi^j$$
$$+2\partial_k[\underset{\sim}{N}P_j\cdot(\tilde{e}^{\,j}\times\tilde{e}^{\,k})], \tag{A26}$$

where $F_{jIJK}:=e^{kI}F_{jkJK}$ and ψ^j is defined in terms of the action $S=\int d^4x\,\mathcal{L}$ as

$$\psi^j:=\left[-\frac{\delta S}{\delta Q_j}+2N^jC_R\right]$$
$$=2(P_0-N^kP_k)\times\tilde{e}^{\,j}-2D_k[\underset{\sim}{N}(\tilde{e}^{\,j}\times\tilde{e}^{\,k})]. \tag{A27}$$

Although $\mathcal{L}'$ is not a total divergence by itself, it reduces to a total divergence under the conditions

$$\psi^j=0, \tag{A28}$$

$$C_B=0, \tag{A29}$$

which are obtained by the variation of the real part of $\mathcal{L}^{\pm}$. To see this, note that Eqs. (A28) and (A29) are equivalent to the following two conditions:

$$P_0=N^jP_j-\partial_jNe^j, \tag{A30}$$

$$Q_{jI}=\tfrac{1}{2}\varepsilon_{IJK}\omega_{jJK}(e). \tag{A31}$$

The latter equation means that the spatial part of the connection is given by the Riemannian connection $\omega_{jIJ}(e)$ corresponding to the triad e^{jI}. Hence the second term on the right-hand side of Eq. (A26) vanishes due to the Bianchi identity. Finally the first term on the same equation is written as a total divergence owing to the identity for Q_{jI} given by Eq. (A31):

$$2\tilde{e}^{\,j}\cdot\dot{Q}_j=(\sqrt{q}\,\varepsilon_{IJK}e_{jI}\partial_ke^{jJ}e^{kK})^{\cdot}$$
$$+\partial_k(\sqrt{q}\,\varepsilon_{IJK}\dot{e}_{jI}e^{jJ}e^{kK}). \tag{A32}$$

This proves the equivalence of the Lagrangian densities (A10) and (A17).

If we regard the chiral Lagrangian density as giving a canonical theory, however, there occurs a problem. First note that the Lagrangian density (A10) gives time-evolution equations for $\tilde{e}^{\,j}$ and P_j but does not for Q_j. Since Q_j is not a Lagrange multiplier unlike P_0, Q_0, N^j, and $\underset{\sim}{N}$, it must be eliminated with the aid of the equation

$$\frac{\delta S}{\delta Q_i}=0. \tag{A33}$$

As stated in the above proof, this condition is equivalent under the constraints $C_B=0$ and $C_R=0$ to the conditions (A30) and (A31). After this elimination of Q_j, the Lagrangian density contains only the canonical variables $\tilde{e}^j$ and P_j, and the genuine Lagrange multipliers Q_0, N^j, and $\underset{\sim}{N}$ to yield the triad ADM canonical Lagrangian.

In contrast, the situation for the complex chiral theory is different. In the above equivalence proof the reality of the triad $\tilde{e}^j$ is implicitly assumed. If we impose this reality condition, the chiral Lagrangian density (A17) does not give a canonical Lagrangian due to the imbalance of the degrees of freedom between $\tilde{e}^j$ and $\mathcal{A}_j^{\pm}$. Unlike the ADM case, the imaginary part of $\mathcal{A}_j^{\pm}$, Q_j, has the time-evolution equation but does not have a conjugate variable. One natural way to remedy this is to consider a Lagrangian density with $\tilde{e}^j$ replaced by the complex variable

$$\mathcal{E}^j=\tilde{e}^j\mp i\chi^j\ , \tag{A34}$$

regard it as a canonical Lagrangian with complex canonical variables $\mathcal{E}^j$ and $\mathcal{A}_j^{\pm}$, and impose the reality condition on $\mathcal{E}^j$ as an additional constraint. Then the consistency of this condition with the time-evolution equations gives

$$2\dot{\chi}^j=\frac{\delta S}{\delta Q_j}=-\frac{\delta S'}{\delta P_j}$$
$$=-\psi^j+N^jC_R=0\quad(\text{mod}\,\chi=0)\ . \tag{A35}$$

Thus under the constraints $C_R=0$ this consistency condition yields a new constraint $\psi^j=0$. This new constraint is consistent with the time evolution. To see it, first note that ψ^j is written as

$$\psi^j=2(P_0+\partial_l Ne^l-N^lP_l+\tfrac{1}{4}\underset{\sim}{N}C_B)\times\tilde{e}^j-2\phi^{jk}\tilde{e}_k\ , \tag{A36}$$

where $\tilde{e}_j=(\tilde{e}_{jI})$ is the inverse of $\tilde{e}^j=(\tilde{e}^{jI})$ and

$$\phi^{jk}:=-\tfrac{1}{2}\psi^{(j}\cdot\tilde{e}^{k)}=-D_l\tilde{e}^{(k}\cdot(\tilde{e}^{j)}\times\tilde{e}^l)\ . \tag{A37}$$

Thus $\psi^j=0$ is equivalent to the condition (A30) and the condition

$$\phi^{jk}=0\ . \tag{A38}$$

Since we are free to set the value of the Lagrange multiplier, we only have to check the consistency of the latter condition. The time evolution of ϕ^{jk} is given by

$$\dot{\phi}^{jk}=\tfrac{1}{2}D_l\underset{\sim}{N}(\tilde{e}^k\cdot\tilde{e}^l)(\tilde{e}^j\cdot C_R)-\tfrac{1}{2}\underset{\sim}{N}(C_R\times\tilde{e}^k)\cdot(\tilde{e}^m\times\tilde{e}^j)\partial_m\ln\sqrt{q}+\tfrac{1}{2}\underset{\sim}{N}(\tilde{e}^l\cdot D_lC_R)(\tilde{e}^j\cdot\tilde{e}^k)$$
$$(\text{mod}\,\chi^j=0, C_B=0, \phi^{jk}=0)\ , \tag{A39}$$

provided that P_0 is given by Eq. (A30). Hence $\dot{\phi}^{jk}=0$ yields no new constraint.

Thus we can construct a complex canonical theory which is equivalent to the ADM canonical theory in the classical level. However, when we try to quantize it, we meet a difficulty. In the complex canonical theory the Poisson brackets among $\mathcal{E}^j$ and $\mathcal{A}_j^{\pm}$ are defined as $\{\mathcal{E}^j(\mathbf{x}),\mathcal{E}^k(\mathbf{y})\}=\{\mathcal{A}_j^{\pm}(\mathbf{x}),\ \mathcal{A}_k^{\pm}(\mathbf{y})\}=0$, $\{\mathcal{E}^{jI}(\mathbf{x}),\ \mathcal{A}_{kJ}^{\pm}(\mathbf{y})\}\ =\delta_k^j\delta_J^I\delta(\mathbf{x},\mathbf{y})$, while the Poisson brackets between them and their complex-conjugate variables are not defined. However, we need the latter Poisson brackets since the constraints $\chi^j\approx0$ and $\phi^{jk}\approx0$ are written as combinations of $\mathcal{E}^j$, $\mathcal{A}_j^{\pm}$, and their complex conjugates. A natural choice for the latter is the one for which the latter Poisson brackets vanish. This choice, however, yields $\{\chi^{jI}(\mathbf{x}),Q_{kJ}(\mathbf{y})\}=\tfrac{1}{2}\delta_k^j\delta_J^I\delta(\mathbf{x},\mathbf{y})$ for which $\{\chi^j,\phi^{kl}\}$ does not vanish. Thus it produces second-class constraints and we cannot quantize it by preserving the polynomiality. Actually the theory obtained by this method is equivalent to the polynomial canonical theory developed in Refs. 20 and 21 where the same difficulty is pointed out. Of course we cannot deny the possibility of some choice of the Poisson brackets for which $\chi^j=0$ and $\phi^{jk}=0$ remain first class, but they would be quite complicated even if they exist.

Another possibility to recover the canonical structure is to assume that the imaginary part of $\mathcal{A}_j^{\pm}$ is implicitly given by Eq. (A31), which is the original form of Ashtekar's theory.[11] Since $\mathcal{C}_G$ is written as $\mathcal{C}_G\equiv C_B\mp iC_R$, it follows from the above argument that $\mathcal{A}_j^{\pm}$ with its imaginary part Q_j replaced by Eq. (A31) also satisfies the time-evolution equation obtained from the complex canonical chiral Lagrangian, modulo C_R which vanishes under the constraint $\mathcal{C}_G=0$. Thus Ashtekar's theory is equivalent to the triad ADM theory in the classical level. However, we meet difficulties again when we try to quantize it. First the reality condition on $\tilde{e}^j$ requires that the real part of $\mathcal{A}_0^{\pm}$ is given by Eq. (A30). Since P_0 does not decouple in Ashtekar's theory unlike the ADM case, it introduces nonpolynomiality into the theory unless $N\equiv1$. Second from Eq. (A35) $\tilde{e}^j$ cannot be an Hermitian operator unless $N^j\equiv0$. Finally even if we impose the Hermiticity of $\tilde{e}^j$ as a constraint on the physical state, the time-evolution equation for $\mathcal{A}_j^{\pm}$ deviates from the canonical equation by terms proportional to C_R as is seen from Eq. (A39). This deviation breaks the canonical structure of the theory. Thus it seems quite difficult to construct a quantum theory from Ashtekar's theory or its variants preserving the polynomiality of the theory.

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