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# Particle production during out-of-equilibrium phase transitions

Jennie H. Traschen
*Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01003*

Robert H. Brandenberger
*Department of Physics, Brown University, Providence, Rhode Island 02912*
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Techniques are developed to calculate the energy production in quantum fields which obtain a mass through the spontaneous symmetry breaking of a second field which is undergoing a phase transition. All fields are assumed to be out of thermal equilibrium and weakly coupled. The energy produced in a field, which is initially in its ground state, is computed for two generic types of time-dependent masses: a roughly monotonic turn on of the mass and an oscillatory mass. The formalism is applied to the questions of particle production and reheating in inflationary universe models. Requirements are found which the couplings in new-inflation-type models must satisfy for efficient reheating to occur.

## I. INTRODUCTION

Let $\phi$ and $\psi$ be two coupled scalar fields. We consider the case when $\phi$ undergoes a spontaneous symmetry-breaking phase transition during which the expectation value of $\phi$ changes continuously from, say, $\langle\phi\rangle=0$ to $\langle\phi\rangle=\sigma$. If the interaction is of the form $\mathcal{L}_I=\frac{1}{2}g\phi^2\psi^2$, then during the phase transition $\psi$ acquires a mass by its coupling to $\phi$. While the phase transition is happening, the new mass $\mu^2(t)=g\langle\phi^2\rangle$ acts like a time-dependent source for the field $\psi$, which causes energy production in $\psi$. It is this production of energy or particles which we wish to compute. In general, $\langle\phi^2\rangle$ depends on space and time; here we treat the dynamics within one spatially homogeneous region.

Processes considered here will typically occur out of thermal equilibrium. The field $\psi$ starts out in its ground state rather than in a thermal ensemble and is excited by the time dependence of $\phi$ during the phase transition. One example is particle production due the time dependence of the inflaton field in inflationary universe models.[1] A second example is meson production in quark jets,[2] a transition from a false-vacuum to a true-vacuum state. Our methods should be useful beyond the context of these examples. They can be applied whenever discussing particle production during a second-order phase transition or during the dynamical evolution of background fields which couple to some quantum system. For example, in Kaluza-Klein cosmology the techniques developed can be used to study particle production during dynamical compactification. There are many potential applications in condensed-matter physics.

Reheating in inflationary universe models has been considered before using a phenomenological equation of motion[3] for $\phi$, the field which drives inflation:

$$\ddot{\phi}+3H\dot{\phi}+\Gamma\dot{\phi}-a^{-2}\nabla^2\phi=-V'(\phi)\ , \tag{1.1}$$

where $H$ is the Hubble constant of a Friedmann-Robertson-Walker metric with a scale factor $a(t)$, and $V'(\phi)$ is the derivative of the potential of $\phi$ with respect to $\phi$. $\Gamma$ is a decay constant which is introduced to mimic the interactions of $\phi$ with other fields which cause $\phi$ to lose energy[4,5] (and hence *reheat* the other fields). The equation is phenomenological since it does not come from a theory of $\phi$ and its interactions with other fields.

Our approach is quite different. We study the equation of motion for $\psi$, and assume that $\phi$ is in a coherent state described by a space-independent expectation value $\langle\phi\rangle(t)$. We compute particle production in $\psi$ using a self-consistent semiclassical theory which gives the evolution of physical operators in the Hilbert space of quantum states of $\psi$ (e.g., the energy density of $\psi$) and also determines the back reaction on $\phi$. $\phi$ is treated as a classical background field with magnitude given by the expectation value $\langle\phi\rangle(t)$. We take $\psi$ to be initially in its ground state.

Thus, in the semiclassical approximation we replace $\phi^2$ by $\langle\phi^2\rangle(t)$ in the interaction. Two broad types of behavior are considered. The first is when $\langle\phi^2\rangle(t)$ changes roughly monotonically from its initial value of zero to a nonzero value of order $\sigma^2$, and the transition may be either fast or slow. The second type of behavior is when $\langle\phi^2\rangle(t)$ oscillates and resonance occurs. For some symmetry-breaking potentials and initial conditions, $\langle\phi^2\rangle$ first changes monotonically while "rolling" along the flat part of the potential, and then oscillates as the field settles into its new minimum. Most energy is produced in the second period when resonances occur, but aside from the inherent interest in particle production during a slow-rolling period, it is useful to study the turn-on period to set the initial conditions for $\psi$ during the phase when $\langle\phi^2\rangle(t)$ oscillates (the "sloshing" phase).

We will develop the formalism both in the Schrödinger and Heisenberg representations. We will discuss approximation schemes which can be applied if $\phi(t)$ changes approximately monotonically or if $\phi(t)$ oscillates rapidly.

 

Some of the methods we use in the Heisenberg representation have been previously discussed in Refs. 6 and 7.

A concrete realization of the class of problems we address in this paper is reheating in inflationary universe models.[1] We consider a theory with two fields $\phi$ and $\psi$. $\phi$ is a scalar field with a nontrivial potential $V(\phi)$ and with initial conditions which are homogeneous and lead to inflation. At the end of the inflationary period, $\phi(t)$ relaxes to a configuration with vanishing energy density. Since any matter energy density which was present before inflation is exponentially diluted during the inflationary phase, the entire present material content of the Universe must be produced at the end of inflation, during the time when $\phi(t)$ relaxes to its equilibrium value. $\phi(t)$ is either directly or indirectly (i.e., via gravity) coupled to the matter fields, which we here represent by a single scalar field $\psi$. Now, quite generally, the state of a quantum field $\psi$ will change due to its interaction with the classical source $\phi(t)$. Starting with $\psi$ in the ground state, we wish to compute the energy density in the final state.

It is well known[8] that in order to avoid generating energy-density perturbations with too large an amplitude, $\phi$ must be a very weakly self-coupled scalar field. Since any coupling between $\phi$ and $\psi$ will give rise at higher-loop order to contributions to the self coupling of $\phi$, it is natural to assume that the coupling between $\phi$ and $\psi$ is weak. Hence, the evolution of $\phi(t)$ and $\psi$ particle production occur out of thermal equilibrium. If the fields were in thermal equilibrium, then they would be described by thermal density matrices, and effective potential techniques could be used. Here, we must take $\psi$ to be initially in its ground state and $\phi$ as a classical background field which evolves according to its zero-temperature potential.

In Sec. II, we outline how to compute the quantities of interest in the Schrödinger picture, and in the Heisenberg picture in Sec. III. In Sec. IV, we compute the energy production during the "turn on" of the new mass. In Sec. V, the energy produced during an oscillating phase is found. Finally, Sec. VI applies the results to the example of new inflation.[9]

## II. FUNCTIONAL SCHRÖDINGER APPROACH

As described in Sec. I, the starting point is a Lagrangian $\mathcal{L}(\phi,\psi)$ for a system of two coupled scalar fields $\phi$ and $\psi$. We assume that initially the state of the system is empty of $\psi$ particles, but there is an energy density in the $\phi$ field; i.e., the $\phi$ field is not in its ground state.

We are interested in the case when the interaction between $\phi$ and $\psi$ "turns on" due to spontaneous symmetry breaking. This occurs as follows. Take $\phi$ to be in a "nearly classical" state, e.g., a coherent state, and use a semiclassical approximation, so that the interaction $\mathcal{L}_i=g\phi^2\psi^2$ is approximated by $g\langle\phi^2\rangle\psi^2$. As $\langle\phi^2\rangle$ evolves, it acts as a time-dependent source for $\psi$. Before some time $t_1$ when the phase transition starts, $\langle\phi^2\rangle=0$, and so there is no interaction between the fields. We take $\psi$ to be initially in its ground state. After $t_1$, $\langle\phi^2\rangle$ becomes nonzero, the interaction between $\phi$ and $\psi$ no longer vanishes, and $\psi$ particles can be produced. We assume that symmetry breaking occurs as an external field or variable, e.g., the temperature changes. For example, in inflationary universe models, $\phi$ is the scalar field which drives inflation. $\langle\phi\rangle(t)$ and $a(t)$, the scale factor of the Universe, must solve the semiclassical Einstein equations.

The action for $\psi$ is

$$S=\int d^4x\sqrt{-g}\,\left(-\tfrac{1}{2}g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi-\tfrac{1}{2}m_0^2\psi^2-\tfrac{1}{2}g\langle\phi^2\rangle\psi^2\right)$$
$$\equiv S_0-\int d^4x\sqrt{-g}\,\tfrac{1}{2}g\langle\phi^2\rangle\psi^2\ . \tag{2.1}$$

We shall restrict our attention to spatially flat Friedmann-Robertson-Walker cosmologies for which the metric $g_{\mu\nu}$ is given by

$$ds^2=-dt^2+a^2\delta_{ij}dx^i dx^j\ . \tag{2.2}$$

We first review the functional Schrödinger approach[10] to study particle production. Readers familiar with this formalism can skip this section. We write down the Schrödinger equation for the wave functional $\Psi[\psi(x,t)]$. This determines the evolution of the expectation value of operators, which we compute for linear and quadratic combinations of the field and momentum operators, and hence for the energy.

The canonical momentum $\Pi$ is defined by

$$\Pi=\sqrt{\gamma}\frac{\delta\mathcal{L}}{\delta\dot\psi}=a^3\frac{\delta\mathcal{L}}{\delta\dot\psi}\ , \tag{2.3}$$

where $\gamma_{ij}$ is the metric for the spatial sections. (That is, one must first choose a slicing of the spacetime, on which one requires the equal-time commutation relations. Let $n^\mu$ be the unit timelike vector field orthogonal to the spatial slices, so $g_{\mu\nu}=-n_\mu n_\nu+\gamma_{\mu\nu}$ with $n^\mu\gamma_{\mu\nu}=0$. Then $\dot\psi$ is the Lie derivative of $\psi$ along the vector field $n^\mu$.)

Thus, the Hamiltonian $H$ can be written as

$$H=\int d^3x(\Pi\dot\psi-\mathcal{L})\ . \tag{2.4}$$

The canonical commutation relations are

$$[\Pi(\mathbf{x}),\psi(\mathbf{y})]=\frac{\hbar}{i}\delta^{(3)}(\mathbf{x}-\mathbf{y})\ . \tag{2.5}$$

Hence, in the Schrödinger representation the canonical momentum is

$$\Pi(\mathbf{x})=\frac{\hbar}{i}\frac{\delta}{\delta\psi(\mathbf{x})} \tag{2.6}$$

and the functional Schrödinger equation becomes, for the metric (2.2),

$$-\frac{\hbar}{i}\frac{d}{dt}\Psi[\psi,t]=H\Psi[\psi,t]$$
$$=\int d^3x\,a^3\left\{-\frac{\hbar^2}{2a^6}\frac{\delta^2}{\delta\psi^2}-\tfrac{1}{2}a^{-2}(\nabla\psi)^2+\tfrac{1}{2}[m_0^2+g\langle\phi^2\rangle(t)]\psi^2\right\}\Psi[\psi,t]\ . \tag{2.7}$$

Since the Lagrangian is quadratic, the Hamiltonian is diagonalized by going to Fourier space. The different Fourier modes decouple and hence the wave functional $\Psi[\psi,t]$ can be written as a product of ordinary harmonic-oscillator wave functions for each mode. Our conventions for Fourier transformation are

$$\psi(\mathbf{x})=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\psi_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2.8}$$

with $\psi_{-\mathbf{k}}=\psi_{\mathbf{k}}^*$ and

$$\Pi(\mathbf{x})=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\Pi_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{2.9}$$

with

$$\Pi_{\mathbf{k}}=(2\pi)^3\frac{\hbar}{i}\frac{\delta}{\delta\psi_{\mathbf{k}}} . \tag{2.10}$$

The variational equation of motion which follows from the action (2.1) can be decoupled into a set of first-order equations for the linear expectation values:

$$\begin{aligned}\frac{d}{dt}\langle\psi_k\rangle&=a^{-3}\langle\Pi_k^*\rangle\\ \frac{d}{dt}\langle\Pi_k^*\rangle&=-a^{+3}\left[\frac{k^2}{a^2}+M^2\right]\langle\psi_{\mathbf{k}}\rangle ,\end{aligned} \tag{2.11}$$

where $M^2=m_0^2+g\langle\phi^2\rangle(t)$. Now let $\rho_{\mathbf{k},a}=(\psi_{\mathbf{k}},\psi_{\mathbf{k}}^*,\Pi_{\mathbf{k}},\Pi_{\mathbf{k}}^*)$. Then (2.11) can be rewritten as

$$\frac{d}{dt}\langle\rho_{\mathbf{k},a}\rangle=\Omega_{\mathbf{k},ab}\langle\rho_{\mathbf{k},b}\rangle . \tag{2.12}$$

Since the $\Omega_{ab}$ are ordinary $c$-number functions of time, it follows that the expectation values of the quadratics evolve according to

$$\frac{d}{dt}\langle\rho_{\mathbf{k},a}\rho_{\mathbf{k},b}\rangle=\Omega_{\mathbf{k},ac}\langle\rho_{\mathbf{k},c}\rho_{\mathbf{k},b}\rangle+\Omega_{\mathbf{k},bc}\langle\rho_{\mathbf{k},a}\rho_{\mathbf{k},c}\rangle . \tag{2.13}$$

[This is seen most easily in the Heisenberg representation by taking the time derivative inside the brackets, using the product rule and applying (2.12) to the individual terms.] Further,

$$\begin{aligned}\frac{d}{dt}(\langle\rho_{\mathbf{k},a}\rangle\langle\rho_{\mathbf{k},b}\rangle)=\Omega_{\mathbf{k},ac}\langle\rho_{k,c}\rangle\langle\rho_{k,b}\rangle\\ +\Omega_{\mathbf{k},bc}\langle\rho_{k,a}\rangle\langle\rho_{k,c}\rangle .\end{aligned} \tag{2.14}$$

Thus, the quadratic $\langle\rho_{\mathbf{k},a}\rho_{\mathbf{k},b}\rangle$ satisfies the same equation of motion as the product of two linear terms. Hence, knowing the general solutions for $\langle\rho_{\mathbf{k},a}\rangle(t)$ will also give us the solution for the quadratics, provided certain algebraic constraints on the arbitrary integration constants are satisfied. So we first solve for the linear expectation values. Combining the two equations of (2.11) we obtain

$$\frac{d^2}{dt^2}\langle\psi_{\mathbf{k}}\rangle+3\frac{\dot a}{a}\frac{d}{dt}\langle\psi_k\rangle+\left[\frac{k^2}{a^2}+M^2(t)\right]\langle\psi_{\mathbf{k}}\rangle=0 . \tag{2.15}$$

Of course this is just the classical equation of motion. In subsequent sections we will discuss approximation schemes for solving (2.15) for two prototypical background field evolutions. In the first case (Sec. IV) the mass $M(t)$ turns on roughly monotonically. In the second case $M(t)$ oscillates about a fixed value (Sec. V). In our application to inflationary universe models, we must combine both scenarios (Sec. VI).

Now, (2.13) and (2.15) give the expectation value of linear and quadratic terms in the fields. However, in many cases one can solve for the full wave functional. For example, given the monotonic type of behavior just mentioned, suppose that there is no interaction for $t<t_0$ and the $\psi$ field is in its ground state. The solution to the Schrödinger equation then is just a product over $k$ of ground-state harmonic-oscillator wave functions, each with width $a^3w_k/\hbar$ and frequency $w_k$, where $w_k^2\equiv k^2/a^2+m_0^2$. Then, as the interaction turns on, it is reasonable to look for solutions which are just products of Gaussians with time-dependent widths and frequencies:[10]

$$\Psi=N\exp\left[-\tfrac{1}{2}\sum_k[A_k(t)\psi_k\psi_k^*+i\Omega_k(t)/\hbar]\right] . \tag{2.16}$$

(We are now working in a finite box in comoving coordinates.) The Schrödinger equation then implies that

$$\frac{i}{\hbar}\dot A_k=\frac{A_k^2}{a^3}-\frac{a^3}{\hbar^3}\left[\frac{k^2}{a^2}+m_0^2+g\langle\phi^2\rangle\right] , \tag{2.17}$$

and

$$\dot\Omega_k=\frac{1}{2}\frac{\hbar}{a^3}A_k . \tag{2.18}$$

The normalization constant is

$$N=\prod_k\left[\frac{\mathrm{Re}\,A_k}{\pi}\right]^{1/4}e^{-\mathrm{Im}\Omega_k/2\hbar} , \tag{2.19}$$

and one can check that this is indeed independent of time.

With the change of variable

$$A_k=\frac{-i}{\hbar}a^3\frac{\dot y_k}{y_k} , \tag{2.20}$$

$y_k$ satisfies

$$\ddot y_k+3\frac{\dot a}{a}\dot y_k+\left[\frac{k^2}{a^2}+m_0^2+g\langle\phi^2\rangle\right]y_k=0 . \tag{2.21}$$

Hence for a Gaussian wave functional, the quantity $y_k$, which is simply related to the width, satisfies the classical equation of motion, as does the expectation value of the field (2.15).

Equation (2.21) must be solved with appropriate boundary conditions. Without expansion, one can check that to have the harmonic-oscillator ground state $A_k=(1/\hbar)w_k$ before the interaction, one needs $y_k=e^{iw_kt}$ for $t<t_0$.

Having solved for the wave functional one has a complete quantum-mechanical description of the system. In particular, in the state (2.16), $\langle\psi_k\rangle=\langle\Pi_k\rangle=0$, and (see also Ref. 10)

$$\langle \psi_k \psi_q \rangle = \delta_{k,-q} \frac{1}{2\,\mathrm{Re}\,A_k} ,$$
$$\langle \Pi_k \Pi_q \rangle = \delta_{k,-q} \frac{\hbar^2}{2\,\mathrm{Re}\,A_k} |A_k|^2 a^6 , \tag{2.22}$$
$$\langle \psi_k \Pi_q + \Pi_q^* \psi_k^* \rangle = -\delta_{k,q} \hbar \frac{\mathrm{Im}\,A_k}{\mathrm{Re}\,A_k} .$$

[Here we are considering the system in a box. To go to the continuum limit, $\delta_{k,-q} \to \delta^{(3)}(k+q)(2\pi)^3$, $\sum_k \to V \int d^3k/(2\pi)^3$, and $\Pi_k \to (2\pi)^3 \Pi_k$.]

So, for some types of behavior of the time-dependent mass, we can find the entire wave functional, and then compute, for example, the energy $\langle H_k \rangle$ in each $k$ mode through (2.22). In other cases, as when resonance occurs, it may not be a good approximation to take the wave function to be a Gaussian, but we can still find the energy in each mode through (2.13)–(2.15). In either case we need the solutions to the classical equation of motion (with boundary conditions appropriate to each case).

The total energy is given by the resulting sum over all $k$ modes of $\langle H_k \rangle$, and is ultraviolet divergent. The prescription we choose for renormalization is to subtract the instantaneous ground-state energy of each mode. The sum is then (typically) finite, and we compute the renormalized expectation values $\langle \psi^2(x) \rangle^{\mathrm{ren}}$, $\langle \Pi^2(x) \rangle^{\mathrm{ren}}$, and $\langle H \rangle^{\mathrm{ren}}$. First we recall the calculation of the same quantity in the Heisenberg picture.

## III. HEISENBERG MODE MIXING APPROACH

We now develop the Heisenberg mode mixing approach. Readers familiar with this approach can skip the section.

We start with the action, metric, and canonical commutation relations given in (2.1)–(2.5). The equation of motion for the classical field $\psi$ is then

$$\Box\psi + [m_0^2 + \mu^2(t)]\psi = 0 , \tag{3.1}$$

where

$$\mu^2(t) = g\langle \phi^2 \rangle (t) . \tag{3.2}$$

We assume that $\mu^2(t)$ changes from zero for $t < t_0$ to a new constant value $\mu_f^2$ for $t > t_1$. For ease in discussion we first only consider the time dependence of the mass term and later only the effects of the expansion of the Universe. Hence, there is no ambiguity of the particle states for $t < t_0$ and $t > t_1$.

Define the inner product between two functions by

$$(f,g) = -i \int_\Sigma d\Sigma^\mu (f \nabla_\mu g^* - g^* \nabla_\mu f) , \tag{3.3}$$

where the integral is over a spatial surface $\Sigma$. If $f$ and $g$ are solutions to the Klein-Gordon equation (3.1), then the inner product is conserved, i.e., independent of $\Sigma$. We assume that there exists a complete set of modes $|u_i\rangle$ to (3.1) which are orthonormal, $(u_i,u_j) = -(u_i^*,u_j^*) = \delta_{ij}$, and $(u_i,u_j^*)=0$. For the spatially flat Robertson-Walker metric we choose $u_i = f_i(t)e^{i\mathbf{k}\cdot\mathbf{x}}$. Without expansion the "in" modes are defined to be solutions which are positive frequency before the interaction:

$$f_k^{\mathrm{in}}(t) \to (2w_k)^{-1/2} e^{-iw_k t} \quad \text{as } t \to -\infty \tag{3.4}$$

(here for $t < t_0$). (In an expanding universe of course this can only be an approximation to the solution over some time interval near $t_0$.) This set of modes then defines the in vacuum $|0\rangle$ and the Fock space of states built on it. We expand the quantum field $\psi$ in the in modes:

$$\psi(\mathbf{x},t) = \sum_k (a_k u_k^{\mathrm{in}} + a_k^\dagger u_k^{*\mathrm{in}}) , \tag{3.5}$$

where $a_k|0\rangle = 0 \forall k$. So the $a_k^\dagger$ and $a_k$ create and annihilate in particles with positive frequency $w_k = (k^2/a^2 + m_0^2)^{1/2}$. Similarly, out modes are defined to be solutions to (3.1) which are positive frequency after interaction:

$$f_k^{\mathrm{out}}(t) \to \frac{1}{\sqrt{2\bar{w}_k}} e^{-i\bar{w}_k t} \quad \text{as } t \to +\infty \tag{3.6}$$

(here for $t > t_1$), where $\bar{w}_k = (k^2/a^2 + m_0^2 + m_f^2)^{1/2}$. The out vacuum $|\bar{0}\rangle$ and out particle states are defined by expanding the field in the out modes:

$$\psi(\mathbf{x},t) = \sum_k (\bar{a}_k u_k^{\mathrm{out}} + \bar{a}_k^\dagger u_k^{*\mathrm{out}}) , \tag{3.7}$$

where $\bar{a}_k|\bar{0}\rangle = 0 \forall k$.

Since the late time observer defines particles with respect to the frequencies $\bar{w}_k$, the late time number operator for the $k$th mode is $\bar{N}_k = \bar{a}_k^\dagger \bar{a}_k$. Having specified the boundary conditions on $u_k^{\mathrm{in}}$ in the far past, these solutions will be a mixture of positive and negative modes in the future, that is, back scattering occurs. It is the back-scattered piece of the incident wave which corresponds to the produced particles. Let

$$u_k^{\mathrm{in}} = \alpha_k u_k^{\mathrm{out}} + \beta_k u_{-k}^{*\mathrm{out}}, \quad \text{so}$$
$$\bar{a}_k = \alpha_k a_k + \beta_{-k}^* a_k^\dagger . \tag{3.8}$$

Then

$$\beta_k = -(u_k^{\mathrm{out}}, u_k^{\mathrm{in}}) = i(f_k^{\mathrm{out}} \dot{f}_k^{\mathrm{in}} - f_k^{\mathrm{in}} \dot{f}_k^{\mathrm{out}}) . \tag{3.9}$$

The coefficients $\alpha_k$ and $\beta_k$ are called Bogoliubov coefficients. Take the state of the system to be the in vacuum $|0\rangle$, so for $t < t_0$ there are no particles present. As the interaction is turned on, particles are produced, in the sense that the in vacuum contains late time particles, $\langle 0|\bar{N}_k|0\rangle = |\beta_k|^2$. We now turn to the computation of particle production and of the energy produced which is $\bar{w}_k|\beta_k|^2$. Note that defining the total energy produced by

$$\langle H \rangle^{\mathrm{ren}} = \sum_k \bar{w}_k |\beta_k|^2 \tag{3.10}$$

includes a renormalization prescription of subtracting the instantaneous ground-state energy, as previously mentioned in Sec. II.

We conclude this section with a comment on the inadequacy of an adiabatic expansion to (approximately) solve for particle production. In this approach[11] one looks for

WKB solutions of the mode equation by a systematic expansion in $1/L$ (hence the name adiabatic), the time rate of change of $\mu^2$. Then, one finds, to order $1/L^2$,

$$f_k^{\rm in}(t) \simeq [2W_{(0)}(t)]^{-1/2} \exp\left[-i\int^t dt'(W_{(0)}+W_{(2)})\right], \tag{3.11}$$

where

$$W_{(0)} = [\omega_k^2 + \mu^2(t)]^{1/2} \tag{3.12}$$

and

$$W_{(2)} = \frac{1}{8\omega_k^3}\left[\frac{5}{4}\frac{(\dot{\mu}^2)^2}{\omega_k^2} - (\ddot{\mu}^2)\right]. \tag{3.13}$$

This gives

$$\beta_k = (f_k^{*\rm out}, f_k^{\rm in}) = \frac{W_{(2)}}{2\omega_k}, \tag{3.14}$$

and therefore $\beta_k = 0$ when $\dot{\mu} = 0$, i.e., in the adiabatic limit. Indeed, this is discussed in an early paper by Schrödinger,[12] before the second-quantized formalism was discovered. Schrödinger observed that there is a mixing of positive and negative frequency field modes, and hence particle production, in an expanding Universe. However, Schrödinger noted that particle number is an adiabatic invariant, and hence that there is no particle production for adiabatic expansion of the Universe.

## IV. ENERGY PRODUCED DURING THE ROLLING PHASE OF SPONTANEOUS SYMMETRY BREAKING

In this section we compute particle and energy production during the "rolling phase" of the spontaneous symmetry-breaking transition. We will first do the calculation in the Heisenberg picture. We then indicate how the calculation is done in the Schrödinger picture, and determine the field quadratics. The results will be used as initial data to evolve the expectation value of the quadratics through the resonance period (see Sec. V).

Let $\psi = f_k(t)e^{i\mathbf{k}\cdot\mathbf{x}}$. The Klein-Gordon equation (3.1) becomes

$$\ddot{f}_k + 3\frac{\dot{a}}{a}\dot{f}_k + [w_k^2 + \mu^2(t)]f_k = 0. \tag{4.1}$$

The above equation contains two interaction terms: the time-dependent mass and the Hubble damping term due to the coupling to the expanding spacetime (most of what is done is quite general, and can easily be specialized to a system without expansion). In inflationary models, one is interested in the case where spacetime changes from de Sitter, with $a(t) = a_1 e^{H_1(t-t_1)}$, to a power-law expansion beginning at some time $t_1$. During the last $e$-folding of the inflationary period the mass changes from zero to its new value of $\mu_f$. This is followed by an oscillatory phase while the scalar field settles into its new vacuum state. In this section we focus on the slow-roll phase. Particles are produced during this phase, both due to the time dependence of the effective mass squared $\mu^2(t)$ and the expanding background.

The solution of the mode equation (4.1) will depend on whether the frequency is high or low compared to the expansion rate $H$. The solution for high frequencies is discussed in Sec. IV A. We solve for the Bogoliubov coefficients using the Born approximation. By the methods of Sec. III [in particular Eq. (3.10)] we can immediately calculate the contribution $\rho^{\rm short}$ to the total energy density of $\psi$ particles contributed by these short-wavelength modes. There are two cases, depending on whether the new mass $\mu_f$ is large or small compared to $H$.

In Sec. IV B we solve the mode equation for low frequencies. Now, the expansion of the Universe is important. However, for sufficiently long wavelengths, we can approximate the transition between de Sitter expansion and the power-law expansion as instantaneous and replace the time dependence of $\mu^2(t)$ by a step function. Then, it is possible to solve the equation exactly in terms of Bessel functions on either side of the step. Thus, we can again calculate the Bogoliubov coefficients (unfortunately the equations are rather lengthy) and deduce the energy density $\rho^{\rm long}$ in long-wavelength excitations of $\psi$.

When applying our analysis to reheating in inflationary universe models in Sec. VI, it turns out that most of the energy is produced not in the slow-rolling period, but in the oscillatory phase. Therefore, it is important to determine the expectation values of field quadratics which are used as initial conditions in Sec. V. These expectation values are calculated in the Schrödinger representation. This is done in Sec. IV C.

After this summary of the long and rather technical Sec. IV, we can turn to the actual calculations.

### A. Short wavelengths

When the mode frequency is high compared to the rate of expansion, the expansion can be ignored in (4.1), and we can focus on the effect of the time-dependent mass. Hence for $w_k \gg H_1$, we approximate $w_k$ by its value at $t_1$, and consider

$$\ddot{f}_k + [w_k^2 + \mu^2(t)]f_k = 0. \tag{4.2}$$

We can compute the back scattering using perturbation theory, which is valid if $w_k^2 \gg \mu_f^2$. Let $L$ be the time scale over which $\mu^2$ changes. For example, in the inflationary universe models considered in Sec. VI, $L = rH_1^{-1}$ with $r \le 1$. In what follows $L$ may be small or large; $L$ goes to zero in the sudden approximation, and $L$ goes to infinity in the adiabatic limit. We will use the Born approximation, which requires $L^2\mu_f^2 \ll 1$. Hence the results of this section are valid for frequencies $w_k > \max(H_1, \mu_f)$ and parameters $\mu_f L \ll 1$.

Rewrite (4.2) as

$$\ddot{f}_k + [w_k^2 + \mu_f^2\theta(t-t_0)]f_k = -[\mu^2(t) - \mu_f^2\theta(t-t_0)]f_k \equiv -V(t)f_k, \tag{4.3}$$

so that $V(t) = 0$ for $t < t_0$ and $t > t_1$. We wish to solve (4.3) for the in modes, i.e., with the boundary conditions that $f_k$ is positive frequency before the interaction.

Equation (4.3) is equivalent to the integral equation

$$f_k = f_{kh} - \int dt' G(t,t') V(t') f_k(t') , \qquad (4.4)$$

where $f_{kh}$ is a solution to the homogeneous equation (with $V=0$). The Green's function with the correct boundary conditions for in modes is

$$G(t,t') = \frac{\sin w_k(t-t')}{w_k}\theta(t_0 - t)\theta(t-t') + \frac{\sin \bar{w}_k(t-t')}{\bar{w}_k}\theta(t-t_0)\theta(t-t') . \qquad (4.5)$$

Hence, substituting (4.5) into (4.4), we obtain the exact result

$$f_k(t) = \frac{1}{\sqrt{2w_k}} e^{-iw_k t}, \quad t < t_0 ,$$

$$f_k(t) = \frac{e^{-i\bar{w}_k t}}{\sqrt{2\bar{w}_k}} \left[ \frac{1}{2}\frac{\bar{w}_k + w_k}{\sqrt{w_k \bar{w}_k}} + \gamma_k(t) \right] + \frac{e^{i\bar{w}_k t}}{\sqrt{2\bar{w}_k}} \left[ \frac{1}{2}\frac{\bar{w}_k - w_k}{\sqrt{w_k \bar{w}_k}} + \sigma_k(t) \right], \quad t > t_0 , \qquad (4.6)$$

where

$$\gamma_k(t) = \frac{1}{2i\bar{w}_k} \int_0^t dt' V(t') e^{i\bar{w}_k t'} f_k(t') \sqrt{2\bar{w}_k} ,$$

$$\sigma_k(t) = \frac{-1}{2i\bar{w}_k} \int_0^t dt' V e^{-i\bar{w}_k t'} f_k(t') \sqrt{2\bar{w}_k} . \qquad (4.7)$$

From (4.6) we can read off the Bogoliubov coefficients

$$\beta_k = -\left[ \frac{1}{2}\frac{\bar{w}_k - w_k}{\sqrt{w_k \bar{w}_k}} + \sigma_k(t) \right] . \qquad (4.8)$$

The Born approximation consists of approximating $f_k(t')$ in the integrand in (4.7) by the solution to the homogeneous equation. In a standard perturbative quantum-field-theory approach, this corresponds to including terms in an amplitude which contain only one propagator. (For an example of this approach in QCD, see Ref. 2.) One can then iterate this expansion. The second-order solution is found by substituting this new solution for $f_k$ in the integrand. It is straightforward to check that one needs to include the second-order terms to get a solution which conserves probability, i.e., such that $|\alpha_k|^2 - |\beta_k|^2 = 1$. However, the "reflection" coefficient $\beta_k$ does not receive a correction at second order, only the transmission coefficient does. Therefore, to find $\beta_k$, it suffices to work to first order.

In computing $\beta_k$, the Fourier transform of the step function in the potential $V$ cancels the constant term in $\beta_k$, and one finds

$$\langle 0|\bar{a}_k^\dagger \bar{a}_k|0\rangle = |\beta_k|^2 = \frac{1}{4\bar{w}_k w_k} \left| \int_0^t dt' \mu^2(t') e^{-i(\bar{w}_k + w_k)t'} \right|^2 \qquad (4.9)$$

for the number of out particles with momentum $k$ present at time $t$. The energy produced in mode $k$ is

$$\langle H_k \rangle^{\text{ren}} = \bar{w}_k |\beta_k|^2 . \qquad (4.10)$$

This renormalization prescription corresponds to subtracting the energy of the instantaneous ground state at $t$ (see discussion in Sec. IV C). The total energy is the sum over all $k$, and is ultraviolet finite if $\mu^2$ is continuous. In the limit where $\mu^2(t)$ is a step function (the sudden approximation), the total number of particles produced is finite, and the total energy diverges logarithmically.

As an example, which will also be a check on our approximate solutions, take $\mu^2(t) = \frac{1}{2}\mu_f^2(1 + \tanh t/L)$. Using (4.9), we obtain

$$|\beta_k|^2 = \frac{1}{16} \frac{(\pi L \mu_f^2)^2}{w_k^2} \frac{1}{\sinh^2 \pi L w_k} . \qquad (4.11)$$

It turns out that for this choice of $\mu^2(t)$ the mode equation is exactly soluble in terms of hypergeometric functions (Ref. 11, Chap. 3). In the limit of the Born approximation ($\mu_f L \ll 1$, $w_k^2 \gg \mu_f^2$) the exact solution gives a result identical to (4.11). Our general formula (4.9) can also be checked in another limiting case, when $\mu^2(t)$ is a step function (the sudden approximation). Then (4.9) gives

$$|\beta_k|^2 = \frac{\mu_f^4}{4\bar{w}_k w_k (\bar{w}_k + w_k)^2} .$$

However, for $\mu^2(t) = \mu_f^2 \theta(t)$, the modes equation can again be solved exactly, and one finds the preceding expression for the number of particles produced. In the sudden approximation, the total number of particles produced is finite, but the total energy is logarithmically divergent. The utility of (4.9) is that it gives the energy production for a general time-dependent mass $\mu(t)$ in the perturbative regime.

Now we can find the total energy produced in short wavelengths for different choices of the time dependence of the mass. We have already mentioned the extreme cases when $\mu^2(t)$ is a step function and when $\mu^2(t)$ is a smooth function proportional to $\tanh t/L$. An intermediate case occurs when $\mu^2(t)$ is a linear function connecting $\mu^2(0) = 0$ and $\mu^2(L) = \mu_f^2$. In this case,

$$|\beta_k|^2 = \frac{1}{32} \frac{\mu_f^4}{L^2} (1 - \cos 2w_k L) w_k^{-6} . \qquad (4.12)$$

In the last two examples, the total energy produced is dominated by the contribution from small wave numbers. Recall that the analysis leading to (4.9) assumes that $w_k \gg \mu_f$ and $w_k \gg H_1$. Therefore the minimum frequency for which (4.11) and (4.12) are applicable is $\mu_f$ if $H_1 < \mu_f < L^{-1}$ and $H_1$ if $\mu_f < H_1 < L^{-1}$.

Therefore, it follows from (4.11) and by integrating (4.10) that the total energy produced in $\psi$ particles in the short wavelengths is

$$\rho^{\text{short}} \simeq \frac{\mu_f^4}{32\pi^2} \begin{cases} |\ln \pi \mu_f L|, & H_1 < \mu_f \ll L^{-1}, \\ |\ln \pi H_1 L|, & \mu_f < H_1 \ll L^{-1}. \end{cases} \qquad (4.13)$$

In the step function example for $\mu^2(t)$, $\rho^{\text{short}}$ diverges.

This ultraviolet divergence is a consequence of the discontinuity of $\mu^2(t)$. In physical models, no such discontinuities will arise and (4.13) will give a good order-of-magnitude estimate of the actual $\rho^{\text{short}}$, as one can verify by finding $\rho^{\text{short}}$ for (4.12).

It is useful to compare the density of particles produced to the energy density in Hawking radiation[13] of $(\pi/2)H_1^4$, which is present in the de Sitter period $t<t_1$. Except in the limit where the arguments of the logarithms actually go to zero, in either of the cases in (4.13) we see that

$$\frac{\rho^{\text{short}}}{\rho^{\text{Hawk}}}\sim\frac{\mu_f^4}{H_1^4} .$$

Therefore, when the new mass is large compared to the Hubble constant, $\mu_f \gg H_1$, particle production due to the time-dependent mass is large compared to particle production due to the de Sitter horizon (and vice versa if $\mu_f \ll H_1$).

### B. Long wavelengths

When the wavelength of the mode is long compared to the Hubble distance, the expansion of the Universe cannot be neglected in the modes equation (4.1). In this case it is useful to rewrite (4.1) in conformal time. Let $dt=a\,d\eta$ and define the rescaled mode function $\tilde{f}_k=af_{\mathbf{k}}$. We will take the initial mass of $\psi$ to be zero. Then

$$\tilde{f}_k''(\eta)+\left[\mathbf{k}^2-\frac{a''}{a}+a^2\mu^2\right]\tilde{f}_k(\eta)=0 , \tag{4.14}$$

where a prime denotes a derivative with respect to conformal time. Again, let $L$ be the time scale for changes in $\mu^2(t)=g\langle\phi^2\rangle$. In the limit where the wavelength $a/k$ is much larger than $L$, we can approximate the scattering potential in (4.14) as a step function. That is, let $\mu^2(t)=\mu_f^2\theta(t-t_1)$, and approximate the scale factor to be an exponential for $t<t_1$ and a power law for $t>t_1$. If $a(t)\propto t^\alpha$, then, in conformal time [with $\eta(t_1)=\eta_1$],

$$a(\eta)=\begin{cases}-\dfrac{1}{H_1\eta}, & \eta<\eta_1<0,\\[2ex] K\left[\eta-\eta_1\dfrac{1}{1-\alpha}\right]^{\alpha/(1-\alpha)}, & \eta_1<\eta,\end{cases} \tag{4.15}$$

where the constant $K$ is

$$K=\frac{1}{H_1}\left[\frac{-1}{\eta_1}\right]^{1/(1-\alpha)}\left[\frac{\alpha}{1-\alpha}\right]^{\alpha/(1-\alpha)} . \tag{4.16}$$

As we shall discuss in the next section, if the stress energy is generated by a scalar field undergoing coherent harmonic oscillations (and hence acting like dust), then one has a self-consistent solution to the Einstein equation with the scale factor proportional to $t^{2/3}$. However, since the results of this section are (qualitatively) independent of the power in a power-law expansion $a\propto t^\alpha$, we will keep $\alpha$ general for now.

The solutions in both periods are spherical Bessel functions. The in modes have the boundary conditions $\tilde{f}_k\sim e^{-ik\eta}$ as $\eta\to-\infty$. Hence,

$$\begin{aligned}\tilde{f}_k^{\text{in}}&=(-i)^{\nu+1}\left[\frac{k}{2}\right]^{1/2}\eta h_1^{(1)}(k\eta), \quad \eta\le\eta_1 ,\\ \tilde{f}_k^{\text{in}}&=(i)^{\nu+1}\left[\frac{\tilde{\omega}_k}{2}\right]^{1/2}\Delta\eta[(-1)^{\nu+1}\alpha_k h_\nu^{(2)}(\tilde{\omega}_k\Delta\eta)\\ &\quad+\beta_k h_\nu^{(1)}(\tilde{\omega}_k\Delta\eta)], \quad \eta\ge\eta_1\end{aligned} \tag{4.17}$$

where

$$\begin{aligned}\Delta\eta&=\eta-\eta_1/(1-\alpha) ,\\ \tilde{\omega}_k^2&=\mathbf{k}^2+ga_1^2\langle\phi_1^2\rangle=\mathbf{k}^2+a_1^2\mu_f^2 ,\end{aligned} \tag{4.18}$$

and

$$\nu=\frac{2\alpha-1}{1-\alpha} . \tag{4.19}$$

$\tilde{\omega}_k$ is the "conformal frequency." Matching the function and its first derivative at $\eta_1$ one finds $\beta_k$.

There are two cases within the long-wavelength regime. First suppose that the new mass is bigger than the Hubble constant. (We will see in the next section that in this case, parametric resonance can occur during the oscillatory phase.) Then

$$|\beta_k|^2\simeq\frac{1}{4}\left[\frac{\alpha}{1-\alpha}\right]^2\frac{\mu_f^2}{H_1^2}\left[\frac{a_1H_1}{k}\right]^4|h_\nu^{(1)}|^2 \quad \text{for } \frac{k}{a_1}\ll L^{-1} \text{ and } \mu_f\gg H_1 , \tag{4.20}$$

where the Hankel function is evaluated at $(\alpha/1-\alpha)(\mu_f/H_1)$. For dust ($\nu=1$) this becomes

$$|\beta_k|^2\simeq\mu_f^2H_1^2\left[\frac{a_1}{k}\right]^4, \quad H_1\ll\mu_f . \tag{4.21}$$

If the new mass is small compared to the Hubble constant, then

$$|\beta_k|^2\simeq2^{2\nu-2}\frac{\Gamma^2(\nu+\frac{1}{2})}{\pi}\left[\frac{1-\alpha}{\alpha}\right]^{2\nu}\times\begin{cases}\left[\dfrac{H_1^2}{\mu_f^2}\right]^\nu\left[\dfrac{a_1H_1}{k}\right]^4, & \dfrac{k}{a_1}\ll\mu_f\ll H_1,\\[2ex] \left[\dfrac{3\alpha-1}{\alpha}\right]^2\left[\dfrac{a_1H_1}{k}\right]^{2\nu+4}, & \mu_f\ll\dfrac{k}{a_1}\ll L^{-1}.\end{cases} \tag{4.22}$$

For dust ($\nu=1$) this becomes

$$|\beta_k|^2\simeq\begin{cases}\dfrac{1}{16}\dfrac{H_1^2}{\mu_f^2}\left[\dfrac{a_1H_1}{k}\right]^4, & \dfrac{k}{a_1}\ll\mu_f\ll H_1,\\[2ex] \dfrac{9}{32}\left[\dfrac{a_1H_1}{k}\right]^6, & \mu_f\ll\dfrac{k}{a_1}\ll L^{-1}.\end{cases} \tag{4.23}$$

So we see that quite generally, the number of particles produced by the transition from an exponential to a power-law expansion goes like $k^{-4}$ in the long-wavelength limit. However, one should be cautious about interpreting this in terms of classical energy-density perturbations. Here we have computed the expectation value of the number of particles with momentum $k/a$ in a quantum state which is homogeneous and isotropic. It is unclear what this has to do with spatial inhomogeneities.[14]

To measure a mode with wavelength $a/k$ we must observe it for a time interval $\Delta t > a/k$. Hence to find the total energy produced in long wavelengths, we cut off the integral at $k_{\min}/a = 1/\Delta t$. Evaluating the resulting integrals at $\Delta t \simeq H_1^{-1}$, we find

$$\rho^{\text{long}} \simeq \frac{2}{\pi^2}\left(\tfrac{2}{3}\right)^{8/3} H_1 \mu_f^3, \quad H_1 \ll \mu_f ,$$
$$\rho^{\text{long}} \simeq \frac{9}{16\pi^2}\frac{H_1^6}{\mu_f^2}, \quad H_1 \gg \mu_f . \tag{4.24}$$

Therefore for the long wavelengths, the energy produced is large compared to the energy density in Hawking radiation $H_1^4$.

The total energy produced is given by summing (4.24) and (4.13). For example, compare the energy density in produced particles to the background energy density $\rho(t_1) = (3\pi/8) m_{\text{Pl}}^2 H_1^2$, for the case $H_1 \ll \mu_f$:

$$\frac{\rho^{\text{part}}}{\rho^{\text{background}}} \simeq \frac{\mu_f^4}{\pi^2 m_{\text{Pl}}^2 H_1^2}\left\{ |\ln \pi \mu_f L| + \frac{H_1}{\mu_f} \right\} . \tag{4.25}$$

### C. Calculating in the Schrödinger picture: Expectation values of field quadratics

In Secs. V and VI we need the values of the expectation values of the field quadratics as initial conditions in order to calculate the rate of particle production during the oscillatory period. These expectation values are most easily computed in the Schrödinger picture to which we now turn.

As in (2.16), let the state be described by a Gaussian wave functional with complex width $A_k(t)$ for each mode, and the initial conditions that $A_k = (1/\hbar) w_k$ before the interaction starts. With the change of variables [see (2.20)]

$$A_k = -\frac{i}{\hbar} a^3 \frac{\dot{y}_k}{y_k} , \tag{4.26}$$

we obtain the modes equation (2.21) for $y_k$. This equation is the same as (4.1), and hence we can use the solutions (4.6). Indeed, if $y_k$ is the complex conjugate of $f_k$ in (4.6), then $A_k$ satisfies the required initial conditions. Hence $A_k$ is known, and the expectation values of the different field quadratics are given explicitly in (2.22). Let us focus on the short-wavelength modes. Then the expectation values can be expanded to leading order in $\mu_f^2/w_k^2$. Now, quantities such as $\langle \psi_k \psi_k^* \rangle$ and $\langle \Pi_k \Pi_k^* \rangle$ have oscillatory time dependence (as for free field theory in Minkowski spacetime), though the combination

$$\langle H_k \rangle = \frac{a^3}{2}\left\langle \frac{1}{a^6}\Pi_k \Pi_k^* + \varpi_k^2 \psi_k \psi_k^* \right\rangle \tag{4.27}$$

only depends on time through the explicit dependence in $a(t)$. So, in computing $\langle \psi_k \psi_k^* \rangle$ we average over a period of oscillation. (Of course, averaging over a period $2\pi/w_k = T_k$ makes sense only if the system is allowed to evolve for a time $\Delta t$ at least as long as $T_k$. For the particular case of the slow-roll phase in inflation, $\Delta t \sim H_1^{-1}$. If $\mu_f > H_1$ this averaging makes sense for all of the high frequencies, $w_k > \mu_f$. If $\mu_f < H_1$, the averaging is more dubious as the frequency of interest approaches $H_1$.) Setting $A_k = (1/\hbar) w_k + \Delta A_k$, one must expand to second order in $\Delta A_k$ to include terms which do not vanish upon time averaging.

The resulting expressions are not yet renormalized; the integral over $k$ diverges like $k^4$ at high momenta. We renormalize the field quadratics by subtracting the instantaneous ground-state contributions. We have already used this prescription to calculate the energy in the Heisenberg picture, when we defined the energy in a particular mode at late times to be the out frequency times the number of out particles produced, $\langle H_k \rangle^{\text{ren}} = \varpi_k |\beta_k|^2$. (One might consider, instead, renormalizing by subtracting the expectation value of the operator before the interaction, and hence computing the change in the quantity. This, however, is divergent; the Hamiltonian has a new term which is proportional to $\mu^2/w_k$ for each $k$.) One finds, for the renormalized expressions,

$$\text{(a)}\ \langle \psi_k \psi_k^* \rangle^{\text{ren}} = \langle \psi_k \psi_k^* \rangle - \frac{1}{2\varpi_k} = \frac{1}{4\varpi_k^3}|\hat{\mu}^2(2\varpi_k)|^2 ,$$
$$\text{(b)}\ \langle \psi_k \Pi_k + \Pi_k^* \psi_k^* \rangle^{\text{ren}} = 0 , \tag{4.28}$$
$$\text{(c)}\ \langle \Pi_k \Pi_k^* \rangle^{\text{ren}} = \langle \Pi_k \Pi_k^* \rangle - \tfrac{1}{2}\varpi_k = \frac{1}{4\varpi_k}|\hat{\mu}^2|^2 .$$

We remind the reader that "ren" means averaged over a period and renormalized, and $\hat{\mu}^2$ denotes the Fourier transform. As a check, $\langle H_k \rangle^{\text{ren}}$ from (a) and (c) is the same as the expression found from the Heisenberg picture calculation, (4.9) and (4.10).

Now, (4.28) is just what is needed as initial data to compute the evolution of the expectation values of the field quadratics during the oscillatory phase of the time-dependent mass. Evaluating (4.28) at the end of the slow-roll period gives the renormalized expectation values, which are used to start the evolution in the resonance period.

## V. PARAMETRIC RESONANCE

In this section we consider situations in which the background field $\phi(t)$ oscillates with a frequency $w$ large compared to the Hubble expansion rate $H$. In this case there are resonance phenomena which greatly increase the rate of particle production over what we discussed in the preceding section.

The outline of this section is as follows: We first discuss the conditions under which the expansion of the Universe can be neglected [(5.2)–(5.5)]. Then, we solve the modes equation and establish the parametric resonance effects which lead to the increased rate of particle production [(5.6)–(5.15)]. Next, we summarize the conditions which must be satisfied in order to be able to apply the previous calculation. When applying our methods to concrete models, it is important to verify that these conditions [(5.16)–(5.20)] are satisfied. We then use the solutions of the modes equation to determine the energy density in $\psi$ particles.

The starting point is Eq. (3.1) with an oscillatory mass term

$$M^2(t)=M_0^2\cos wt+m_0^2 \ . \tag{5.1}$$

In the absence of the expansion of the Universe, (3.1) with the above mass term is the Mathieu equation which is well known[15] to admit instabilities—parametric resonance. It is intuitively clear that these resonance phenomena should persist even in an expanding Universe, provided that the time scale for resonance is much shorter than the expansion time scale, and provided that the amplitude of the forcing term is sufficiently large. To make these conditions precise, we change fields to

$$f_k=a\psi_k \tag{5.2}$$

and introduce conformal time $\eta$ by

$$d\eta=a^{-1}dt \ . \tag{5.3}$$

Denoting the derivative with respect to $\eta$ by a prime, Eq. (3.1) for $\psi_k$ becomes

$$f_k''+\left[k^{2+}M^2(t)a^2(t)-\frac{a''}{a}\right]f_k=0 \ . \tag{5.4}$$

Assuming that $w>H$, we can treat the expansion of the Universe adiabatically. Thus, at any given time, its effect is a shift in the oscillator frequency

$$k^2\to k^2-\frac{a''}{a} \tag{5.5}$$

and an adiabatic increase in the amplitude of the driving force. As will be discussed below, there is resonance only in narrow frequency bands with width proportional to $M_0^2$. A condition for the realization of parametric resonance in an expanding Universe is that the adiabatically changing frequency remains inside the resonance band for a time period long compared to the period of increase of $f_k$.

Neglecting the expansion of the Universe, (5.4) becomes the well-known Mathieu equation

$$f_k''+(w_k^2+M_0^2\cos wt)f_k=0 \ , \tag{5.6}$$

with $w_k^2=k^2+m_0^2$. We shall consider the last term in the above equation as a time-dependent oscillatory perturbation. To make this clear, we replace $M_0^2$ by $\epsilon M_0^2$, where $\epsilon$ is treated as the expansion parameter.

It is well known that outside of narrow resonance bands centered at the frequencies

$$w_k^{(n)}=\frac{n}{2}w \tag{5.7}$$

the solutions of the unperturbed equation ($\epsilon=0$) are strongly stable[15] and, therefore, there is no particle production. However, inside the resonance bands there are instabilities in the sense that the flow of the dynamical system given by (5.6) amplifies the initial separation between two points in phase space without bound.

For small $\epsilon$ we can analyze the instability perturbatively. It can be shown[15] that for $w_k^{(n)}$ the instability arises only at the $n$th order in perturbation theory. Hence we shall focus attention on the first instability band.

The width of the first instability band depends linearly on the amplitude of the perturbation. If $w_k=w/2+\delta_k$, then $w_k$ is in the instability band if

$$\delta_k\in\left[-\frac{M_0^2}{2w},\frac{M_0^2}{2w}\right] \ . \tag{5.8}$$

We see this as follows: For frequencies in this band, we can write down the following ansatz for a perturbative solution:

$$f_k(t)=a(t)\cos\frac{w}{2}t+b(t)\sin\frac{w}{2}t \ . \tag{5.9}$$

Inserting this into the Mathieu equation (5.6), we obtain the following set of equations for the coefficient functions $a(t)$ and $b(t)$ by equating the coefficients of the $\cos(w/2)t$ and $\sin(w/2)t$ terms, and dropping the higher harmonics:

$$\begin{aligned}\ddot a+w\dot b-\left[\frac{w}{2}\right]^2a&=-w_k^2a-\frac{M_0^2}{2}a \ ,\\ \ddot b-w\dot a-\left[\frac{w}{2}\right]^2b&=-w_k^2b-\frac{M_0^2}{2}b \ ,\end{aligned} \tag{5.10}$$

with $w_k$ as above. It is a self-consistent ansatz to assume that $\dot a\sim\epsilon$ and $\dot b\sim\epsilon$. Hence $\ddot a$ and $\ddot b$ are of higher order in $\epsilon$ and can be neglected. (5.10) reduces to a system of linear first-order differential equations which can be solved by the ansatz

$$a(t)\sim e^{st} \text{ and } b(t)\sim e^{st} \ . \tag{5.11}$$

The solutions for $s$ are $\pm s_k$ where

$$s_k=\frac{M_0^2}{2w}\left[1-\frac{2w\delta}{M_0^2}\right]^{1/2} \ . \tag{5.12}$$

We see that precisely for frequencies in the resonance band (5.8) there is an exponential instability. The exponent is proportional to $\epsilon$ which shows that the above analysis is self-consistent. The growing and decaying solutions can be written down explicitly:

$$\begin{aligned}f_k^+&=\frac{e^{s_kt}}{\sqrt{w\nu_k}}\left[\nu_k\cos\frac{w}{2}t-\sin\frac{w}{2}t\right] \ ,\\ f_k^-&=\frac{e^{-s_kt}}{\sqrt{w\nu_k}}\left[\nu_k\cos\frac{w}{2}t+\sin\frac{w}{2}t\right] \ ,\end{aligned} \tag{5.13}$$

where

$$\nu_k=1+\frac{2\delta w_k}{M_0^2} \tag{5.14}$$

and the normalization has been chosen such that

$\dot{f}_k^+ f_k^- - \dot{f}_k^- f_k^+ = -1$. We shall come back to these solutions when discussing the evolution of quadratics.

To summarize the above discussion, we conclude that for frequencies in the lowest resonance band the amplitude of $f_k$ increases exponentially:

$$|f_k(t)| \sim e^{s_k t}|f_k(0)| \ , \quad (5.15)$$

with $s_k$ given by (5.12).

The parametric resonance analysis is only applicable if several conditions are satisfied. The first condition is that perturbation theory be valid, $M_0^2 \ll w_k^2$. At resonance, $w_k \simeq \frac{1}{2}w$, so this condition becomes

$$M_0 \ll \tfrac{1}{2}w \ . \quad (5.16)$$

Note that if we recall the original interaction process, in which the $\phi$ field produces $\psi$ particles, one needs that the energy in the $\phi$ state is greater than the mass of the $\psi$, for the process to occur. For a coherent state of $\phi$ particles at rest, this means $m_\phi = w > m_\psi \sim M_0$, so condition (5.16) makes sense.

The second condition is that the expansion of the Universe can be neglected, which requires

$$w \gg H_1 \ . \quad (5.17)$$

Indeed, this is a necessary condition for resonance to occur. Third, we need the time scale $\Delta t$ on which the growing mode dominates to be smaller than $H_1^{-1}$, so that expansion is unimportant. This requires

$$\frac{H_1 w}{M_0^2} \ll 1 \ . \quad (5.18)$$

There is also the condition that the frequency does not redshift out of the resonance band in a time interval shorter than the amplification period $s^{-1}(w/2)$. For nonvanishing $m_0$ this condition becomes

$$s^{-1}\left[\frac{w}{2}\right]\frac{d}{d\eta}w_k < \frac{M_0^2}{2m} \quad (5.19)$$

or, using $w_k^2 = k^2 + m_0^2 a^2 - a''/a$,

$$2m_0^2 a' a - \left[\frac{a''}{a}\right]' < \frac{M_0^4}{2w} \ . \quad (5.20)$$

Finally, there is a condition if one uses the Born approximation. Now, resonance occurs at $w_k \simeq \frac{1}{2}w$. If $wL \ll 1$, then this means that resonance is at a "long wavelength," in our previous categorization. If $wL \gg 1$, resonance occurs at a "short wavelength."

We will shortly show how to analyze energy production by parametric resonance in the semiclassical framework treating $\psi$ as a quantum field. First, however, we can quickly derive an expression for the energy considering $\psi$ as a classical field.

If $t=0$ is the time when the driving force in (5.6) starts, then the energy at time $t>0$ is

$$E(t) = \int d^3\mathbf{k}\tfrac{1}{2}[w_k^2|f_k(t)|^2 + |\dot{f}_k(t)|^2] \simeq \int d^3k\, w_k^2|f_k|^2 \ . \quad (5.21)$$

To first order in $\epsilon$, only the first resonance band contributes. Hence, for $m_0=0$,

$$E(t) \simeq 4\pi\left[\frac{w}{2}\right]^4 \frac{M_0^2}{w}|f_k(0)|^2 e^{(M_0^2/w)t} \ , \quad (5.22)$$

where $f_k(0)$ is evaluated in the center of the resonance band. If all the modes start out with an amplitude $|f_k(0)|$ given by equipartition of energy for a harmonic oscillator, then

$$|f_k(0)|^2 = \frac{V}{w_k} = \frac{2V}{w} \ , \quad (5.23)$$

where $V$ is the volume of space, and

$$\rho(t) \simeq \pi/2w^2 M_0^2 e^{(M_0^2/w)t} \ . \quad (5.24)$$

Also within the context of our consistent semiclassical analysis, parametric resonance leads to an increase in the energy of the $\psi$ field. For each $k$ mode, the energy is the sum of the quadratic terms $\langle H_k\rangle = \frac{1}{2}\langle|\pi_k|^2\rangle + \frac{1}{2}\varpi_k^2\langle|\psi_k|^2\rangle$. But we now know how this evolves in time, having found the solutions $f_k^+, f_k^-$. From (2.13) and (2.14), it follows that the general solution for the quadratics is

$$\begin{aligned}\chi_1 &\equiv \langle\psi\psi^*\rangle = b_i b_j'^* f^i f^j \ ,\\ \chi_2 &\equiv \langle\Pi^*\psi^* + \psi\Pi\rangle = (b_i b_j'^* + b_j b_i'^*) f_i \dot{f}_j \ ,\\ \chi_3 &\equiv \langle\Pi^*\Pi\rangle = b_i b_j'^* \dot{f}^i \dot{f}^j \ ,\end{aligned} \quad (5.25)$$

for a set of coefficients $b_i$ and $b_j'$ (where we have dropped the index $k$). $i$ runs over $+$ and $-$.

We can rewrite (5.25) in the more compact form:

$$\chi_n(t) = F_{nm}(t) B_m \ , \quad (5.26)$$

where $n$ and $m$ run from 1 to 3, the $B_n$ are defined by $B_1 = b_+ b_+'^*$, $B_2 = b_+ b_-'^* + b_- b_+'^*$, and $B_3 = b_- b_-^{*\prime}$, and the $F_{nm}(t)$ are given in terms of $f_+(t)$ and $f_-(t)$ by (5.25). Note that $b_+ b_-^{*\prime} - b_- b_+^{*\prime} = i\hbar$.

Thus, we know the evolution of the quadratics in terms of the solutions of the Mathieu equation. At late times $t \gg t_1$ we obtain

$$\begin{aligned}\chi_1(t) &\simeq f_+^2(t) B_1 \ ,\\ \chi_2(t) &\simeq 2f_+(t)\dot{f}_+(t) B_1 \ ,\\ \chi_3(t) &\simeq \dot{f}_+^2(t) B_1 \ .\end{aligned} \quad (5.27)$$

The key point is that the coefficients $B_i$ are determined by the values of $\chi_i$ at the start of the parametric resonance period, $\chi_i(t_1)$. To find $B_i$ one must invert (5.27). The result is

$$\begin{aligned}B_1 &= w_k^2\langle|\psi_k|^2\rangle(t_1)\left[\frac{F_{11}^{-1}(t_1)}{w_k^2} + F_{13}^{-1}(t_1)\right]\\ &= w_k^2\langle|\psi_k|^2\rangle(t_1)\frac{1}{w\nu_k}\left[\left[\frac{w}{2w_k}\right]^2 + \nu_k^2\right] \ ,\end{aligned} \quad (5.28)$$

where $\nu_k$ is given by (5.14). At the center of the resonance band,

$$B_1 = \frac{2}{w}w_k^2\langle|\psi_k|^2\rangle(t_1) \ . \quad (5.29)$$

Using the equations derived in Sec. IV, we can write down $B_1$ for long and short wavelengths.

At late times the contribution to the energy is dominated by the growing mode

$$\langle H_k \rangle \simeq \tfrac{1}{2} B_1 (w_k^2 f_+^2 + \dot{f}_+^2) \simeq B_1 w_k^2 f_1^2 \ , \qquad (5.30)$$

to leading order in the perturbation expansion parameter $M_0^2/w$. Averaging over a period of oscillation $2\pi/w$, evaluating the result at the center of the resonance band, and using (5.29), one finds

$$\langle H_k \rangle^{\rm ren} \simeq \left[ \frac{w}{2} \right]^2 \langle |\psi_k|^2 \rangle^{\rm ren}(t_1) e^{(M_0^2/w)(t-t_1)} \ , \qquad (5.31)$$

for $|w_k - w/2| < M_0^2/2w$. This is our desired formula. (5.31) gives the energy at time $t$ due to the parametric resonance amplification in terms of the initial values $\langle |\psi_k|^2 \rangle^{\rm ren}$ at $t_1$. If one uses renormalized initial data, then the total energy at any later time is also renormalized. To get the total energy produced, one sums (5.31) over all $k$ values inside the resonance band, and integrates over the time during which the process continues. In an expanding universe, a particular comoving wave vector will redshift out of the resonance band, and new ones will redshift into the band. The process continues until the oscillations of the $\phi$ field have been damped by back reaction.

In the case when the oscillatory phase is preceded by a monotonic turn on of the mass term, the initial data are determined by what has been produced during the turn on, i.e.,

$$\langle |\psi_k|^2 \rangle^{\rm ren}(t_1) = \frac{1}{a^3 \overline{w}_k} |\beta_k|^2 \ . \qquad (5.32)$$

Hence,

$$\langle H_k \rangle^{\rm ren} \simeq \frac{w}{2} \frac{1}{a^3} |\beta_k|^2 e^{(M_0^2/w)(t-t_1)} \ , \qquad (5.33)$$

where the appropriate value for $|\beta_k|^2$ must be used depending on whether resonance occurs at a "short" or "long" wavelength. We will work this out explicitly in the next section in the context of inflation.

There is another choice for $\langle \psi_k^2(t_1) \rangle$ which could be used in computing the energy in (5.31). Indeed, in the classical calculation we used

$$\langle |\psi_k^2|(t_1) \rangle = \frac{V}{2w_k} \ . \qquad (5.34)$$

In the quantum calculation, this piece was the ground-state energy which was subtracted off in the normalization prescription. However, ground-state fluctuations can be important. For example, Hawking radiation[16,13] and fluctuations in inflationary models[8] can be viewed as being driven by vacuum fluctuations.

To first order in the coupling, the expansion of the Universe is important in that a frequency which starts in the resonance band will get redshifted out. Also, the amplitude $M^2(t)$ decreases as a power of the scale factor, since the background energy density does. In the next section we will give an example of the application of parametric resonance to inflation, and include these features.

To next order in the coupling, one could compute the back reaction on the Higgs field and the cosmological scale factor. In the semiclassical Einstein equation $G_{ab} = 8\pi G \langle T_{ab} \rangle$ one would include the contribution to $\langle T_{ab} \rangle$ from the interaction $g\phi_{(0)}^2 \psi^2$, where $\phi_{(0)}$ denotes the solution with $g=0$. To lowest order, $\langle \psi^2 \rangle$ is $O(g^2)$, and its expectation value has been found here. So this next correction is indeed higher order in $g$. This will modify the evolution of $\phi$ and $a(t)$.

One perhaps expects that the energy in the Higgs field decreases as it is pumped into new particles, and indeed this is what one means by saying the Higgs field settles into its new minimum. However, the interactions we have considered so far could just as well be transfer energy from $\psi$ back to $\phi$ (as with masses connected by linear springs). So to correctly compute the decay of the amplitude of oscillations of $\phi$, one must consider self-interactions of the $\psi$, and redshift of momentum, which make the rates of the reverse reactions small.

In the following section we use the parametric resonance approach to compute energy production after inflation.

## VI. REHEATING IN INFLATIONARY UNIVERSE MODELS

As an example of particle production in out-of-equilibrium phase transitions we shall consider reheating in inflationary universe models, the problem outlined in the Introduction. In most inflationary universe models there are two phases. During the first phase, the scalar field $\phi(t)$ which generates the inflationary equation of state evolves slowly and monotonically towards its ground-state value. In the second phase it oscillates rapidly about the equilibrium value. We shall see how during the first phase a small background of energy in all modes of the $\psi$ field is generated according to the mechanism discussed in Sec. IV. In the second phase, the initial amplitude of the $\psi$ field is amplified by parametric resonance for modes in the resonance bands.

In this section, we first review some of the essential equations of our inflationary model [(6.1)–(6.12)]. Then, we identify the parameters which enter the parametric resonance analysis [(6.13)–(6.16)] and discuss the conditions under which the approximations developed in earlier sections of this paper are applicable. We then calculate the energy production during reheating (6.23). Finally, we discuss the conditions for efficient reheating.

We will discuss a toy model for new inflation.[9] (Note that reheating in this model has recently also been considered in Ref. 17, albeit with methods quite different from the ones used here.) Chaotic inflation[18] will be considered in a later publication. The potential $V(\phi)$ for the real scalar field $\phi$ is

$$V(\phi) = \tfrac{1}{4}\lambda(\phi^2 - \sigma^2)^2 \ . \qquad (6.1)$$

Note, however, that this model in general does not lead to a period of inflation long enough ($\gg H^{-1}$) to solve the flatness and horizon problems. To obtain a model which gives enough inflation, $V(\phi)$ must be much flatter near $\phi=0$. One way to achieve this is by using a Coleman-Weinberg-type potential. Since the basic physics of particle production is model independent, we shall use (6.1) for simplicity.

Our previous general analysis was presented in terms of the time scale $L$ for the "turn on" of the new mass, with value $\mu_f$, and the sloshing frequency $\omega$. So first we must identify these parameters in the inflationary model.

For $\phi \ll \sigma$ the quartic term in $V(\phi)$ is negligible and the equation of motion for $\phi(t)$ reads

$$\ddot{\phi}+3H\dot{\phi}=\lambda\sigma^2\phi \ . \tag{6.2}$$

According to standard (but unjustified) dogma[9] of new inflation we assume that $\phi(\mathbf{x},t)$ is homogeneous and starts out displaced from the origin only by quantum fluctuations, i.e., $\phi(0)=O(H)$. (6.2) has a growing mode solution

$$\phi(t)=\phi(0)e^{\alpha t} \quad \text{with } \alpha=-\frac{3H}{2}+\tfrac{1}{2}(9H^2+4\lambda\sigma^2)^{1/2} \ . \tag{6.3}$$

The Hubble constant $H$ is

$$H=\left[\frac{2\pi}{3}\right]^{1/2}\lambda^{1/2}\sigma^2 m_{\rm Pl}^{-1} \ , \tag{6.4}$$

where $m_{\rm Pl}=G^{-1/2}$ is the Planck mass. For $4\lambda\sigma^2>9H^2$ or equivalently $\sigma<(2/3\pi)^{1/2}m_{\rm Pl}$, we get

$$\alpha\simeq\lambda^{1/2}\sigma \ . \tag{6.5}$$

Note that this exponent differs from what would be obtained making the naive "slow-rolling" approximation, namely neglecting $\ddot{\phi}$ in (6.2) and solving the resulting first-order differential equation. For

$$4\lambda\sigma^2<9H^2, \quad \text{i.e.,} \quad \left[\sigma>\left[\frac{2}{3\pi}\right]^{1/2}m_{\rm Pl}\right] ,$$

we get

$$\alpha\simeq\tfrac{1}{3}\lambda\frac{\sigma^2}{H}=\left[\frac{1}{6\pi}\right]^{1/2}\lambda^{1/2}m_{\rm Pl} \ , \tag{6.6}$$

in agreement with what would be obtained neglecting $\ddot{\phi}$.

The period $\tau$ of inflation can be estimated by

$$e^{\alpha\tau}\phi(0)=\sigma \ . \tag{6.7}$$

For $\sigma \ll m_{\rm Pl}$ and $\phi(0)=H$ we get

$$\tau H=\left[\frac{2\pi}{3}\right]^{1/2}\frac{\sigma}{m_{\rm Pl}}\ln\left[\left[\frac{2\pi}{3}\right]^{-1/2}\lambda^{-1/2}\frac{m_{\rm Pl}}{\sigma}\right] , \tag{6.8}$$

which in general is much smaller than 1. Most of the increase in $\phi(t)$ takes place over a much shorter time period $L$. To be specific, we take $L$ to be the period over which $\phi(t)$ increases from $10^{-1}\sigma$ to $\sigma$. Then

$$L\sim\frac{1}{\alpha} \quad \text{and } LH\sim\frac{\sigma}{m_{\rm Pl}}\ll 1 \ . \tag{6.9}$$

After the slow-roll period, the Higgs oscillates about its new minimum. An oscillating homogeneous scalar field acts like pressureless dust. Explicitly, we want to solve the Friedmann equations with source $\phi(t)$:

$$\left[\frac{\dot{a}}{a}\right]^2=\tfrac{8}{3}\pi G\cdot\tfrac{1}{2}[\dot{\phi}^2+m_\varphi^2(\phi-\sigma)^2] \ ,$$
$$\ddot{\phi}+3\frac{\dot{a}}{a}\dot{\phi}+m_\varphi^2(\phi-\sigma)=0 \ . \tag{6.10}$$

Then for $H_1 \ll m_\varphi$, the solution is

$$\phi=\sigma+A_1\left[\frac{a_1}{a}\right]^{3/2}\cos m_\varphi(t-t_1) \ ,$$
$$a(t)=a_1(\tfrac{3}{2})^{2/3}(H_1t-H_1t_1+\tfrac{2}{3})^{2/3} \ , \tag{6.11}$$

and

$$\rho=\tfrac{1}{2}m^2A_1^2\left[\frac{a_1}{a(t)}\right]^3+O\left[\frac{H_1^2}{m_\varphi^2}\right] , \tag{6.12}$$

where the oscillatory phase starts at $t_1$ and $H_1$ is the Hubble constant at time $t_1$.

There are the background fields during the oscillatory phase to be used in the equation of motion for $\psi$.

Note that in the model of new inflation, the frequency of oscillation is

$$m_\varphi=\sqrt{2}\lambda^{1/2}\sigma \ . \tag{6.13}$$

We shall now consider in detail particle production in the toy model for new inflation. First we must consider the conditions under which the approximation schemes discussed in Secs. IV and V are applicable. Now, parametric resonance only occurs if the expansion of the Universe is unimportant at the resonance frequency $m_\phi$. By (6.4) and (6.13) we see that for $\sigma \ll m_{\rm Pl}$ this is satisfied, i.e., $m_\phi \gg H$. To be able to use the Born approximation, we need to satisfy

$$L\mu_f \ll 1 \tag{6.14}$$

[see the discussion following (4.2)]. If we assume that the $\psi$ field is massless in the absence of interactions (i.e., $m_0=0$), then

$$\mu_f^2=g\phi^2(t_f)\simeq g\sigma^2 \ . \tag{6.15}$$

Since $L=\ln(10\lambda^{-1/2}\sigma^{-1})$ it follows that (6.14) is satisfied if

$$\lambda \gg g \ . \tag{6.16}$$

This condition is reasonable, but not necessary. Naturalness would only require $g \ll \lambda^{1/2}$.

If (6.16) is satisfied, we can calculate the energy in the high-frequency modes at the beginning of the oscillatory phase using the Born approximation. Equations (4.11) and (4.13) give the particle and energy production for short wavelengths. In these general expressions, we now set $L\simeq\lambda^{-1/2}\sigma^{-1}$ and insert $\mu_f^2$ and $H$ from (6.15) and (6.4). Then, for $\omega_k^2 \gg g\sigma^2$,

$$|\beta_k|^2=\frac{\pi^2}{16}\frac{g^2\sigma^2}{\lambda\omega_k^2}\left[\sinh\left[\frac{\pi\omega_k}{\sqrt{\lambda}\sigma}\right]\right]^{-2} . \tag{6.17}$$

For definiteness, we consider only the case $H_1 \ll \mu_f$, i.e.,

$$\frac{\sigma}{m_{\rm Pl}} \ll \left[\frac{g}{\lambda}\right]^{1/2} . \tag{6.18}$$

Then, in the long-wavelength regime, $|\beta_k|^2$ is given by (4.21), and the total energy $\rho$ produced during the turn-on period is [from (4.25)]

$$\frac{\rho}{\rho^{\rm back}}=\frac{g^2}{8\pi^3\lambda}\left[\frac{1}{2}\ln\frac{\lambda}{\pi^2 g}+\left[\frac{\lambda}{g}\right]^{1/2}\frac{\sigma}{m_{\rm Pl}}\right] . \tag{6.19}$$

Note that the energy density produced during turn on, when calculated in perturbation theory, is very small compared to the background density. We will next see that the energy density produced by parametric resonance can be large even when calculated perturbatively.

From Sec. V we know the expectation value of the Hamiltonian density in the first resonance band. Integrating (5.31) over the resonance band, a shell of approximate volume $2\pi\delta_k\omega^2 a^3(t)$, we obtain

$$\rho(t)_R \simeq \pi^2\mu_f^2\omega^2 e^{2\delta_k(t-t_1)}|\beta_k|^2\frac{a^3(t)}{a_1^3} , \tag{6.20}$$

where the subscript stands for "resonance." For our toy model of inflation, $\overline{\omega}_k=\frac{1}{2}m_\varphi$ and

$$\delta_k=\frac{\mu_f^2}{2\omega}=\frac{g}{2\sqrt{\lambda}}\sigma . \tag{6.21}$$

Since we are restricting ourselves to the case $\lambda > g$, resonance occurs at short wavelengths. Hence, we can use the Born approximation to evaluate $|\beta_k|^2$. Substituting (6.3) into (4.9), and evaluating at resonance, we find

$$|\beta_k|^2=\frac{1}{5}\left[\frac{g}{\lambda}\right]^2\left\{1+O\left[\lambda^2\left[\frac{\sigma}{m_{\rm Pl}}\right]^2\right]\right\} . \tag{6.22}$$

Hence, from (6.20),

$$\rho(t)_R \simeq \pi\frac{g^3}{\lambda}\sigma^4 e^{g\sigma\lambda^{-1/2}(t-t_1)}\left[\frac{a(t)}{a_1}\right]^3 . \tag{6.23}$$

We stress that (6.23) has been derived using the initial conditions from the turn-on phase. If we use as initial conditions the ground-state fluctuations

$$\langle|\psi_k|^2(t_1)\rangle=\frac{1}{2\omega_k V} , \tag{6.24}$$

then we obtain an energy density $\rho_G(t)$:

$$\rho_G(t)\simeq\left[\frac{\lambda}{g}\right]^2\rho_R(t) . \tag{6.25}$$

Although $\rho_G$ is larger than $\rho_R$, whether there is efficient reheating or not does not depend on the choice since the energy increases exponentially during parametric resonance, and in comparison the ratio between $\rho_G$ and $\rho_R$ is insignificant.

Note also that we obtain approximately the same value for $\rho_R(t)$ if we use

$$g\langle\varphi^2(t)\rangle=\tfrac{1}{2}\mu_f^2(1+\tanh t/L) \tag{6.26}$$

or the linear interpolating expression discussed in Sec. IV. Thus, we see that $\rho_R(t)$ depends essentially only on the scales $\mu_f$ and $L$.

Equation (6.23) is valid only as long as any given mode remains in the resonance band. The modes will gradually redshift out of the band. The time interval $\Delta t$ for which the mode remains in the band is

$$\Delta t \simeq 2\frac{\delta_k}{H_1}\frac{1}{m_\varphi}=\frac{g}{\lambda^{3/2}}\frac{m_{\rm Pl}}{\sigma^2} . \tag{6.27}$$

Hence, for $\lambda > g$, $\Delta t$ is smaller than the Hubble expansion time:

$$H_1\Delta t=\left[\frac{\mu_f}{m_\varphi}\right]^2 \ll 1 . \tag{6.28}$$

(This was in fact one of our criteria for parametric resonance.)

As modes leave the resonance bands, new ones will enter it. As long as the total time is small compared to $H_1^{-1}$, the total energy produced during a time interval $N\Delta t$ is approximately $N\rho_1$, where $\rho_1$ is obtained by evaluating (6.23) at time $t_1+\Delta t$:

$$\frac{\rho_R(t_1+N\Delta t)}{\rho^{\rm back}}\simeq N\frac{g^3}{\lambda^2}\exp\left[\left[\frac{g}{\lambda}\right]^2\frac{m_{\rm Pl}}{\sigma}\right] , \tag{6.29}$$

where $\rho^{\rm back}$ is the background (false-vacuum) energy density.

Reheating is efficient if this ratio becomes of order one in a time less than the Hubble time. Otherwise, a significant fraction of the original energy density is redshifted away—the Universe "reheats" to a temperature less than its preinflationary temperature. The ratio (6.29) equals unity for

$$N\simeq\frac{\lambda^2}{g^3}e^{-g^2/\lambda^2(m_{\rm Pl}/\sigma)} . \tag{6.30}$$

Therefore, the condition $N\Delta t < H^{-1}$, for efficient reheating, becomes

$$\frac{g^2}{\lambda^2}\frac{m_{\rm Pl}}{\sigma} > 2\ln(g^2/\lambda) . \tag{6.31}$$

For example, if $g/\lambda=10^{-1}$, efficient reheating requires

$$\frac{m_{\rm Pl}}{\sigma} > 10^2\ln\lambda^{-1} , \tag{6.32}$$

which depends rather weakly on the value of $\lambda$. This shows that efficient reheating can happen even for extremely small values of $\lambda$ such as $10^{-12}$.

## VII. CONCLUSIONS

We have developed a consistent semiclassical formalism with which to calculate energy production in symmetry-breaking transitions during which a field $\phi$ acquires a mass. $\phi$ is weakly coupled to matter fields $\psi$ which start out in their ground state. We compute the energy production in the quantum field $\psi$ as $\langle \phi^2 \rangle$ obtains a nonzero expectation value and oscillates about the minimum of the potential. Two generic types of behavior of $\langle \phi^2(t) \rangle$ are considered. The first is a monotonic increase; the second is oscillating behavior. We develop formulas for the particle spectrum and the total energy production in terms of quite general parameters which describe the behavior of $\langle \phi^2(t) \rangle$.

We apply the formalism to the example of new inflation. As one would expect, for weak coupling not much energy is produced during the turn-on phase. However, in the oscillatory phase a lot of energy can be produced by parametric resonance. We derive a condition for efficient reheating and show that such efficient reheating can occur even with the small coupling constants usually assumed for inflationary universe models.

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