Quantum creation of a generic universe

Robert Graham and Peter Szépfalusy*
Fachbereich Physik, Universität Gesamthochschule Essen, D 4300 Essen, Federal Republic of Germany
(Received 22 May 1990)

The quantum fluctuations of an anisotropic Bianchi type-IX metric and its tunneling into a quasiclassical inflationary phase are analyzed by solving the Wheeler-DeWitt equation within a minisuperspace description. Within this model and the adopted statistical interpretation of the "wave function of the Universe" the appearance of an extremely isotropic universe from quantum fluctuations occurs with necessity.

I. INTRODUCTION

Classical general relativity and observed cosmological data imply that a singularity of the gravitational field has occurred at a finite time in the past.[1] As a form reflecting asymptotically close to a singularity the most essential features of a generic solution the Bianchi type-IX metric was proposed,[2-4] which was analyzed by Misner[5] in an early attempt to explain the isotropy of the Universe, an attempt which later turned out unsuccessful. The Bianchi type-IX metric is highly anisotropic and approaches the singularity in an infinite sequence of oscillations which is a property of a generic singularity.[6-8] These oscillations have been shown to be chaotic and their statistical properties have been investigated in detail.[7-14]

However, sufficiently close to the singularity quantum effects should dominate. One (admittedly far from perfect) way to discuss these in the absence of a quantum field theory of gravitation consists in resorting to a description of the gravitational field by a finite (and small) number of degrees of freedom in "minisuperspace."[15-32] There are two basic hopes underlying this approach. (i) Because of the extreme smallness of the Universe near the initial singularity it may be possible to treat all fields as spatially homogeneous. (ii) The dynamics of a few global parameters (such as the global scale factor of the Universe, or its two global anisotropy parameters) may at least approximately be separable from the dynamics of all the remaining degrees of freedom, whose net effect may then be lumped into a matter term or a cosmological term of an effective potential in the Lagrangian.

The quantized Bianchi type-IX metric has first been analyzed in this framework by Misner,[16-18] and more recently in a numerical study of wave-packet dynamics in Ref. 19. These authors studied the associated Schrödinger equation, called the Wheeler-DeWitt equation.[1,15,33] Recently a number of authors have taken up this problem again within a path-integral approach.[22-26] Their aim was to analyze the consequences of a proposal by Hartle and Hawking[20] that asymptotically close to the singularity the Universe approaches the path-integral equivalent of a quantum-mechanical ground state. However, the path-integral approach to quantum gravity, even in its minisuperspace version, is far from well established. In addition to the usual Euclidean rotation necessary for the definition of all quantum-mechanical path-integrals, an additional analytical continuation is required in order to tackle the fact that the gravitational action is unbounded from above and from below. In addition the statistical interpretation of the resulting wave function is not at all obvious. Finally, the proposal of Hartle and Hawking for the initial state consists in a rather formal prescription for the path integral whose physical meaning is far from obvious. For this reason it seems desirable to further explore the alternative description based on the Wheeler-DeWitt equation. This is the purpose of the present paper. Building on the work of Misner[17] we set up, in Sec. II, the Wheeler-DeWitt equation for the Bianchi type-IX metric including a cosmological term. In Sec. III we determine its ground state asymptotically close to the singularity (for a small-scale parameter), in an analytically solvable approximation, and show that there the quantum fluctuations of anisotropy are large, i.e., of the order of the scale factor. In Sec. IV we then show that the quantum fluctuations of anisotropy are strongly suppressed dynamically in the wave function for scale parameters larger than the Planck length and derive analytical expressions for the mean square of the anisotropy fluctuations, their probability distribution, and wave function tunneling from the quantum domain to an inflationary quasiclassical domain. In the concluding section we compare the most commonly used statistical interpretation of the "wave function of the Universe" with the statistical interpretation adopted here, which is based on the Wheeler-DeWitt equation. We conclude that within the adopted model and its statistical interpretation the quasiclassical phase of the Universe appears with certainty and with negligible anisotropy.

II. QUANTIZATION OF A BIANCHI TYPE-IX METRIC

Let us consider the Universe close to the initial singularity implied by classical general relativity. It has been argued (for a review see Refs. 3 and 4) that a generic form of the metric is then of Bianchi type IX:[1,2]

$$ds^2 = -dt^2 + g_{ij}(t)\omega^i\omega^j \tag{2.1}$$

which, as most previous workers, we take as homogeneous, for simplicity. Here t is standard cosmic time. Restricting ourselves to closed nonrotating universes, ω^i are a basis of one-forms on the three-sphere:

$$\begin{aligned}
&\omega^1=\cos\psi\, d\theta+\sin\psi\sin\theta\, d\phi\ ,\\
&\omega^2=\sin\psi\, d\theta-\cos\psi\sin\theta\, d\phi\ ,\\
&\omega^3=d\psi+\cos\theta\, d\phi\ ,\\
&0\le\theta<\pi,\quad 0\le\phi<2\pi,\quad 0\le\psi<4\pi\ ,
\end{aligned}\tag{2.2}$$

and

$$\begin{aligned}
&g_{ij}(t)=a^2(t)(e^{2\beta(t)})ij\ ,\\
&\boldsymbol{\beta}=\mathrm{diag}(\beta_+ +\sqrt{3}\beta_-,\beta_+-\sqrt{3}\beta_-,-2\beta_+)
\end{aligned}\tag{2.3}$$

with $\mathrm{Tr}\boldsymbol{\beta}=0$. In the following let $g=\mathrm{Det}g_{ij}$ and R denote the scalar curvature of the metric (2.1). The Einstein-Hilbert action[1,2]

$$\begin{aligned}
&I_g=\frac{1}{16\pi G}\int L(t)dt\ ,\\
&L(t)=(4\pi)^2R(t)\sqrt{g(t)}
\end{aligned}\tag{2.4}$$

can then be reexpressed in the generalized coordinates a, β_+, β_-,

$$\frac{1}{12\pi^2}L=a^3\left[-\frac{\dot a^2}{a^2}+\dot\beta_+^2+\beta_-^2\right]-a[V(\beta_+,\beta_-)-1]\tag{2.5}$$

with

$$V(\beta_+,\beta_-)=\tfrac{1}{3}\mathrm{Tr}(1-2e^{-2\beta}+e^{4\beta})\ .\tag{2.6}$$

Redefining the time coordinate, $dt=N(t')dt'$, e.g., by

$$dt=12\pi^2a^3dt_f\ ,\tag{2.7}$$

L is changed, in a corresponding way; e.g., in the case (2.7),

$$L\to L_f=-\left[\frac{\dot a}{a}\right]^2+\dot\beta_+^2+\dot\beta_-^2-\left[\frac{a}{a_0}\right]^4[V(\beta_+,\beta_-)-1]\ ,\tag{2.8}$$

where the dot now denotes the derivative with respect to t_f and $a_0=1/2\pi\sqrt{3}$. Thus, the metric tensor $\tilde{\mathbf{G}}$ in "minisuperspace" apparent in the kinetic term $\dot{\mathbf{g}}\cdot\tilde{\mathbf{G}}\cdot\dot{\mathbf{g}}$ of L [where $\dot{\mathbf{g}}=(\dot g_A,\dot g_B,\ldots,\dot g_C)$ is the velocity vector in minisuperspace] is subject to conformal transformations $\tilde{\mathbf{G}}'=N^{-1}\tilde{\mathbf{G}}$ by such redefinitions of time, and, moreover, the choice (2.7) leads to a flat realization of minisuperspace.[17]

Let us now quantize this dynamical system. We follow Misner[17] and demand that the free choice of the time coordinate be preserved in the quantum theory; i.e., all conformally equivalent realizations of minisuperspace must be quantum-mechanically equivalent. This is most easily achieved, in the present case, by quantizing in the flat realization, writing the action of the Schrödinger equation as

$$\begin{aligned}
&I_g=\frac{1}{16\pi G}\int L_f(t_f)dt_f\ ,\\
&L_f=\int\frac{da}{a}d\beta_+d\beta_-\left[\left|\frac{\partial\psi}{\partial\ln a}\right|^2-\left|\frac{\partial\psi}{\partial\beta_+}\right|^2-\left|\frac{\partial\psi}{\partial\beta_-}\right|^2-\left[\frac{a}{a_0}\right]^4[V(\beta_+,\beta_-)-1]|\psi|^2-\lambda\left[\frac{a}{a_0}\right]^6|\psi|^2\right]
\end{aligned}\tag{2.9}$$

with the corresponding wave equation

$$\frac{\partial^2\psi}{\partial\Omega^2}-\frac{\partial^2\psi}{\partial\beta_+^2}-\frac{\partial^2\psi}{\partial\beta_-^2}+e^{-4\Omega}[V(\beta_+,\beta_-)-1]\psi+\lambda e^{-6\Omega}\psi=0\ ,\tag{2.10}$$

where we introduced $\Omega=-\ln(a/a_0)$ and added a matter term which we assume to change adiabatically and to act like a cosmological term with cosmological constant $\lambda>0$. Conformal invariance in the present three-dimensional minisuperspace dictates that the wave function ψ_c corresponding to the use of standard cosmic time t as in Eq. (2.1) is related to ψ for general time $dt'=dt/N$ via $\psi=N^{1/4}\psi_c$, i.e., in the case (2.7),

$$\psi_c=(12\pi^2a^3)^{-1/4}\psi\ .\tag{2.11}$$

For later use we note that

$$\begin{aligned}
dw&=-\frac{i}{2}(\mathrm{Det}\tilde{\mathbf{G}})^{1/2}\left[\psi^*\frac{\partial\psi}{\partial\mathbf{g}}-\psi\frac{\partial\psi^*}{\partial\mathbf{g}}\right]\cdot\tilde{\mathbf{G}}^{-1}\cdot d^{(2)}\boldsymbol{\Sigma}\\
&=\frac{i}{2}\left[\psi^*\frac{\partial\psi}{\partial\Omega}-\psi\frac{\partial\psi^*}{\partial\Omega}\right]d\beta_+d\beta_-\\
&=\frac{i}{2}\left[\psi_c^*\frac{\partial\psi_c}{\partial\Omega}-\psi_c\frac{\partial\psi_c^*}{\partial\Omega}\right]d\beta_+d\beta_-
\end{aligned}\tag{2.12}$$

but not

$$\begin{aligned}
dw'&=(\mathrm{Det}\tilde{\mathbf{G}})^{1/2}|\psi|^2d^{(3)}\Omega\\
&=|\psi|^2d\Omega\, d\beta_+d\beta_-\\
&=(12\pi^2a^3)^{1/2}|\psi_c|^2d\Omega\, d\beta_+d\beta_-
\end{aligned}\tag{2.13}$$

is invariant under the latter transformation (and conformal transformations, in general). Here we used the metric-independent volume element

$$d^{(n)}\Omega=\frac{1}{n!}\epsilon_{AB\cdots C}dg^Adg^B\cdots dg^C$$

and timelike hypersurface element

$$d^{(n-1)}\Sigma_A=\frac{1}{(n-1)!}\epsilon_{AB\cdots C}dg^B\cdots dg^C$$

in minisuperspace, where $\epsilon_{AB\cdots C}$ is the numerical antisymmetric symbol $\epsilon_{AB\cdots C}=0,\pm1$. Therefore dw', while positive, cannot be used as a probability measure without violating the underlying symmetry.

Let us now analyze Eq. (2.10). Two limits can be understood quite completely: the case $(-\Omega)\to-\infty$ near the initial singularity, and the case $(-\Omega)>1$ [i.e., $a>(e/2\pi\sqrt{3})$], which are considered in Secs. III and IV, respectively. In both cases the coordinate transformation

$$\begin{aligned}\beta_+&=\ln\frac{1}{\rho}\sinh\zeta\cos\phi\ ,\\ \beta_-&=\ln\frac{1}{\rho}\sinh\zeta\sin\phi\ ,\\ \Omega&=\ln\frac{1}{\rho}\cosh\zeta\end{aligned}\tag{2.14}$$

is useful, after which

$$-\frac{1}{\rho(\ln\rho)^2}\frac{\partial}{\partial\rho}[\rho(\ln\rho)^2]\frac{\partial\psi}{\partial\rho}+\frac{1}{\rho^2(\ln\rho)^2}[\Delta_{\rm LB}-U(\rho,\zeta,\phi)]\psi=0\ ,\tag{2.15}$$

where

$$\Delta_{\rm LB}=\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left[\sinh\zeta\frac{\partial}{\partial\zeta}\right]+\frac{1}{\sinh^2\zeta}\frac{\partial^2}{\partial\phi^2}\tag{2.16}$$

is the Laplace-Beltrami operator and

$$U(\rho,\zeta,\phi)=(\ln\rho)^2[\rho^{4\cosh\zeta}(V-1)+\lambda\rho^{6\cosh\zeta}]\ .\tag{2.17}$$

We note the correspondence for fixed ζ:

$$\begin{aligned}&-\infty<-\Omega<0\Longleftrightarrow 0<\rho<1\ ,\\ &0<-\Omega<\infty\Longleftrightarrow 1<\rho<\infty\ .\end{aligned}\tag{2.18}$$

III. PRIMORDIAL QUANTUM BILLIARD AND ITS GROUND STATE

Let us now consider Eq. (2.15) asymptotically for $\rho\to+0$. Then the potential U vanishes inside and is plus infinity outside the triangular domain bounded by[17]

$$\tanh\zeta=-\tfrac{1}{2}\sec\left[\phi+m\frac{2\pi}{3}\right],\quad m=0,\pm1\ .\tag{3.1}$$

This makes it possible to factorize the solution of Eq. (2.15) as

$$\psi=\Phi(\rho)\Theta(\zeta,\phi)\tag{3.2}$$

and one has to consider first the eigenvalue problem

$$-\Delta_{\rm LB}\Theta_i(\zeta,\phi)=\lambda_i\Theta_i(\zeta,\phi)\tag{3.3}$$

with the boundary conditions $\Theta(\zeta,\phi)=0$ at the walls specified above.

Two reparametrizations of Eq. (2.16) will be useful in the following. The transformation

$$r=\tanh\frac{\zeta}{2}\tag{3.4}$$

leads to a quantum billiard problem on the Bolyai-Lobachevsky plane, i.e., on the plane with constant negative Gaussian curvature (-1) (for a review of the classical and quantum theory of motion on the pseudosphere see Refs. 34 and 36). The region displayed in Fig. 1 is an infinite equilateral triangle of geodesics with zero angles (correspondingly its area is equal to π). The three straight lines in Fig. 1 exhibit the obvious symmetry of the problem and are drawn for future purposes. (Note that these lines are also geodesics.) Furthermore a horocycle is shown with dotted lines. Classically the trajectories are broken geodesics reflected at the boundary and the motion is known to be chaotic.[9,10]

Finally it is useful to go over to the Poincaré half-plane with the help of the fractional linear transformation

$$x+iy=\frac{3^{1/2}}{2}\frac{-ize^{i\pi/6}+i}{ze^{i\pi/6}+1},\quad z=re^{i\phi}\ .\tag{3.5}$$

Note that the rotation by $\pi/6$ in the Bolyai-Lobachevsky plane and the scale change in the Poincaré half-plane are introduced for practical purposes. The shaded region in Fig. 2 is the image of that in Fig. 1.

Equation (3.3) transforms to

$$-y^2\left[\frac{\partial^2Z}{\partial x^2}+\frac{\partial^2Z}{\partial y^2}\right]=\lambda Z\ .\tag{3.6}$$

The general solution of (3.6) can be written as

$$Z_\omega(x,y)=\sum_{n=1}^{\infty}A_n\sin[n\pi(x-\tfrac{1}{2})]y^{1/2}K_{i\omega}(n\pi y)\ ,\tag{3.7}$$

where $K_{i\omega}$ stands for the Hankel function of imaginary argument[37] and ω is defined by

$$\lambda=\tfrac{1}{4}+\omega^2\ .\tag{3.8}$$

Note that by symmetry reasons one may choose the solutions of (3.6) such that $A_n=0$ either for N even or for N odd. In the case of the ground state, which is our main

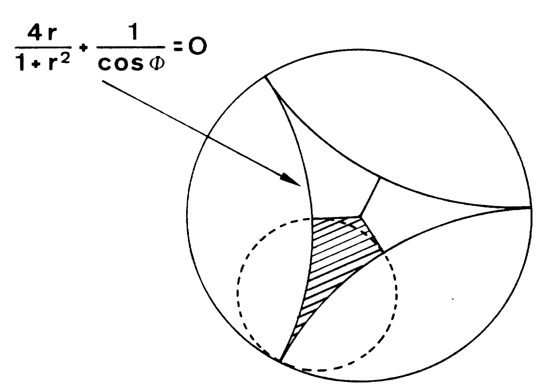

FIG. 1. The billiard on the Bolyai-Lobachevsky plane.

interest here, the boundary condition at the bottom line should be of von Neumann type and $A_n=0$ for even N. The problem is greatly simplified if we replace the boundary geodesics at the bottom by the straight line drawn in Fig. 2 as dotted line. In the representation on the Bolyai-Lobachevsky plane it means that the two straight line segments are replaced by the arc of the horocycle in Fig. 1. Since around the center the ground-state wave function is nearly constant the inaccuracy caused by this change of the boundary curve is small. [Note that with this boundary condition the system becomes integrable; it is no longer chaotic classically. For the quantum ground state this changed behavior of the classical system is not of great importance. However, all highly excited states and even those low-lying states of the form (3.7) which satisfy neither Dirichlet nor von Neumann conditions on the bottom line cannot be approximated within this integrable description.] For states satisfying Dirichlet or von Neumann boundary conditions on the shifted bottom line the solution separates as

$$Z_n(x,y)=A_n\sin[n\pi(x-\tfrac{1}{2})]y^{1/2}K_{i\omega_{n-1}}(n\pi y) . \quad (3.9)$$

For the ground state $n=1$ and $(\partial Z_1/\partial y)|_{y=1}=0$. For the corresponding eigenvalue one finds numerically $\omega_0=4.21$ which yields $\lambda_0=18.0$.

The line $x=0$ on the Poincaré half-plane corresponds to the line $\phi_0=4\pi/3$, which is a line of symmetry of the shaded region in Fig. 1. Along this line $\Theta(\zeta,\phi)$ is given by

$$\Theta(\hat{\beta})\sim\left[\frac{1+\hat{\beta}}{1-\hat{\beta}}\right]^{1/4}K_{i\omega_0}\left[\frac{3^{1/2}\pi}{2}\left[\frac{1+\hat{\beta}}{1-\hat{\beta}}\right]^{1/2}\right], \quad \tfrac{1}{7}\le\hat{\beta}\le 1 , \quad (3.10)$$

$$\Theta(\hat{\beta})=\text{const}, \quad 0\le\hat{\beta}\le\tfrac{1}{7}$$

as a function of

$$\hat{\beta}\equiv\frac{(\beta_1^2+\beta_2^2)^{1/2}}{\Omega}=\tanh\zeta , \quad (3.11)$$

which is a measure of the anisotropy. The distribution $|\Theta(\hat{\beta})|^2$ is plotted in Fig. 3. As expected it has its maximum value around zero anisotropy, but one can see that large quantum fluctuations of anisotropy (of the order of the scale factor) occur due to the zero-point fluctuations in the underlying quantum billiard.

It remains to solve the ρ-dependent part of Eq. (2.15) for $U=0$ which reads

$$-\frac{1}{\rho(\ln\rho)^2}\frac{\partial}{\partial\rho}[\rho(\ln\rho)^2]\frac{\partial\Phi}{\partial\rho}-\frac{\omega_0^2+\frac{1}{4}}{\rho^2(\ln\rho)^2}\Phi=0 \quad (3.12)$$

and has solutions $\sim|\ln\rho|^{-1/2}\exp(\pm i\omega_0\ln|\ln\rho|)$. The total wave function for the ground state of the billiard is therefore of the form

$$\psi=\frac{\Theta_0(\zeta,\phi)}{\left[\ln\frac{1}{\rho}\right]^{1/2}}(Ae^{-i\omega_0\ln|\ln\rho|}+Be^{i\omega_0\ln|\ln\rho|}) . \quad (3.13)$$

The invariant (2.12) transformed to the present case becomes

$$dw=(|A|^2-|B|^2)|\Theta_0(\zeta,\phi)|^2\sinh\zeta\,d\zeta\,d\phi \quad (3.14)$$

and can be used as a probability measure if we require $|A|^2>|B|^2$. We shall see later that the ratio A/B is determined at least in principle by the observed fact that only an outgoing wave is present in the quasiclassical domain. These considerations and the requirement that

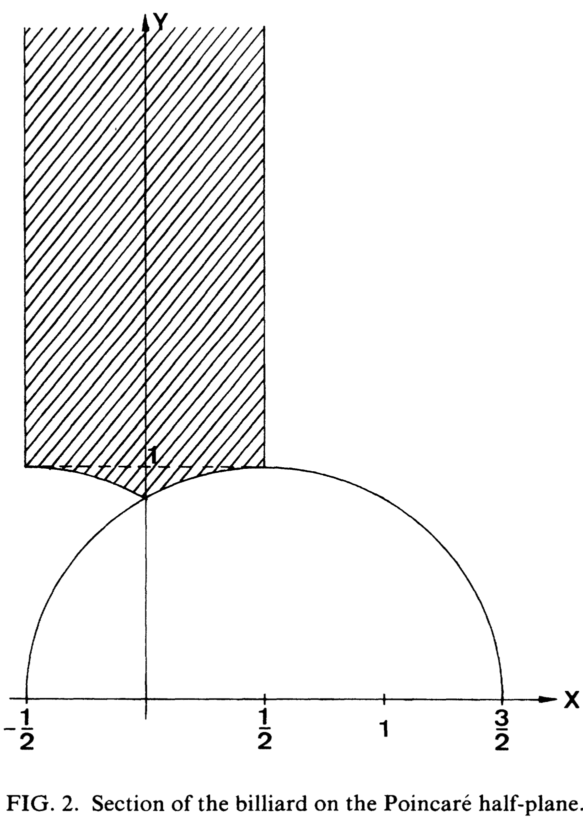

FIG. 2. Section of the billiard on the Poincaré half-plane.

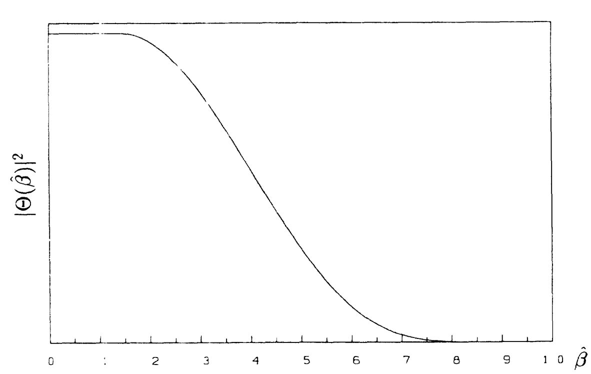

FIG. 3. Probability distribution of quantum fluctuations of anisotropy in the ground state in arbitrary units.

the billiard is in its ground state therefore serve to completely fix the wave function for $\rho \to 0$.

IV. SUPPRESSION OF ANISOTROPY AND TUNNELING

Let us now turn back to Eq. (2.15) and consider the case $\rho \gg 1$. Then the potential term in Eq. (2.15) becomes very important. It will be helpful, for some purposes, to introduce the variable

$$u = \frac{1}{\ln \frac{1}{\rho}} . \tag{4.1}$$

The domain $\rho \gg 1$ then corresponds to the region $-1 \ll u < 0$, while $\rho \ll 1$ corresponds to $0 < u \ll 1$. Equation (2.15) then takes the form

$$-\frac{\partial^2 \psi}{\partial u^2} + \frac{1}{u^2}[\Delta_{\mathrm{LB}} - U(u,\zeta,\phi)]\psi = 0 \tag{4.2}$$

with

$$U(u,\zeta,\phi) = \frac{1}{u^2}[e^{-(4/u)\cosh\zeta}(V-1) + \lambda e^{-(6/u)\cosh\zeta}] \tag{4.3}$$

which shows that in the region considered here the anisotropy term acquires a small effective mass, of order u^2, and the u variable, and hence also ρ, may be treated adiabatically, making the ansatz

$$\psi(\rho,\zeta,\phi) = \tilde{\Theta}_i(\zeta,\phi,\rho)\Phi(\rho) . \tag{4.4}$$

The wave function of the anisotropy for fixed ρ satisfies

$$-\Delta_{\mathrm{LB}}\tilde{\Theta}_i + U(\rho,\zeta,\phi)\tilde{\Theta}_i = \lambda_i(\rho)\tilde{\Theta}_i , \tag{4.5}$$

where now, for $\rho \gg 1$,

$$U(\rho,\zeta,\phi) \simeq \begin{cases} \infty \quad \text{for } \sinh^2\zeta > \dfrac{8}{(\ln\rho)^2}, \\ (\ln\rho)^2 \left[\rho^{4\cosh\zeta} \left[-1 + \dfrac{(\ln\rho)^2}{8}\sinh^2\zeta \right] + \lambda\rho^{6\cosh\zeta} \right] \quad \text{for } \sinh^2\zeta \le \dfrac{8}{(\ln\rho)^2}. \end{cases} \tag{4.6}$$

Here we were allowed to approximate the potential V near the origin $\beta_+ = \beta_- = 0$ because the support of the wave function, according to Eq. (4.6), becomes restricted to the domain $\sinh^2\zeta < 8/(\ln\rho)^2$ which goes to zero; in short the anisotropy becomes very small. Expanding the potential for small ζ and solving the Schrödinger equation for the resulting two-dimensional harmonic oscillator with mass $m = \frac{1}{2}$ and frequency

$$\Omega_0(\rho) = \frac{1}{\sqrt{2}}\rho^2(\ln\rho)^2 , \tag{4.7}$$

we find, e.g., for the ground state,

$$\tilde{\Omega}_0(\zeta,\rho) = \frac{1}{2\pi\zeta_0^2(\rho)} \exp\left[-\frac{\zeta^2}{4\zeta_0^2(\rho)} \right] \tag{4.8}$$

with

$$\zeta_0^2(\rho) = \frac{2}{\Omega_0(\rho)} . \tag{4.9}$$

Similarly all excited states $\tilde{\Theta}_i$ can be written down. Thus in the limit considered the zero-point fluctuations of the anisotropy acquire extremely high frequency (justifying once more the adiabatic approximation with respect to ρ) and are of extremely small amplitude.

The eigenvalues $\lambda_i(\rho)$ ($i = m,n$) of Eq. (4.5) depend on ρ explicitly:

$$\lambda_{m,n}(\rho) \simeq (\ln\rho)^2 \left[\frac{n+m+1}{\sqrt{2}}\rho^2 - \rho^4 + \lambda\rho^6 \right] . \tag{4.10}$$

We note that the first term in large parentheses has the same dependence on ρ as a radiation term included in the energy-momentum tensor. However, while the latter describes the cumulative gravitational effect of infinitely many modes, the fluctuations of the anisotropy are due to only two collective modes described by β_+, β_- or ζ, ϕ, of extremely high frequency whose enormous size prevents these oscillations from ever becoming classical. However, in the domain $\rho \gg 1$ where Eq. (4.10) is valid, the contribution of the first term in (4.10) is of minor importance. Equation (2.15) is now reduced to

$$-\frac{1}{\rho(\ln\rho)^2}\frac{\partial}{\partial\rho}[\rho(\ln\rho)^2]\frac{\partial\Phi}{\partial\rho} - \frac{\lambda_i(\rho)}{\rho^2(\ln\rho)^2}\Phi = 0 \tag{4.11}$$

or, equivalently,

$$-\frac{d^2\Phi}{du^2} - \frac{\tilde{\lambda}_i(u)}{u^2}\Phi = 0 \tag{4.12}$$

with $\lambda_i(\rho) = \tilde{\lambda}_i(u)$. It describes tunneling in the effective potential [with $i = (n,m)$]

$$U_{\mathrm{eff}}^{(i)}(\rho) = -\frac{\lambda_i(\rho)}{\rho^2(\ln\rho)^2} = \left[-\frac{m+n+1}{\sqrt{2}} + \rho^2 - \lambda\rho^4 \right] \quad (\rho \gg 1) , \tag{4.13}$$

$$U_{\mathrm{eff}}(\rho) = -\frac{\omega_0^2 + \frac{1}{4}}{\rho^2(\ln\rho)^2} \quad (\rho \ll 1) ,$$

where the second equation applies to the regime of the billiard and follows from Eq. (3.12), assuming that the billiard is in its ground state. A schematic plot of $U_{\mathrm{eff}}^{(i)}(\rho)$

for fixed $i=(n,m)$ is given in Fig. 4.

In the quasiclassical domain $\rho > \rho_2$ with $\rho_2 \simeq \lambda^{-1/2}$ the solution of Eq. (4.11) is well approximated by the WKB result [most easily derived from Eq. (4.11)]

$$\psi = \sum_j C_j \frac{\tilde{\Theta}_j(\zeta,\rho)}{|\ln\rho|^{1/2}[\lambda_j(\rho)]^{1/4}} \times \exp\left[-i\int_{\rho_2}^{\rho} d\rho \left[\frac{\lambda_j(\rho)}{\rho^2(\ln\rho)^2}\right]^{1/2}\right] \tag{4.14}$$

with coefficients C_j to be determined. Here we assume that only an outgoing wave is present. The weight dw, Eq. (2.12), rewritten in the present coordinates for small ζ

$$dw = \frac{i}{2}\rho(\ln\rho)^2 \left[\psi^* \frac{\partial\psi}{\partial\rho} - \psi\frac{\partial\psi^*}{\partial\rho}\right]\zeta\, d\zeta\, d\phi\ , \tag{4.15}$$

and evaluated in WKB approximation for the ground state becomes

$$dw = \frac{1}{2\pi\zeta_0^2(\rho)}\exp\left[-\frac{\zeta^2}{2\zeta_0^2(\rho)}\right]\zeta\, d\zeta\, d\phi\ , \tag{4.16}$$

i.e., a positive measure.

However, a general state (4.14) does not lead to a positive dw, and therefore cannot be statistically interpreted within the framework we have adopted here. On the other hand the wave function (4.14) describes quantum oscillations of anisotropy uncoupled to any other degrees of freedom (except the scale factor) which are, indeed, in this form, unobservable. Coupling the anisotropy to other degrees of freedom, e.g., massless radiation fields, will

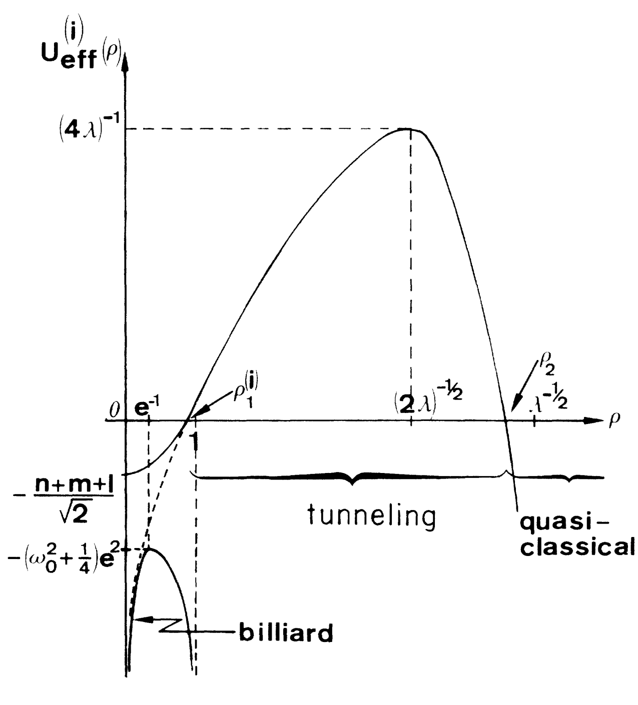

FIG. 4. Effective potential in the regions of the quantum billard, quantum tunneling, and quasiclassical inflationary expansion.

make them observable, in principle, but will at the same time lead to a loss of coherence between the levels of the oscillator destroying interferences in the linear superposition (4.14). Assuming therefore the coefficients C_i to be randomly phased, the application of (4.15) together with a phase-average (corresponding to taking the trace over the radiative degrees of freedom) again leads to a positive dw given by

$$dw = \sum |C_i|^2 |\tilde{\Theta}_i(\zeta,\rho)|^2\ . \tag{4.17}$$

Note that dw remains normalized. Note also that by the same mechanism an interference between the outgoing wave (4.14) and a reflected incoming wave present in a recollapsing universe will not be observable.

It is certainly unusual that the degrees of freedom of the "meter" (e.g., the radiation field coupled to the anisotropy oscillations) must be included and traced out before a statistical interpretation even becomes possible. However, it should be recalled that the total wave function of the Universe in principle must include also any "meter" and is therefore neither observable (there is no observer left) nor is a statistical interpretation meaningful (there is no ensemble). Only after averaging over the degrees of freedom of the "meter" is a statistical description of the reduced system required and meaningful. Further remarks related to this point will be made in the concluding section.

Our approximation, so far, is restricted to the domain $\rho \gg 1$ (the region of very small anisotropy) and $\rho \ll 1$ (the region of the primordial billiard, where the anisotropy is large). We now try to interpolate between these two regions by the same adiabatic ansatz (4.4) assuming that $\tilde{\Theta}_i$ satisfies Eq. (4.5) with the full potential $U(\rho,\zeta,\phi)$. Clearly the resulting eigenvalues $\lambda_i(\rho)$ will interpolate between the two regimes we have considered in detail, and will lead to some interpolating effective potential, in Eq. (4.11)

$$U^{(i)}_{\rm eff}(\rho) = -\frac{\lambda_i(\rho)}{\rho^2(\ln\rho)^2} \tag{4.18}$$

shown schematically as a dashed line in Fig. 4. While we cannot solve Eq. (4.5) analytically, in the general case, it is quite clear how the eigenvalue $\lambda_i(\rho)$ must behave: For $\rho \to 0$ the problem reduces to the billard and $\lambda_i(\rho) \to \omega_i^2 + \frac{1}{4}$.

Increasing ρ from zero the walls of the billard described by the potential U gradually lose their steepness and $\lambda_i(\rho)$ decreases. At $\rho = 1$ the walls of the billard have disappeared completely and U vanishes identically. Thus $\lambda_i(\rho) \sim (\ln\rho)^2$ for $\rho \to 1$ and $\lambda_i(1) = 0$. Increasing ρ further the potential increases in the domain $V > 1$ and decreases in the domain $V < 1$ near the origin where U develops an increasingly sharp and deep minimum and we are finally in the regime where Eq. (4.10) holds. As we can infer from this discussion, in the effective potential (4.18) the $(\ln\rho)^2$ factor cancels for $\rho \to 1$. Thus $U^{(i)}_{\rm eff}(\rho)$ changes smoothly between the regimes $\rho \ll 1$ and $\rho \gg 1$ and changes sign for $\rho = \rho_1^{(i)}$. However, there is a neighborhood of $\rho = 1$ where the adiabatic assumption on which this simple picture is based lacks self-consistency and must break down. Indeed, since $U=0$ for $\rho=1$ the quantum states of anisotropy in the potential U for $\rho\to 1$ are arbitrarily closely spaced and will be mixed by even the slightest ρ dependence of U. Therefore, the anisotropy oscillations for $\rho\gg 1$ will not emerge in the ground state but in some linear combination of excited states of the form (4.14) determined by the details of the passage of U through zero at $\rho=1$. However, regardless of the detailed form of this linear combination, the amplitude of the anisotropy oscillations scales with $\zeta_0(\rho)$ and becomes extremely small for $\rho\gg 1$.

In the domain $\rho\le\rho_0^{(0)}<1$ (where $i=0$ means $n=m=0$) the Wheeler-DeWitt equation (4.11) can also be solved in WKB approximation. Assuming that the billard for $\rho\to 0$ is in its ground state we find

$$\psi\simeq\frac{\Theta_0(\zeta,\rho)}{|\ln\rho|^{1/2}[\lambda_0(\rho)]^{1/4}}\left\{\frac{1}{T_{12}}\cos\left[\int_\rho^{\rho_1^{(0)}}\left[\frac{\lambda_0(\rho)}{\rho^2|\ln\rho|^2}\right]^{1/2}d\rho-\frac{\pi}{4}\right]+iT_{12}\sin\left[\int_\rho^{\rho_1^{(0)}}\left[\frac{\lambda_0(\rho)}{\rho^2|\ln\rho|^2}\right]^{1/2}d\rho-\frac{\pi}{4}\right]\right\}, \tag{4.19}$$

where T_{12} is the tunneling amplitude through the barrier. If the adiabatic approximation would hold also near $\rho=1$ we could again apply the WKB approximation to obtain

$$T_{12}=\exp\left[-\int_{\rho_1^{(0)}}^{\rho_2}d\rho\left[-\frac{\lambda_0(\rho)}{\rho^2|\ln\rho|^2}\right]^{1/2}\right], \tag{4.20}$$

where we used the boundary condition (4.14) with $C_i=\delta_{i0}$ for $\rho\ge\rho_2$. However, because of the breakdown of the adiabatic approximation near $\rho=1$, Eq. (4.20) can only be qualitatively correct, at best. In any case T_{12} is an extremely small number, which can, in principle, be calculated. This suffices to see that the wave function for $\rho<\rho_1^{(0)}$ is completely determined by the two conditions that (i) the billiard is initially in its ground state and (ii) there is only an outgoing wave (an expanding universe) beyond the tunneling regime. For $\rho\to 0$ we have $\lambda_0(\rho)\to\omega_0^2+\frac{1}{4}$ and the WKB result very nearly reduces to the exact result (3.13) (since $\omega_0^2\gg\frac{1}{4}$), but now the coefficients A and B are determined by T_{12}. Both A and B are proportional, essentially, to T_{12}^{-1} and extremely large, while their difference is proportional to T_{12} and extremely small. Thus for $\rho<\rho_1^{(0)}$ the wave function is an extremely strong standing wave in the ρ direction with a tiny surplus of the amplitude in the outgoing direction due to tunneling through the potential barrier. The weight dw remains, of course normalized to unity for all ρ.

V. CONCLUSION

The Wheeler-DeWitt equation is a second-order differential equation with respect to the timelike coordinate in superspace and for this reason the statistical interpretation of the wave function which satisfies this equation has been a matter of debate since its discovery. The basic point of view which we adopt in the present work is that the total wave function of the Universe (as, indeed, the wave function of any completely isolated quantum system) has no intrinsic statistical interpretation as there is neither an ensemble nor an outside observer. This fact is reflected by the nonpositivity of the only general conserved weight dw, Eq. (2.12), which can be associated with the Wheeler-DeWitt equation. The statistical interpretation arises, and, indeed, becomes necessary, only if a separation of the total Universe into an observing part (the measuring device or "meter") and an observed part (the rest of the quantum universe) is made. The "meter" may be thought to include all the degrees of freedom not present in the explicit description of the rest of the quantum universe. From this point of view the appropriate description of the rest of the quantum universe is an example of the description of a single quantum system, which is observed continuously in time. It is then interesting to remark that there have been important recent advances on this general class of problems, both experimentally, and theoretically. Experimentally, it has recently become possible for the first time to isolate single atoms (rather than an ensemble of atoms) and to observe continuously in time the quantum jumps of their optically active electrons in an externally applied light field.[38,39] Theoretically, advances have been made in the dynamical description of observed quantum systems[40] by mixed states (density matrices) satisfying master equations, which, at least in principle, can be derived from the Schrödinger equation of the total system not yet separated into observing and observed parts. This kind of description has recently also been applied to continuously observed quantum systems, which, like the model discussed in the present work, are chaotic in their classical limit.[41]

For the Wheeler-DeWitt equation a derivation from first principles of the dynamics of the continuously observed rest of the quantum universe has not yet been given. As a substitute we have argued after Eq. (4.14) that the coefficients C_i, in general depending on all the variables of the meter, will as a result be randomly phased. Averaging over unobserved meter variables, and hence the phases of C_i, automatically introduces a mixed-state description which, in the example under study, led to the positive probability measure (4.17). In general, however, it seems clear that not all separations of the total Universe into two parts will lead to a positive dw and the general conditions for such a separation to quality as a measurement ($dw>0$) remain unknown.

Finally, we note that the statistical interpretation adopted by us renders meaningless the notion of a probability smaller than 1 for tunneling through the barrier. Such tunneling probabilities have sometimes been defined on the basis of the probability measure dw' which we have discarded above because of its lack of conformal invariance. The numerical values obtained for such tunneling probabilities were found to be fantastically small[20] [e.g., $\exp(-10^{120})$] due to the immense height of the barrier. By contrast, in our description tunneling occurs with certainty because the solution of Eq. (2.15) must be nonzero outside the barrier, and our statistical interpretation ensures that the probability is normalized to unity for each value of ρ. The basic principle difference between the probability measure (2.12) [which is a probability measure only after including the modification discussed in connection with Eq. (4.17)] and (2.13) is the presence of a timelike parameter (Ω, u or ρ) in (2.12) and the absence of such a parameter in (2.13) (where Ω, u or ρ is an independent variable along with β_+,β_- or ζ,ϕ).

Therefore (2.13) seems appropriate when describing a system extrinsically with reference to an external and independent time parameter, which is of course the usual case of quantum theory, while (2.12) describes the Universe intrinsically correlating its properties, e.g., anisotropy, with another intrinsic property, the scale factor.

In this context an interesting question can be raised. If the creation of the Universe from quantum fluctuations is a necessity in our theoretical framework, why do we not observe a continuous nucleation of new universes from local quantum fluctuations of the metric? On the basis of our discussion of the difference between dw and dw' this question can be answered. For such fluctuations "we" play the role of external observers equipped with our own local (classical) time coordinate. Therefore we argue that the usual statistical interpretation (2.13) based on the absolute square of the wave function should be applied and then it makes sense to define and calculate tunneling rates through the cosmological barrier. (As mentioned above these rates will come out as tiny which is one reason why these processes are not observed. Another reason might be the appearance of a horizon shielding these new universes from our observation.) The fact that we have to interpret the wave function of our own quantum universe intrinsically from its inside makes this case special and we propose that this is the physical reason for the necessity to modify the standard statistical interpretation (2.13) in this particular case.

ACKNOWLEDGMENTS

This work was supported by the Deutsche Forschungsgemeinschaft through Sonderforschungsbereich 237 "Unordnung und grosse Fluktuationen." The authors would like to thank Ralph Hübner for the preparation of Fig. 3 and numerical assistance.

*Permanent address: Institute for Solid State Physics, Eötvös University, H-1088 Budapest, Hungary.

[1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[2] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields,* 4th ed. (Pergamon, Oxford, 1975).
[3] V. A. Belinsky, I. M. Khalatnikov, and E. M. Lifshitz, Adv. Phys. **19**, 525 (1970).
[4] V. A. Belinsky, I. M. Khalatnikov, and E. M. Lifshitz, Adv. Phys. **31**, 639 (1982).
[5] C. W. Misner, Phys. Rev. Lett. **22**, 1071 (1969).
[6] V. A. Belinsky and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **56**, 1700 (1969) [Sov. Phys. JETP **29**, 911 (1969)].
[7] E. M. Lifshitz, I. M. Lifshitz, and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **59**, 322 (1970) [Sov. Phys. JETP **32**, 173 (1971)].
[8] I. M. Khalatnikov, E. M. Lifshitz, K. M. Khanin, I. Shchur, and Ya. G. Sinai, J. Stat. Phys. **38**, 97 (1985).
[9] D. M. Chitre, University of Maryland Technical Report No. 72-125, 1972 (unpublished).
[10] J. D. Barrow, Phys. Rep. **85**, 1 (1982).
[11] D. F. Chernoff and J. D. Barrow, Phys. Rev. Lett. **50**, 134 (1983); J. D. Barrow, in *Classical General Relativity,* edited by W. B. Bonnor, J. N. Islam, and M. A. H. MacCallum (Cambridge University Press, Cambridge, England, 1984), p. 24.
[12] D. H. Mayer, Phys. Lett. A **121**, 390 (1987).
[13] D. Mayer and G. Roepstorff, J. Stat. Phys. **47**, 149 (1987); **50**, 331 (1988).
[14] A. Csordás and P. Szépfalusy, Phys. Rev. A **40**, 2221 (1989).
[15] B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).
[16] C. W. Misner, Phys. Rev. **186**, 1319 (1969).
[17] C. W. Misner, in *Magic Without Magic: J. A. Wheeler,* edited by J. Klander (Freeman, San Francisco, 1972).
[18] M. Ryan, *Hamiltonian Cosmology* (Springer, Berlin, 1972).
[19] T. Furusawa, Prog. Theor. Phys. **75**, 59 (1986); **76**, 67 (1986).
[20] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).
[21] S. W. Hawking, Nucl. Phys. **B239**, 257 (1984).
[22] S. W. Hawking and J. C. Luttrell, Phys. Lett. **143B**, 83 (1984).
[23] I. G. Moss and W. A. Wright, Phys. Rev. D **29**, 1067 (1984).
[24] W. A. Wright and I. G. Moss, Phys. Lett. **154B**, 115 (1985).
[25] P. Amsterdamski, Phys. Rev. D **31**, 3037 (1985).
[26] B. K. Berger, Phys. Rev. D **39**, 2426 (1989).
[27] A. Vilenkin, Nucl. Phys. **B252**, 141 (1985).
[28] *Quantum Gravity,* proceedings of the Fourth Seminar, Moscow, USSR, 1987, edited by M. A. Markov, V. A. Berezin, and V. P. Frolov (World Scientific, Singapore, 1988).
[29] L. P. Grishchuk and Yu. V. Sidorov, Zh. Eksp. Teor. Fiz. **94**, 29 (1988) [Sov. Phys. JETP **67**, 1533 (1988)].
[30] G. W. Gibbons and L. P. Grischuk, Nucl. Phys. **B313**, 736 (1989).
[31] R. Bergamini and G. Giampieri, Phys. Rev. D **40**, 3960 (1989).
[32] K. V. Kuchar and M. P. Ryan, Jr., Phys. Rev. D **40**, 3982 (1989).
[33] E. Alvarez, Rev. Mod. Phys. **61**, 561 (1989).
[34] N. L. Balazs and A. Voros, Phys. Rep. **143**, 109 (1986).
[35] M. J. Gutzwiller, Physica D **7**, 341 (1983).
[36] Y. Colin De Verdière, in *Chaos and Quantum Physics,* edited by M. J. Giannoni, A. Voros, and J. Zinn-Justin (Elsevier, New York, 1990).
[37] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).
[38] W. Nagourney, J. Sandberg, and H. Dehmelt, Phys. Rev. Lett. **56**, 2727 (1986).
[39] Th. Sauter, W. Neuhauser, R.Blatt, and P. F. Toschek, Phys. Rev. Lett. **57**, 1696 (1986).
[40] M. D. Srinivas and E. B. Davies, Opt. Acta **28**, 981 (1981).
[41] T. Dittrich and R. Graham, Europhys. Lett. **11**, 589 (1990).