Long-range spin-orbit potential, Gromes's consistency condition, Richardson's potential, and properties of heavy quarkonium systems

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The energies, fine-structure splittings, leptonic widths, and dipole electromagnetic transition rates are calculated for the Υ and charmonium systems using a potential model based on Richardson's interpolating form for the running coupling constant. Using the definitions of the spin-dependent potentials of Eichten and Feinberg and arguments based on lattice gauge calculations, we have determined the long-range spin-orbit potential from Gromes's consistency condition. It is also shown that the sign difference between the long-range spin-orbit potential and the perturbative spin-orbit potential is important in explaining the measured values of the fine-structure ratio in both the Υ and charmonium P states. The question of whether the parameter Λ of Richardson's potential is supposed to play the role of the universal QCD scale is addressed. It is argued that the agreement of the Υ and charmonium leptonic widths with experiment is convincing support for Richardson's form for the running coupling constant since these numbers reflect directly the values of the various S-state wave functions at the origin. Certain problems with the magnitudes of the hyperfine splittings and dipole transition rates of charmonium are discussed.

I. INTRODUCTION

By careful consideration of the leading QCD relativistic corrections, Eichten and Feinberg[1] (EF) were able to derive expressions for the spin-dependent potentials of an interacting heavy-quark–antiquark ($Q\bar{Q}$) pair. They expressed the two spin-orbit potentials, the spin-spin potential, and the tensor potential in terms of the expectation values of bilinear combinations of the color-electric and color-magnetic fields. Evaluating EF's expressions in the context of a lattice gauge calculation, Huntley and Michael[2,3] and Campostrini, Moriarity, and Rebbi[4] presented important evidence that one of the spin-orbit potentials supported a long-range, or nonperturbative, component. They did not see any evidence of long-range behavior in the other spin-orbit potential, the spin-spin potential or the tensor potential.[5] Thus, from lattice gauge calculations, one may infer that the behavior of these potentials is dominated by short-range components, where perturbation theory should be an adequate approximation. Such an inference is consistent with the hypothesis of electric confinement,[1] where one assumes that the color-electric field alone is responsible for the nonperturbative effects associated with quark confinement and hence that perturbation theory is an adequate tool to treat color-magnetic effects.

In an effort to create a context for understanding why this particular spin-orbit potential has a long-range component, I have[6,7] suggested a generalization of the concept of electric confinement to that of electromagnetic confinement. This new terminology is intended to convey the essentials of an observation made by Buchmüller,[8] that is, color electromagnetic effects would be expected in the center-of-mass frame because of the rotation of the tube of color-electric flux responsible for confining the $Q\bar{Q}$ pair. The new terminology is not intended to suggest that the magnetic field has long-range correlations with itself, since this would be inconsistent with the results of lattice gauge theory.

As explained in detail in Ref. 6, one can use Gromes's consistency condition[9,10] to determine the nonperturbative spin-orbit potential. This method has a solid foundation in QCD, in contrast with an approach where one makes an *ad hoc* assumption about the nature of the confining potential.[11–16] It is interesting to note that the argument based on Gromes's consistency condition leads to the same destructive interference between perturbative spin-orbit effects and long-range effects as the *ad hoc* assumption that the confining potential is a Lorentz scalar. In Ref. 6 it was pointed out that this destructive interference is important in reducing the fine-structure ratio r below 0.8, the value resulting from one-gluon exchange (OGE). Observations[17] of the fine-structure ratio yield $r \simeq 0.60-0.70$ for the $1P$ and $2P$ states of the Υ system and $r \simeq 0.45-0.50$ for the $1P$ states of charmonium. All of these values are below the OGE value, which is important experimental support for the destructive interference between perturbative and nonperturbative spin-orbit effects.

Most of the successful calculations of the properties of heavy quarkonium[11–16,18] have some mechanism for softening the strong-interacting coupling constant α_s at small distances, as required by the hypothesis of asymptotic freedom, and include the effects of a linear confining potential. One may obtain the characteristic logarithmic dependence of the running coupling constant on momentum transfer, namely,

$$\alpha_s(|q^2|)=\frac{12\pi}{(33-2n_f)\ln(|q^2|/\Lambda^2)}\ , \tag{1.1}$$

by summing the contributions of all the vacuum-polarization chain bubble graphs as a geometric series.[19] In Eq. (1.1) the quantity n_f denotes the number of quark degrees of freedom and Λ the QCD scale parameter. Richardson[20] made the clever observation that if one solved the singularity problem which arises with Eq. (1.1) as values of $|q|^2 \to \Lambda^2$ by making the substitution

$$\frac{|q^2|}{\Lambda^2} \to 1 + \frac{|q^2|}{\Lambda^2} , \tag{1.2}$$

then the momentum-transfer dependence of the potential at small spacelike momentum transfers could be interpreted as that of a linear confining potential. Richardson's potential contained only a single parameter Λ since the coefficient of the linear term could be expressed as $8\pi\Lambda^2/(33-2n_f)$. However, the substitution of Eq. (1.2) blurred the connection between the parameter Λ and the QCD scale parameter, which should be a universal constant for all heavy quarkonium systems.[21] Richardson's initial calculations[20] showed that his potential had considerable promise in accounting for the location of the spin-averaged levels of charmonium and the Υ system when incorporated into a nonrelativistic model.

In a calculation where they explored the effects of relativistic corrections, Moxhay and Rosner[18] (MR) relaxed the restriction contained in the relationship between the string constant and Λ of Richardson's potential and treated the string constant as a parameter. By carefully taking the nonrelativistic limit of the Dirac spinors, they obtained explicit expressions for the spin-dependent potentials. Since the lattice gauge results of Campostrini, Moriarity, and Rebbi[4] were not available to MR, it was unclear to them which of the spin-dependent potentials would support long-range contributions. To restrict the effects of the transverse degrees of freedom to short distances, they introduced a damping factor. MR also carefully considered the effects of spin-independent relativistic corrections.[22-24]

Improvements in the precision of the measurements of the fine structure of the Υ system[25-27] and a better understanding of the implications of Gromes's consistency condition[6] for potential model calculations present an opportunity to subject Richardson's form for the potential to a more careful scrutiny. In particular, if we assume that perturbative QCD is adequate to treat color-magnetic effects, then the expectation values of the perturbative spin-orbit, spin-spin, and tensor potentials are measures of the accuracy of the wave functions generated with Richardson's potential. It is the purpose of this work to add that dimension to the experimental tests of Richardson's parametric form for the running coupling constant. We will also present results for the leptonic widths and the dipole transition rates of the Υ system and charmonium.

Some of the goals of our present calculation can be summarized as follows.

(1) To show that Gromes's consistency condition can be used to determine the nonperturbative spin-orbit potential regardless of the form of the central potential.

(2) To emphasize the importance of the fine-structure splittings as measures of the accuracy of the wave functions generated by Richardson's parametrization of the running coupling constant.

(3) To explore the question of whether the parameter Λ used in Richardson's parametrization of the running coupling constants plays the same role as that of the universal QCD scale parameter.

(4) To see whether including the spin-independent relativistic corrections to the central potential makes the calculation with Richardson's running coupling constant more or less successful.

II. THE POTENTIAL MODEL

Our calculations are based on a Hamiltonian that contains a kinetic energy operator K, a central potential $E(r)$ and the spin-dependent potential $V_{\rm SD}$; that is,

$$H = K + E(r) + V_{\rm SD}(r) . \tag{2.1}$$

We follow the spirit of earlier calculations done with nonrelativistic potential models[6,11,18] and keep only three terms in the expansion of the kinetic energy operator of a quark-antiquark pair in mutual orbit in their center-of-mass frame, namely,

$$K = 2m + p^2/m - p^4/4m^3 + \cdots . \tag{2.2}$$

The major part of our central potential is based on Richardson's interpolating form for the running coupling constant:

$$\alpha_s(|q^2|) = \frac{12\pi}{(33-2n_f)\ln(1+|q^2|/\Lambda^2)} . \tag{2.3}$$

This form has the interesting feature that it leads to a $|\mathbf{q}|^{-4}$ behavior for the potential at small spacelike momentum transfers, which may be interpreted as a linear confining potential, and the correct logarithmic dependence at large values of $|q^2|$. However, we have found that allowing the string constant A to be a parameter opens up the prospect of better agreement with experiment. Thus, our form for the central potential is

$$V(r) = Ar - \frac{8\pi}{(33-2n_f)r} f(\Lambda r) , \tag{2.4}$$

where[20]

$$f(t) \equiv \frac{4}{\pi}\int_0^\infty \frac{\sin tx}{x}\left[\frac{1}{\ln(1+x^2)} - \frac{1}{x^2}\right]dx . \tag{2.5}$$

By using contour deformation techniques in the complex plane, it is straightforward to verify that $f(t)$ may be written

$$f(t) = 1 - 4\int_1^\infty \frac{dy}{y}\frac{e^{-yt}}{\ln^2(y^2-1)+\pi^2} , \tag{2.6}$$

which is more convenient for the purposes of numerical computation.

From the form of Eq. (2.4), it is clear that the two leading terms in the large-r behavior of the potential are given by

$$V(r) = Ar - \frac{8\pi}{(33-2n_f)r} + \cdots . \tag{2.7}$$

The powers of r that appear in Eq. (2.7) are consistent with one's expectations based on the results of recent lattice gauge calculations.[4,28,29] However, the magnitude of the coefficient of the Coulomb term in Eq. (2.7) is 0.93 (assuming $n_f=3$), which is considerably different from coefficient (~ 0.24) obtained in these lattice gauge calculations. This difference is probably not a cause for serious concern since the lattice gauge calculations have been done in the quenched, or pure gauge, approximation where one does not consider the effects of vacuum polarization. In fact, using a model developed by Poggio and Schnitzer[30] to consider the effects of vacuum polarization, Olsson and Suchyta[31,32] have shown that it is reasonable to expect a modification of the coefficient of the Coulomb term by at least a factor of 2. Moreover, since the magnitude of the effect in the Poggio-Schnitzer model depends on the fourth power of the effective mass of the quark-antiquark pair in the vacuum polarization loop,[32] modification of the Coulomb coefficient of the pure gauge calculation by a factor larger than 2 is certainly possible. In this regard, we also note that the results of the recent lattice (pure) gauge calculation of Ding, Baillie, and Fox[33] suggest that the earlier pure gauge results for the inverse r term were a factor of 2 too small.

The central potential includes spin-independent (SI) corrections as well as our modified form of Richardson's potential, that is

$$E(r)=V(r)+V_{\rm SI}(r) , \tag{2.8}$$

To obtain the correct form for the SI corrections, one has to be careful to take the nonrelativistic limit (NRL) in a manner that is consistent with his/her assumptions about the nature of the confining potential. As stated above, we have assumed that the effects of confinement arise mainly from the color-electric degrees of freedom, including only those color-magnetic effects necessary to satisfy Gromes's consistency condition. Thus, we keep the parts of the SI correction that arise from the NRL of the time components of the Dirac spinors separate from those that arise from the space components. In fact, as Olsson and Miller[23] point out, to consistently incorporate the effects of a Lorentz invariant interaction in the center-of-mass frame, one must also consider the effects of small timelike momentum transfers in addition to the usual spacelike momentum transfers. Considering all of these factors, our form for the SI correction[22] is

$$V_{\rm SI}=\frac{\nabla^2 V}{4m^2}+\frac{1}{4m^2}\left[2(g-rg')p^2+2ig''\mathbf{r}\cdot\mathbf{p}+\tfrac{1}{2}\nabla^2(g+rg')+\frac{2g'}{r}L^2\right] , \tag{2.9}$$

where V is the potential of Eq. (2.4), L is the angular momentum operator, and g is the potential resulting from the exchange of transverse gluons. The momentum and angular momentum operators in Eq. (2.9) operate only to the right. The operators ∇^2 in Eq. (2.9) result from factors of the momentum transfer and thus operator only on the potentials. If we assume that lowest-order perturbation theory is adequate to treat the transverse degrees of freedom, then the expression for g is that from one-gluon exchange, that is, $-4\alpha_s/3r$ and the SI correction takes the form

$$V_{\rm SI}=\frac{\nabla^2 V}{4m^2}-\frac{4\alpha_s}{3m^2 r}p^2-\frac{4\alpha_s}{3m^2 r^2}\frac{\partial}{\partial r}+\frac{2\alpha_s}{3m^2 r^3}L^2 . \tag{2.10}$$

We have verified that Eqs. (2.9) and (2.10) agree with the relevant parts of Eq. (7) in Moxhay and Rosner[18] when allowance is made for the difference between their way of handling the transverse degrees of freedom and ours.

We assume that all of the relativistic corrections are small enough to be calculated with first-order perturbation theory. Thus, our unperturbed Hamiltonian is given by

$$H_0=2m+p^2/m+V(r) . \tag{2.11}$$

Following the procedure outlined in Ref. 32, this Hamiltonian is used as the basis of a numerical solution to the Schrödinger equation which generates unperturbed energies E_0 and wave functions ψ_0. The energies are then corrected with the expression from first-order perturbation theory, that is,

$$E(nLJ)=E_0(nL)+\langle JMLSn|H'|JMLSn\rangle , \tag{2.12}$$

where n is the radial quantum number, L is the orbital angular momentum, J is the total angular momentum, and S denotes the spin. The perturbation H' includes the spin-independent corrections of Eq. (2.10), the spin-dependent potential of Eq. (2.1) and the last term of Eq. (2.2).

In applying Eqs. (2.11) and (2.12) to the calculation of the energies and splittings of the Υ system, we will find it possible to obtain an excellent fit to the data without adding an arbitrary constant to the potential. However, our calculations of the properties of charmonium will not be as successful, and we will find it necessary to add an arbitrary constant to V to attempt to improve the agreement.

Finally, a comment about the limitations of using first-order perturbation theory. We cannot expect agreement with experiment to be any better than the size of the second-order perturbation theory corrections. Using the measured fine-structure splittings as a way of assessing the strength of the matrix elements and the spacing between levels as a means of estimating the energy denominators, we can make a rough estimate of the size of the second-order corrections. Hence,

$$\begin{aligned}\Delta E_b^{(2)}&\simeq(50\ \text{MeV})^2/500\ \text{MeV}=5\ \text{MeV} ,\\ \nabla E_c^{(2)}&\simeq(100\ \text{MeV})^2/500\ \text{MeV}=20\ \text{MeV} ,\end{aligned} \tag{2.13}$$

where the first result pertains to the Υ system and the second to charmonium. Although the approximations used to obtain these two numbers are very crude, they are nevertheless useful as rough guides to the kind of accuracy we may expect. Thus, whenever our results for the Υ system are within about 5 MeV of the experimental numbers, we shall speak of good agreement with experiment.

The 20-MeV result of Eq. (2.13) can be used for the same purpose in evaluating our calculated results for charmonium.

III. SPIN-DEPENDENT POTENTIALS

Using their approach for treating relativistic corrections, EF found[1] that the spin-dependent potential of a heavy Q-$\bar{Q}$ pair could be written as the sum of three terms, each of which contains a factor of m^{-2}, that is,

$$V_{\rm SD}=\frac{\mathbf{L}\cdot\mathbf{S}}{m^2 r}\left[\frac{1}{2}\frac{dV}{dr}+\frac{dV_1}{dr}+\frac{dV_2}{dr}\right]+\frac{\mathbf{S}_1\cdot\mathbf{S}_2}{3m^2}V_4 + \frac{1}{m^2}(\hat{\mathbf{r}}\cdot\mathbf{S}_1\hat{\mathbf{r}}\cdot\mathbf{S}_2-\tfrac{1}{3}\mathbf{S}_1\cdot\mathbf{S}_2)V_3 , \tag{3.1}$$

where $\mathbf{S}=\mathbf{S}_1+\mathbf{S}_2$, V_1 and V_2 are the spin-orbit potentials, and V_3 and V_4 carry the radial dependence of the spin-spin and tensor potentials. EF presented explicit expressions for each of the spin-dependent potentials in terms of expectation values of bilinear combinations of color-electric and -magnetic fields and the Wilson loop factor. In particular, the potentials V_3 and V_4 involve correlations of the color-magnetic field at different points in space and time, and the potential V_2 involves correlations of the color-electric field with the color-magnetic field at different points in space and time. However, the potential V_1 has a different qualitative behavior; its bilinear combination of color-field operators is evaluated at the same point in space. Thus, the assumption that all color-magnetic interactions are short ranged and amenable to treatment by perturbation theory is not a barrier which prevents the potential V_1 from developing a nonperturbative component. Indeed, the lattice gauge calculations of Michael, Campostrini, and co-workers[2-4] shows that only V_1 has a long-range part.

The central potential and spin-orbit potentials must satisfy Gromes's consistency condition

$$\frac{d}{dr}[V(r)+V_1(r)-V_2(r)]=0 . \tag{3.2}$$

In Refs. 6 and 7, I have argued that since only V and V_1 have nonperturbative parts and since Eq. (3.2) is satisfied by each order of perturbation theory,[34] Eq. (3.2) can be used to determine the nonperturbative part of V_1. However, this argument must be modified when Richardson's form is used for the running coupling constant because it is no longer clear how to carry out the separation into perturbative and nonperturbative parts. Thus, we use Gromes's relation to determine V_1 directly in terms of V and V_2, that is

$$V_1(r)=V_2(r)-V(r) , \tag{3.3}$$

and the spin-orbit contribution to Eq. (3.1) can be written

$$V_{\rm SO}=\frac{\mathbf{L}\cdot\mathbf{S}}{m^2 r}\left[2\frac{dV_2}{dr}-\frac{1}{2}\frac{dV}{dr}\right] . \tag{3.4}$$

Since all of the correlation functions appearing in the definitions of V_2, V_3, and V_4 involve two different points in space and at least one factor of the magnetic field, we will assume that perturbative QCD suffices to determine these potentials. Under this assumption the form of Eq. (3.4) makes it clear that the *nonperturbative effects in the spin-orbit potential contribute with a sign opposite to the perturbative effects.* Using the lowest-order expressions to consider the color-magnetic effects, we have that[1]

$$V_2=-\frac{4}{3}\frac{\alpha_s}{r},\quad V_3=\frac{4\alpha_s}{r^3} , \qquad V_4=2\nabla^2 V_2=\frac{32\pi}{3}\alpha_s\delta(\mathbf{r}) , \tag{3.5}$$

where α_s is the coupling constant that controls the sizes of the fine-structure and hyperfine-structure effects. Unlike the expression of Eq. (2.3), it does not run. It is the same coupling constant that appears in Eq. (2.10) since there we were careful to separate the effects of the magnetic degrees of freedom from those of the electric degrees of freedom. Thus, the spin-dependent potential is given by

$$V_{\rm SD}=\frac{\mathbf{L}\cdot\mathbf{S}}{m^2 r}\left[\frac{8}{3}\frac{\alpha_s}{r^2}-\frac{1}{2}\frac{dV}{dr}\right]+\frac{32\pi\alpha_s}{9m^2}\mathbf{S}_1\cdot\mathbf{S}_2\delta(\mathbf{r}) + \frac{4\alpha_s}{m^2 r^3}(\hat{\mathbf{r}}\cdot\mathbf{S}_1\hat{\mathbf{r}}\cdot\mathbf{S}_2-\tfrac{1}{3}\mathbf{S}_1\cdot\mathbf{S}_2) . \tag{3.6}$$

It is instructive to consider Eq. (3.6) in the OGE limit. Then the central potential is also given by the lowest-order perturbation theory limit, $V(r)\to-4\alpha_s/3r$. The matrix elements of both the spin-orbit and the tensor potentials are thus those of r^{-3}. In the OGE limit, the fine-structure ratio r, which is defined by

$$r=\frac{M(^3P_2)-M(^3P_1)}{M(^3P_1)-M(^3P_0)} , \tag{3.7}$$

is equal to 0.8, a universal constant for all P states. Because of the minus sign in the first parentheses of Eq. (3.6), long-range spin-orbit effects tend to reduce the size of the spin-orbit matrix element. The effect of this reduction is to give a value of r less than the OGE value. Thus, the observed values of r in charmonium and the Υ system find a very natural explanation in the destructive interference between the nonperturbative and perturbative spin-orbit contributions.

IV. RESULTS

Our calculation of the energies, leptonic widths, and dipole transition rates of the Υ system requires values for four parameters: Λ, m, α_s, and A. We first consider variations of Λ and A and concentrate on fitting the differences between the centers of gravity of the $1P$ and $2P$ states and the $1\,^3S_1$ state.[35] These energy differences also presuppose a value of the perturbative coupling constant α_s since this parameter determines the strength of the triplet-singlet splitting according to Eq. (3.6). We select a value of α_s which gives reasonable values for the fine-structure splittings of the P states. After choosing $\alpha_s=0.33$, we determined that $\Lambda=0.431$ GeV and that $A=0.159$ GeV^2. Then the mass was varied to obtain

agreement of the $1\,^3S_1$ state with the measured value of 9460 MeV. This yields $m=4.896$ GeV.

Our value of Λ is about 8% larger than Richardson's original value[36] of 398 MeV. Our value for A is about 10% smaller than the value one would expect from using the expression $8\pi\Lambda^2/(33-2n_f)$. Our value of Λ is about 6% smaller than that of Moxhay and Rosner.[18] The difference between our value and MR's can probably be traced to different ways of handling the effects of the transverse QCD degrees of freedom.

Our results for the energies of the Υ system are listed in the fifth column of Table I. There they are compared with the results of our earlier calculation than I did,[6] the results of Gupta, Repko, and Suchyta[13] (GRS), the results of Schmitz, Beavis, and Kaus[14] and with experiment. The agreement with experment is excellent. In no case does the difference between the calculated results and experiment exceed the 5-MeV criterion that we discussed at the end of Sec. II. The close agreement of the fine-structure splittings with experiment, and especially the agreement of the calculated values of r with the measured values, is an important indication of the validity of our treatment of the color-magnetic effects and our means of selecting the long-range part of the spin-orbit potential.

The comparison with my earlier calculation is of special interest. This earlier calculation was based on a linear confining potential and all of the perturbative diagrams to the full one-loop radiative level. There the expression for the running coupling constant included only the lowest-order vacuum-polarization corrections, that is,

$$\alpha_s(r)=\alpha_s\left[1-\frac{3\alpha_s}{2\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)(\ln\mu r+\gamma_E)\right], \tag{4.1}$$

where μ is the renormalization scale and γ_E is Euler's constant. Our present results show improvements over the full one-loop results in two important respects. Our present values for the fine-structure splittings are larger, and in the $1P$ case closer to experiment. The discrepancy of about 17 MeV in the $2\,^3S_1$ energy of the earlier calculation has been reduced to an acceptable level of about 4 MeV.

Some of the most pronounced differences between the two calculations arise from the values of the radial wave function at the origin. The present calculation yields a value of $|R(0)|^2=7.48$ GeV^3, which is to be compared with the earlier result $|R(0)|^3=5.00$ GeV^3. This 50% increase has important implications for the leptonic widths listed in the fifth column of Table II. These results were obtained with the formula[37,38]

$$\Gamma_{ee}=\frac{4\alpha^2e_Q^2}{M^2(Q\bar{Q})}|R(0)|^2\left[1-\frac{16\alpha_s}{3\pi}\right], \tag{4.2}$$

where α is the fine structure, e_Q is the quark charge, and M^2 denotes the mass of the Q-$\bar{Q}$ state that decays. The factor in the large parentheses arises from the radiative corrections and the value of the coupling constant α_s there requires some further consideration. We do not expect that its value should be the same as that describing the color-magnetic scatterings of Q-$\bar{Q}$ pairs, as in Eqs. (3.5) and (3.6), because the four momentum transferred in the annihilation is much larger and timelike. Thus, we choose $\alpha_s=0.18$ for Eq. (4.2), a value from the analysis of quarkonium decays.[38,39] Such a value reduces the leptonic width of Eq. (4.2) by about 30%. Leptonic widths from my earlier one-loop calculation are listed in the fourth column of Table II. These values were not corrected for the radiative correction of Eq. (4.2). Thus, including the radiative corrections would reduce these values about 30%. Since the radiative correction should

TABLE I. Energies of the low-lying S and P states of the Υ system (MeV). The parameters used for the present calculation are $\Lambda=0.431$ GeV, $A=0.159$ GeV^2, $\alpha_s=0.33$, and $m=4.896$ GeV.

State	SBK (Ref. 14)	Fulcher (Ref. 6)	GRS (Ref. 13)	Fulcher (present work)	Expt.
$1\,^3S_1$ (Υ)	9461	9460	9460	9461	9460.3±0.2 [a]
$1\,^1S_0$ (η_b)		9420	9412	9369	
$2\,^3S_1$	10023	10006	10016	10019	10023.3±0.3
$2\,^1S_0$		9983	9993	9975	
$3\,^3S_1$		10355	10358	10357	10355.3±0.5
$3\,^1S_0$		10336	10340	10324	
$1\,^3P_2$ (χ)	9908	9908	9914	9912	9913.2±0.6
$1\,^3P_1$	9880	9895	9894	9893	9891.9±0.7
$1\,^3P_0$	9849	9874	9862	9865	9859.8±1.3
$1\,^1P_1$ (h_b)		9901	9900	9900	
$2\,^3P_2$	10292	10268	10270	10270	10269.0±0.7
$2\,^3P_1$	10270	10256	10254	10254	10255.2±0.4
$2\,^3P_0$	10245	10239	10229	10232	10235.3±1.1
$2\,^1P_1$		10262	10259	10261	
r_{1P}	0.90	0.63	0.65	0.67	0.66
r_{2P}	0.88	0.63	0.65	0.70	0.69

[a]Particle Data Group (Ref. 17).

TABLE II. Leptonic widths of the low-lying S states of the Υ system (keV).

State	MR (Ref. 18)	GRS (Ref. 13)	Fulcher (Ref. 6)	Fulcher (present work)	Expt.
$\Upsilon(1S)$		1.21	1.32[a]	1.37	1.34±0.05[b]
$\Upsilon(2S)$	$\left\lvert\frac{2S}{1S}\right\rvert^2=0.41$	0.55	0.65[a]	0.58	0.60±0.04
$\Upsilon(3S)$	$\left\lvert\frac{3S}{1S}\right\rvert^2=0.29$	0.41	0.50[a]	0.41	0.44±0.03

[a]Does not include the radiative correction.
[b]Particle Data Group (Ref. 17).

be included in the leptonic width formula, a comparison of columns 4 and 5 of Table II allows one to assess the relative merits of the modified Richardson's potential of Eq. (2.4) and the potential used in Ref. 6 in producing reasonable values for the wave function near the origin. The measured leptonic widths give a clear preference for the results obtained with the modified Richardson's potential. For the purposes of comparison, leptonic widths from the recent GRS calculation and MR's calculation are also listed in Table II.

The rather large difference in the values of $|R(0)|^2$ between these two calculations can mostly be traced to a problem with the running coupling constant of Eq. (4.1). As one would expect, the expression in the large parentheses there softens the coupling constant whenever the factor $(\ln\mu r+\gamma_E)$ is negative. However, for very small values of r, the coupling constant is softened too much, or "oversoftened" as I have said earlier,[6,32,40] because the expression in large parentheses can change sign. In the region of oversoftening, the perturbative central potential becomes repulsive and reduces the wave function at the origin. Clearly, the change of sign in Eq. (4.1) is an artifact of the use of the lowest-order vacuum-polarization correction because it does not occur when one includes the next term or when one uses a geometric series to sum all the contributions of the vacuum-polarization bubble graphs.

Our result for the Υ energies presented in Table I agree with experiment as well as those of Gupta, Radford, and Suchyta,[13] whose work was built upon a tradition of excellent agreement between theory and experiment.[41,42] Our results for the leptonic widths are slightly closer to the measured values than those of GRS. It is instructive to compare the theoretical underpinnings of GRS's latest calculation with those of the present work. Concentrating on the effects of hard gluons, these authors were able to eliminate some of the troublesome singular effects of the perturbative potentials by considering improvements to the usual quasistatic approximation.[43,44] Since their improvement of the quasistatic approximation is based on observations about the softening of the running coupling constant at high-momentum transfers, GRS's recent calculation includes some of the effects of higher-order renormalization-group improvements of the running coupling constant. In this sense, the improvements of their most recent calculation over some of their earlier work are quite similar to our own progress in going from the running coupling constant of Eq. (4.1) to that of Eq. (2.3). Our results for the fine-structure splittings are somewhat smaller than those of Schmitz, Beavis, and Kaus[14] and are closer to the measured values than theirs. Our predictions for the locations of the $1D$ states are listed in Table III, where they are compared with results of Kwong and Rosner[38] and of GRS.

Our results for the dipole transition rates presented in Table IV were obtained with the expressions

$$\Gamma(^3S_1\rightarrow{}^3P_J)=\tfrac{4}{9}\frac{2J+1}{3}\alpha e_Q^2\omega_{\rm fi}^3|\langle f|r|i\rangle|^2\ , \tag{4.3}$$

$$\Gamma(^3P_J\rightarrow{}^3S_1)=\tfrac{4}{9}\alpha e_Q^2\omega_{\rm fi}^3|\langle f|r|i\rangle|^2\ , \tag{4.4}$$

where $\omega_{\rm fi}$ denotes the energy difference of the initial and final state. Agreement with experiment for the $3S\rightarrow1P$ and the $3S\rightarrow2P$ transitions is very good, and agreement with the $2S\rightarrow1P$ transitions is satisfactory since the calculated results lie within 2 standard deviations of the measured values. Our results for the matrix elments and transition rates are also compared with those of Kwong and Rosner[38] and of GRS. The excellent agreement with the work of Kwong and Rosner is very gratifying because their wave functions were obtained with a potential based on the inverse-scattering method. Our results for the $2S\rightarrow1P$ and the $3S\rightarrow2P$ transitions are somewhat closer to the measured values than those of Gupta, Radford, and Suchyta.[13]

In an attempt to see if we could find independent experimental support for the form of the SI relativistic correction in Eq. (2.10), we also carried out a calculation of the properties of the Υ system without including this term. The parameter set determined in this manner did not differ too much from that used to obtain the results of Tables I–IV; that is, the new values are $\Lambda=0.483$ GeV, $A=0.147$, GeV^2, $\alpha_s=0.33$, $m=4.900$ GeV. Our results obtained with this set of parameters were very close to those presented in Tables I, II, and IV. For example, in most cases the energies did not differ by more than 1 or 2 MeV. An exception was the $3\,^3S_1$ energy, where the difference with experiment was 9 MeV, somewhat larger than the 2-MeV difference in Table I. Thus, from the perspective of agreement with experiment our results

TABLE III. Predictions for D-state energies (MeV).

State	KR (Ref. 38)	GRS (Ref. 13)	Fulcher (present work)	Expt.
$1\,^3D_3$	10 160	10 163	10 172	?
$1\,^3D_2$	10 156	10 153	10 169	
$1\,^3D_1$	10 150	10 141	10 163	
$1\,^1D_2$	10 157	10 154	10 169	
r_{1D}	0.71	0.96	0.55	

without the SI corrections of Eq. (2.10) are almost as good as our results with it. Nevertheless, we feel that the SI corrections should be included because their theoretical foundation is solid.[23]

To investigate the prospects of flavor independence in the potentials we first did a calculation of the energies and leptonic widths of the charmonium system with the same potential parameters as those used for the Υ system. This calculation encountered several problems. For example the fine-structure splittings were about a factor of 2 too small, the leptonic widths were about 30% too large and the energies of the $2S$ and $1P$ states were about 50 MeV too low. We were able to take care of this set of problems by readjusting parameters. Increasing α_s to 0.50 cleared up the problem of the size of the fine structure. We were able to move both the $2S$ and $1P$ energies closer to experiments by increasing the string constant A to 0.190 GeV^2. To solve the leptonic width problem we were forced to reduce the charmonium mass to $m_c=1.30$ GeV. Thus, we had to add an arbitrary constant of 310 MeV to the central potential. However, we were able to effect these improvements while keeping Λ fixed at 0.431 GeV. Using the same value of Λ for both the charmonium and Υ systems represents a significant accomplishment because it is consistent with the existence of a universal QCD scale.[21]

Our results for the charmonium energies and leptonic widths are presented in Tables V and VI. All of the numbers presented in Table V meet our criterion of 20 MeV discussed at the end of Sec. II. In particular the agreement of the calculated fine-structure ratio r with experiment is very significant because it supports our argument about the destructive interference between the long-range spin-orbit potential and the OGE spin-orbit potential. It is also understandable why our calculated leptonic widths might be slightly larger than the experimental results. These were based on Eq. (4.2) with $\alpha_s=0.18$ in the radiative correction as in the Υ case. Since the momentum transferred in charmonium decay is substantially less than in Υ decay, one should expect a larger value of α_s to be appropriate. A 15% increase in α_s would decrease the leptonic widths about 7%. This reduction would place the $1S$ result at 5.00 keV, in agreement with experiment. However, our use of the value 0.50 for α_s in the color-

TABLE IV. Dipole matrix elements and electromagnetic transition rates for the Υ system. All transitions are between triplet states.

	$\langle r\rangle$ (GeV^{-1})		Γ (keV)			
Transition	KR (Ref. 38)	Fulcher (present work)	KR (Ref. 38)	GRS (Ref. 13)	Fulcher (present work)	Expt.
$2S_1\rightarrow 1P_2$			2.14	1.86	2.07	2.90±0.59 [a]
$1P_1$	−1.646	−1.623	2.18	1.68	2.11	2.96±0.60
$1P_0$			1.39	0.74	1.35	1.89±0.44
$3S_1\rightarrow 1P_2$			0.03	0.36 (?)	0.03	0.06±0.05 [b]
$1P_1$	0.023	0.023	0.02	0.05	0.02	0.04±0.03
$1P_0$			0.01	0.01	0.01	
$3S_1\rightarrow 2P_2$			2.78	2.20	2.64	3.33±0.8 [c]
$2P_1$	−2.672	−2.628	2.52	2.08	2.46	3.12±0.7
$2P_0$			1.65	1.03	1.41	1.25±0.3
$1P_2\rightarrow 1S_1$			37.8	31.6	36.4	
$1P_1$	1.098	1.079	32.8	28.4	31.6	
$1P_0$			26.1	25.1	25.2	
$2P_2\rightarrow 1S_1$			9.75	9.5	9.81	
$2P_1$	0.240	0.242	9.31	6.6	9.35	
$2P_0$			8.48	3.1	8.70	
$2P_2\rightarrow 2S_1$			18.7	14.7	18.1	
$2P_1$	1.911	1.870	15.9	13.0	15.3	
$2P_0$			11.3	11.2	11.8	

[a]Based on a total $\Upsilon(2S)$ width of 44±9 keV and the branching ratios of Ref. 17.
[b]From Ref. 39.
[c]Based on a total $\Upsilon(3S)$ width of 26±6 keV and the branching ratios of Ref. 27.

TABLE V. Energies of the low-lying S and P states of the charmonium system (MeV). The parameters used in the present calculation are $\Lambda=0.431$ GeV, $A=0.190$ GeV^2, $\alpha_s=0.50$, and $m_c=1.30$ GeV. c.o.g. stands for center of gravity.

State	SBK (Ref. 14)	GRS (Ref. 13)	Fulcher (present work)	Expt.
$\psi_{\text{c.o.g.}}(1S)$	3076	3068	3074	3067.5±1.7 [a]
$\psi'_{\text{c.o.g.}}(2S)$		3672	3651	3663±? [a,b]
$1\,^3P_2$ (χ)	3561	3554	3561	3556.3±0.4
$1\,^3P_1$	3490	3507	3506	3510.6±0.5
$1\,^3P_0$	3412	3412	3407	3415.1±1.0
$1\,^1P_1$ (h_c)		3519	3525	
r_{1P}	0.91	0.49	0.56	0.48

[a]Determined from the spin averages of the triplet and singlet states.
[b]Based on the value of 3594.0 for the η_c state as listed in Ref. 45.

magnetic scattering potentials of Eqs. (3.5) and (3.6) did lead to a problem with the hyperfine splittings, where our calculated values are about 50–70 % too large. It may be that our decision to use a single coupling constant for all the magnetic potentials is at fault here. Perhaps if one allows the coupling constant to run to a smaller value for the spin-spin potential this problem would be solved. Of course, such a proposal would also have an effect on the hyperfine splittings in the Υ system.

Our results for the dipole transition rates of the charmonium system are shown in Table VII. The matrix elements used to determine these numbers are also presented here. Results from Moxhay and Rosner's calculation and GRS's calculation are also listed. The discrepancy between our calculated results and experiments is a factor of 2 or so in most cases, which represents the biggest problem for our calculation. Removal of this discrepancy will probably require some new physics since the matrix elements listed in Table VII do not change radically as the parameters are varied by reasonable amounts. Relativistic corrections to the dipole formulas of Eqs. (4.3) and (4.4), which have been studied by Grotch and coworkers,[46–48] present on opportunity to remove this discrepancy.

V. CONCLUSIONS

Our calculation of the energies, fine-structure splittings, leptonic widths, and dipole transition rates has been very successful. Since a linear confining potential and Richardson's form for the running coupling constant are the main ingredients used to generate the wave functions, the agreement with experiment is impressive support for both. In particular, the agreement of our calculated values for the leptonic widths of the Υ and charmonium S states with experiment shows that our modified form of Richardson's potential generates good values for these wave functions at the origin, since the radiative corrections are included in the leptonic width formulas. Using Gromes's consistency condition and arguments based on lattice gauge theory, we have determined the long-range spin-orbit potential and demonstrated that it has a sign opposite to the leading perturbative spin-orbit potential. Convincing experimental support for this destructive interference is garnered from the fine-structure ratios of both the charmonium and upsilon P states.

We have carried out an extensive comparison with an earlier calculation of the properties of the Υ system that was based on a linear confining potential and all of the perturbative diagrams of QCD to the full radiative one-loop level. In most respects, the results obtained with Richardson's interpolating form for the running coupling constants are superior. In particular, the leptonic widths, the magnitudes of the fine-structure splittings, and the spacing between the ^3S states are all in better agreement with experiment.

Our calculation of the properties of charmonium,

TABLE VI. Leptonic widths of the low-lying S states of charmonium (keV).

State	MR (Ref. 18)	SBK (Ref. 14)	GRS (Ref. 13)	Fulcher (present work)	Expt.
$1S$			5.57	5.34	4.72±0.35 [a]
$2S$	$\left[\frac{2S}{1S}\right]=0.44$	$\left[\frac{2S}{1S}\right]=0.50$	2.87	2.56	2.15±0.21

[a]Particle Data Group (Ref. 17).

TABLE VII. Dipole matrix elements and electromagnetic transition rates for charmonium. All transitions are between triplet states.

Transition	$\langle r \rangle$ (GeV^{-1})	Γ (keV) MR (Ref. 18)	GRS (Ref. 13)	Fulcher (present work)	Expt.
$2S_1 \rightarrow 1P_2$		41	25	35	19.0 ± 3.4 [a]
$1P_1$	−2.583	48	29	52	21.1 ± 3.7
$1P_0$		37	20	64	22.6 ± 4.0
$1P_2 \rightarrow 1S_1$		609	326	633	351^{+162}_{-122}
$1P_1$	2.128	460	250	462	< 355
$1P_0$		226	117	210	95 ± 37

[a]Particle Data Group (Ref. 17).

where relativistic effects are expected to be more important scored several important successes but also encountered some problems. The magnitudes of the hyperfine splittings were 50–70 % too large. These splittings depend on the values of the radial functions at the origin. Since the agreement of our calculated values of the leptonic widths with the experimental values is an indication of the accuracy of these wave functions, it is likely that the resolution of this discrepancy will require the consideration of higher-order perturbative effects. The dipole electromagnetic rates were too large, and the resolution of this problem will probably require some new physics since most of the dipole matrix elements are not very sensitive to reasonable changes in the values for the parameters.

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