PHYSICAL REVIEW D VOLUME 42, NUMBER 7 1 OCTOBER 1990

Angular distribution of photons in the parity-changing one-photon radiative decays of quarkonia

M. A. Doncheski and H. Grotch
Department of Physics, The Pennsylvania State University, 104 Davey Laboratory, University Park, Pennsylvania 16802

K. J. Sebastian
Department of Physics and Applied Physics, University of Lowell, One University Avenue, Lowell, Massachusetts 01854
(Received 13 October 1989; revised manuscript received 4 December 1989)

We derive a general expression, correct to order v^2/c^2 and valid in any potential model, for the angular distribution of photons in the parity-changing one-photon radiative decays of quarkonia, in terms of its different multipole contributions $E1$, $M2$, and $E3$. The expression is given in terms of Clebsch-Gordan coefficients and six reduced matrix elements. We make detailed numerical calculations for the radiative decays of charmonium using Buchmüller-Tye and Gupta-Radford-Repko potential models and find that there is not much difference in the angular distributions between the two models. It is interesting to note that the $E3$ contributions to the decays $\psi' \to \chi_2 + \gamma$ and $\chi_2 \to \psi + \gamma$ are nonvanishing only if ψ' and ψ have admixtures of D states in them.

I. INTRODUCTION

The electric dipole ($E1$) radiative decays of charmonium, and also of the $b\bar{b}$ system, have been studied extensively, both theoretically[1–6] and experimentally,[7–15] but so far most of the work has focused on the decay rates and the branching ratios of these decays. The general conclusion of the many theoretical papers[3–6] is that the relativistic corrections to the $E1$ decay rates are important, and that they tend to depress the nonrelativistic rates by a considerable fraction. More information on the potential, the states, and the $E1$ transition operator can be obtained by looking at the angular distribution of the photons in the radiative decays. In the nonrelativistic limit, the transition operator involved is a pure $E1$ operator, and it gives a characteristic electric dipole angular distribution which depends only on the initial and final angular momentum states. It is completely independent of the dynamics and in particular on the potential used. But when first-order relativistic corrections to the radiative transition operator are introduced, the transition amplitude involved is not pure $E1$, but a coherent mixture of $E1$, $M2$, and $E3$ amplitudes. As a result the angular distributions change considerably. The relative strengths of these $M2$ and $E3$ parts in the transition amplitude, and hence in the angular distribution, depend on the dynamics, in particular on the potential used and how the initial and final states are constructed. One of the challenging problems in charmonium physics has been the explanation of the rather large leptonic annihilation rate of ψ'' (3769.9 MeV). One way to solve the problem is to introduce D-state mixing in the wave function of ψ' (which is supposed to be predominantly $2S$) and S-state mixing into the wave function of ψ'' (which is supposed to be predominantly $1D$) (Ref. 3). Such mixing is, of course, theoretically possible, especially if the $2S$ and $1D$ eigenstates of the nonrelativistic charmonium Hamiltonian are nearly degenerate. However, it is very difficult to predict the exact mixing coefficients since the $1D$-$2S$ nonrelativistic energy difference depends sensitively on the potential model used. We find that when we introduce the relativistic corrections to the one-photon transition operator, the angular distribution of the photon in the ψ' and ψ'' radiative decays to χ_J depend on the D- and S-state mixing coefficients. It is especially interesting to note that the $E3$ multipole transition amplitude of $\psi' \to \chi_2 + \gamma$ will be zero if ψ' has no D-state mixing. So the $E3$ amplitude in $\psi' \to \chi_2 + \gamma$ is directly proportional to the D-state mixing coefficient. In the radiative decay $\psi' \to \chi_1 + \gamma$ there can be no $E3$ amplitude because of angular momentum conservation. By the same reasoning, $\psi' \to \chi_0 + \gamma$ is pure $E1$. The interesting question now is whether the same mixing coefficients which explain the leptonic annihilation rate will also explain the angular distributions of the photons. Another interesting aspect of the study of angular distributions is that any deviation from the pure $E1$ distribution is a hint that the relativistic corrections to the $E1$ transition operator are important. A precise determination of these angular distributions will give us an estimate of the relative strengths of these relativistic corrections.

In the past, several authors[16–18] have studied the question of angular distributions in the $E1$ radiative decays of charmonium. But their studies have been mostly on the kinematics of the problem, where the coefficients of the higher multipole contributions to the angular distribution are treated as free parameters and are not related to the dynamics of the problem. Here we show how these higher multipoles will arise naturally when we introduce relativistic corrections to the interactions of the radiation field with the quarkonium. We also express these coefficients in terms of radial integrals involving the radial wave functions of the charmonium states, and numerically evaluate these coefficients for two potential models: namely, the Buchmüller-Tye potential[2] and the Gupta-Repko-Radford (GRR) potential.[6] We have plotted these angular distributions for various mixing coefficients of D and S states.

The format of the rest of the paper is as follows. In Sec. II we give the outline of our calculation and give a general result for the angular distribution of any $E1$ one-photon transition (correct to first order in relativistic corrections) in terms of Clebsch-Gordan (CG) coefficients and various reduced matrix elements, which is valid for all initial and final angular momentum states. It should be noted that we give angular distributions after summing over final photon polarizations and the final spin states of charmonium. In Sec. III we give the details of the numerical calculations for the radiative decays in charmonium. The reduced matrix elements are first expressed in terms of four radial integrals which can be calculated once the potential is specified. We also plot the numerical results on the angular distribution for various decays and for different mixing coefficients in Figs. 1–9. Finally, in Sec. IV we make some concluding remarks.

II. OUTLINE OF THE CALCULATION

We begin by writing down the odd-parity one-photon transition amplitude in the c.m. frame, correct to order v^2/c^2. Using the result of Ref. 19 we have

$$T_{\mathrm{odd}}(t_0)=\frac{c}{\sqrt{V}}\left[\frac{2\pi}{\omega}\right]^{1/2}\left[\int_0^{t_0} e^{i(\omega-\omega_{AB})t}dt\right]k_0\left\{\hat{\boldsymbol{\epsilon}}_\alpha\cdot\left\langle A\left|\sum_{\mu=1}^{2} e_\mu \boldsymbol{\rho}_\mu\right|B\right\rangle + \hat{\boldsymbol{\epsilon}}_\alpha\cdot\left\langle A\left|-i\sum_{\mu=1}^{2}\frac{e_\mu}{4m_\mu c}[\boldsymbol{\pi}_\mu(\mathbf{k}\cdot\boldsymbol{\rho}_\mu)^2+(\mathbf{k}\cdot\boldsymbol{\rho}_\mu)^2\boldsymbol{\pi}_\mu]\right|B\right\rangle + \hat{\boldsymbol{\epsilon}}_\alpha\cdot\left\langle A\left|\sum_{\mu=1}^{2}\left[\frac{-ie_\mu}{m_\mu c}(\mathbf{k}\cdot\boldsymbol{\rho}_\mu)(\boldsymbol{\sigma}_\mu\times\mathbf{k})-\frac{ke_\mu}{2m_\mu^2c^2}(\boldsymbol{\sigma}_\mu\times\boldsymbol{\pi}_\mu)\right]\right|B\right\rangle\right\}, \quad (1)$$

where $\omega=|\mathbf{k}|$ is the photon energy, $\mathbf{k}(\hat{\boldsymbol{\epsilon}}_\alpha)$ the photon momentum (polarization) three-vector, $\omega_{AB}=k_0$ the measured energy difference, and e_μ, $\boldsymbol{\rho}_\mu$, m_μ, $\boldsymbol{\sigma}_\mu$, $\boldsymbol{\pi}_\mu$ are the charge, position, mass, spin, and momentum of the fermionic components of the bound state in the c.m. frame.

The derivation of Eq. (1), which is somewhat involved, is given in Ref. 19. To arrive at this equation we have used the relativistic c.m. and internal variables of Krajcik and Foldy.[20] The states $|A\rangle$ and $|B\rangle$ are eigenstates of the *internal* Hamiltonian[19,20] correct to order v^2/c^2. The interaction Hamiltonian used in first-order perturbation theory to derive Eq. (1) was generated by the minimal replacement $p_i \to [p_i-(e_i/c)\mathbf{A}_i]$ in the Hamiltonian of the isolated quarkonium along with the addition of certain spin-dependent terms. These spin-dependent terms are dictated by the requirement that when the internal interaction goes to zero, the Hamiltonian of the quarkonium in the presence of an external electromagnetic (e.m.) field should reduce to the sum of two Foldy-Wouthuysen reduced Hamiltonians for single Dirac particles in the same external fields. The resulting Schrödinger equation is not only gauge invariant but it leads to one- and two-photon transition amplitudes which satisfy all the correct Lorentz-transformation conditions[19] to order v^2/c^2. All the terms obtained by the minimal replacement which are linear in $\mathbf{A}_i$ can be written as $\mathbf{A}_i\cdot\partial H/\partial\mathbf{p}_i = -i\,\mathbf{A}_i\cdot[\mathbf{r}_i,H]$. These terms include the bulk of relativistic corrections in the interaction Hamiltonian H_I when the quarkonium Hamiltonian H includes leading relativ-

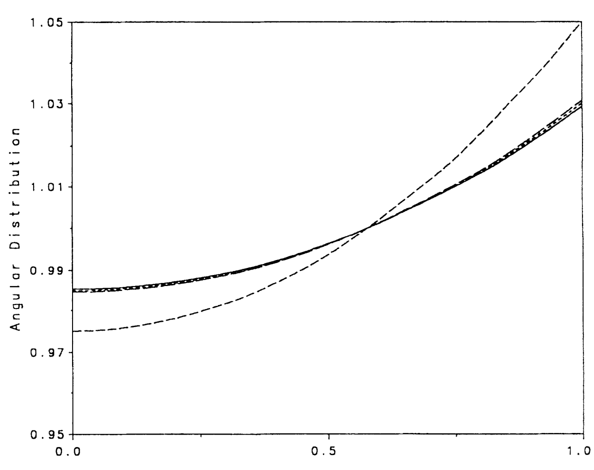

FIG. 1. Angular distribution for $\psi'\to\chi_2+\gamma$, with $M=1$. Solid line is -10% D-state mixing; short dashed line is -5% D-state mixing; medium dashed line is pure $E1$; long dashed line is 0% D-state mixing.

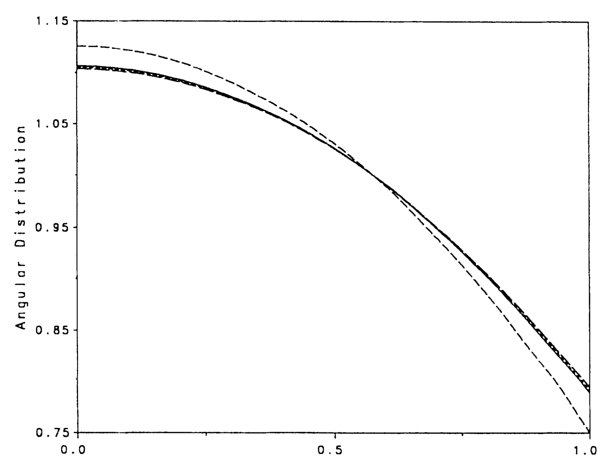

FIG. 2. Same as Fig. 1, for $\psi'\to\chi_1+\gamma$, with $M=1$.

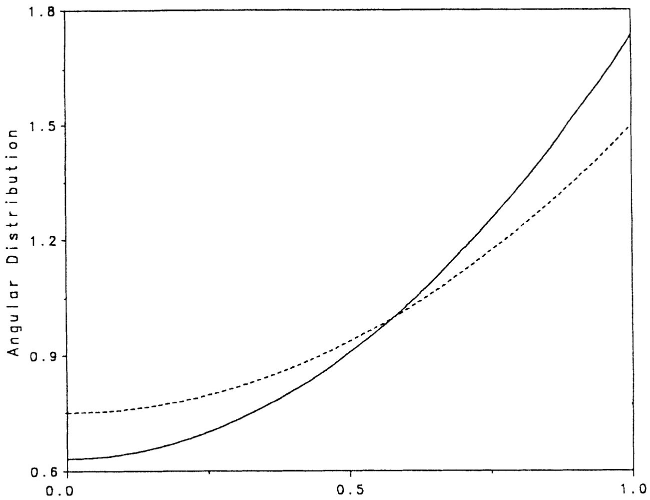

FIG. 3. Angular distribution for $\chi_2 \to \psi + \gamma$, with $M=2$. Solid line is -1.5% D-state mixing; dashed line is pure $E1$.

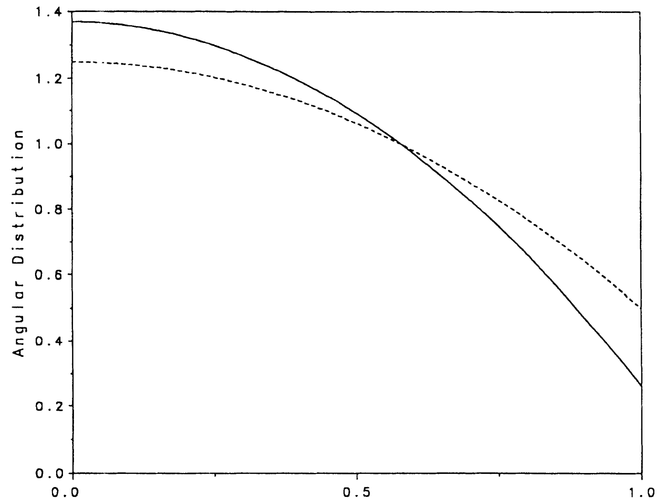

FIG. 5. Same as Fig. 3, for $\chi_2 \to \psi + \gamma$, with $M=0$.

istic corrections. After expressing the exponential $e^{i\mathbf{k}\cdot\mathbf{r}_i}$ in $\mathbf{A}_i$ in terms of relativistic internal and c.m. variables[20] we expand $e^{i\mathbf{k}\cdot\boldsymbol{\rho}_i}$ in powers of $(\mathbf{k}\cdot\boldsymbol{\rho}_i)$ and keep only terms up to order $(\mathbf{k}\cdot\boldsymbol{\rho}_i)^2$. The symbol $\boldsymbol{\rho}_i$ represents the relativistic internal position operator as defined by Krajcik and Foldy.[19,20] Since the matrix element of $(\mathbf{k}\cdot\boldsymbol{\rho}_i)$ is of order v/c between bound states, this expansion is consistent in our approximately relativistic scheme. The first term in Eq. (1) comes from the above-mentioned commutator term after we place $e^{i\mathbf{k}\cdot\boldsymbol{\rho}_i}$ by 1. The second term in Eq. (1) also comes from $\mathbf{A}_i\cdot[\mathbf{r}_i,H]$, but here we keep the second-order term in the expansion of the exponential. Finally, the third term of Eq. (1) is from the spin-dependent terms in the interaction Hamiltonian H_I mentioned above. Two things must be noticed about Eq. (1). First, a large part of relativistic corrections will be contained in the first term when we use relativistically corrected eigenstates $|A\rangle$ and $|B\rangle$ in the matrix element. Second, we have only included the parity-changing terms of H_I in the matrix elements of Eq. (1) as we are interested only in the parity-changing one-photon transitions.

Now we specialize Eq. (1) to the quarkonium problem. As we are dealing with bound states of a particle and its antiparticle, we let $e_1=-e_2=e_q$ and $m_1=m_2=m_q$. Also, in this case, $\boldsymbol{\rho}_1=-\boldsymbol{\rho}_2=\mathbf{r}/2$ and $\boldsymbol{\pi}_1=-\boldsymbol{\pi}_2=\mathbf{p}$. Then (we let $c=1$)

$$
\begin{aligned}
T_{\text{odd}}(t_0) &= \frac{1}{\sqrt{V}}\left[\frac{2\pi}{\omega}\right]^{1/2} k_0\hat{\epsilon}_\alpha\cdot\langle A|\mathbf{T}_0+\mathbf{T}_1|B\rangle \\
&\quad\times\int_0^{t_0} e^{i(\omega-\omega_{AB})t}\,dt\,, \\
\mathbf{T}_0 &= e_q\mathbf{r} = \text{electric dipole operator}\,, \\
\mathbf{T}_1 &= -\frac{ie_q}{4m_qk}[(\mathbf{k}\cdot\mathbf{r})^2\mathbf{p}-i(\mathbf{k}\cdot\mathbf{r})\mathbf{k} \\
&\quad +2(\mathbf{k}\cdot\mathbf{r})(\mathbf{S}\times\mathbf{k})-k^2(\mathbf{S}\times\mathbf{r})]\,, \\
\mathbf{S} &= \boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2\,.
\end{aligned}
\tag{2}
$$

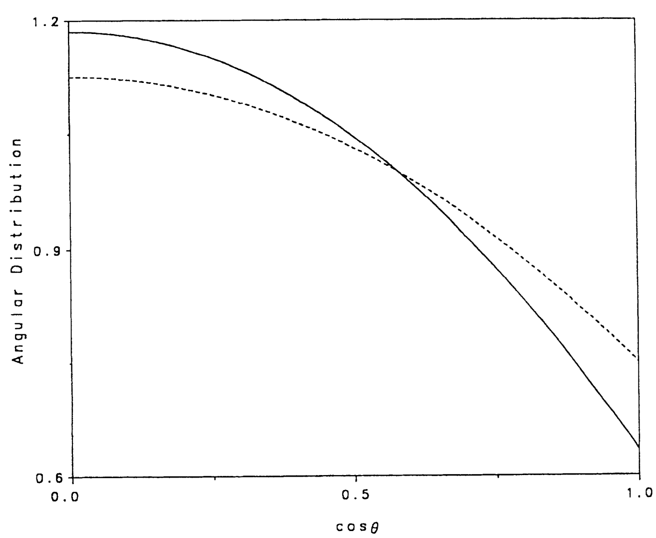

FIG. 4. Same as Fig. 3, for $\chi_2 \to \psi + \gamma$, with $M=1$.

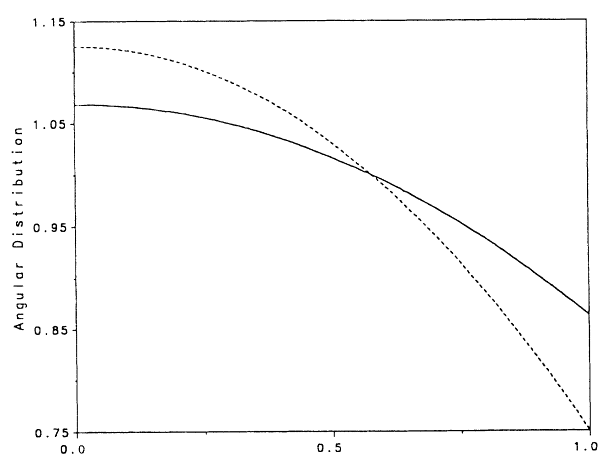

FIG. 6. Same as Fig. 3, for $\chi_1 \to \psi + \gamma$, with $M=1$.

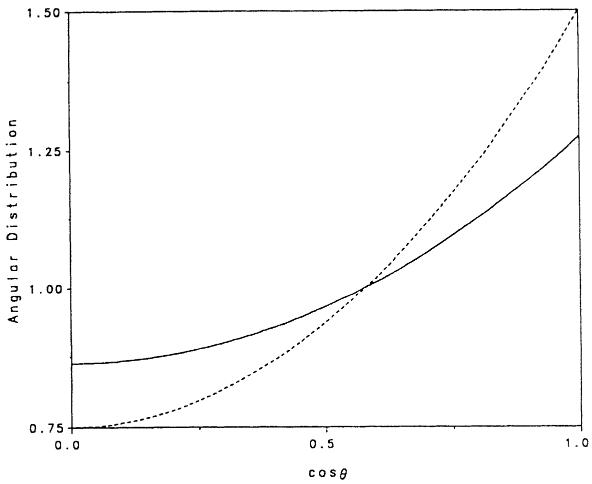

FIG. 7. Same as Fig. 3, for $\chi_1 \to \psi + \gamma$, with $M=0$.

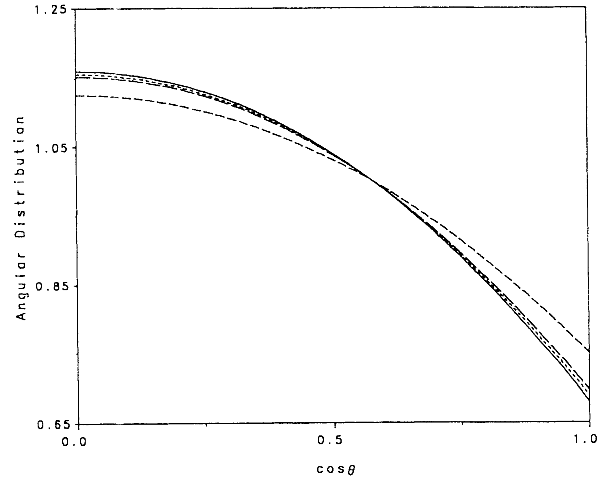

FIG. 9. Same as Fig. 8, for $\psi'' \to \chi_1 + \gamma$, with $M=1$.

Because we are dealing with particles of equal mass and opposite charge, only $\sigma_1 + \sigma_2$ terms survive in the odd-parity transition operator. Finally, after summing over the photon polarization, the term of interest is the angular distribution of the photons $d\Gamma/d\Omega$:

$$\frac{d\Gamma}{d\Omega} \propto \left[\langle A|\mathbf{T}_0|B\rangle^* \cdot \langle A|\mathbf{T}_0|B\rangle - \frac{1}{k^2}\langle A|\mathbf{k}\cdot\mathbf{T}_0|B\rangle^* \langle A|\mathbf{k}\cdot\mathbf{T}_0|B\rangle \right.$$
$$\left. + 2\,\mathrm{Re}\left[\langle A|\mathbf{T}_0|B\rangle^* \cdot \langle A|\mathbf{T}_1|B\rangle - \frac{1}{k^2}\langle A|\mathbf{k}\cdot\mathbf{T}_0|B\rangle^* \langle A|\mathbf{k}\cdot\mathbf{T}_1|B\rangle \right] \right] . \tag{3}$$

We can reexpress components of all three-vectors in the above expression in terms of irreducible vector operators:

$$A_+ = -\frac{1}{\sqrt{2}}(A_x + iA_y) ,$$
$$A_- = \frac{1}{\sqrt{2}}(A_x - iA_y) , \tag{4}$$
$$A_3 = A_z .$$

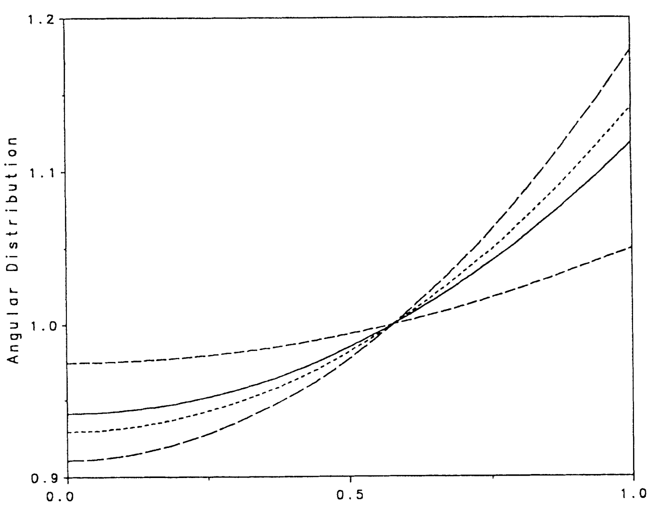

FIG. 8. Angular distribution for $\psi'' \to \chi_2 + \gamma$, with $M=1$. Solid line is 10% S-state mixing; short dashed line is 5% S-state mixing; medium dashed line is pure $E1$; long dashed line is 0% S-state mixing.

On doing this we note that all terms in Eq. (3) can be expressed in terms of eight irreducible tensors whose highest-weight components are given by

$$T_{33} = x_+^2 p_+ , \quad T_{22} = x_+ (\mathbf{r}\times\mathbf{p})_+ ,$$
$$T_{11} = (\tfrac{2}{5})^{1/2} r^2 p_+ , \quad T'_{11} = (\tfrac{2}{5})^{1/2} x_+ (\mathbf{r}\cdot\mathbf{p}) ,$$
$$X_{11} = x_+ , \quad S_{22} = x_+ S_+ , \tag{5}$$
$$S_{11} = (\mathbf{r}\times\mathbf{S})_+ , \quad S_{00} = \mathbf{r}\cdot\mathbf{S} .$$

Other components of each tensor operator can be obtained from commutation with J_- by the relation

$$[J_-, A_{k.q}] = \sqrt{k(k+1)+q(1-q)}\, A_{k,q-1} . \tag{6}$$

The states $|A\rangle$ and $|B\rangle$ are eigenstates of the angular momentum operators and they can be labeled as $|\beta' J' M'\rangle$ and $|\beta J M\rangle$, respectively. Furthermore, by use of the Wigner-Eckart theorem we can express matrix elements of the above tensors as

$$\langle \beta' J' M' | A_{kq} | \beta J M \rangle = \langle J' M' | kq ; JM \rangle \langle \beta' J' \| A_k \| \beta J \rangle , \tag{7}$$

where $\langle \beta' J' \| A_k \| \beta J \rangle$ is the reduced matrix element which does not depend on M or M'.

We present the angular distribution of photons after summing over the magnetic quantum number M' of the final-quarkonium state and when the magnetic quantum number M of the initial-quarkonium state is specified. After a lengthy bit of algebra, the general form of the angular distribution of photons in the decay of quarkonia, resulting from Eq. (3), is found to be

$$\frac{d\Gamma}{d\Omega} \propto \mathrm{Re}\sum_{M'} \Bigg\{ \langle \beta' J' \| X_1 \| \beta J \rangle^* \left\langle \beta' J' \left\| \left[X_1 + \frac{ik}{2m\sqrt{10}}(-2T_1+T_1') \right] \right\| \beta J \right\rangle$$
$$\times \left[\frac{\cos^2\theta+1}{2}(\langle J'M'|11;JM\rangle^2 + \langle J'M'|1-1;JM\rangle^2) + (1-\cos^2\theta)\langle J'M'|10;JM\rangle^2 \right]$$
$$- \frac{ik}{2m}\langle \beta' J' \| X_1 \| \beta J \rangle^* \left\langle \beta' J' \left\| \left[\frac{iS_2}{\sqrt{2}} + \frac{T_2}{3} \right] \right\| \beta J \right\rangle$$
$$\times [(3\cos^2\theta-1)(\langle J'M'|1-1;JM\rangle\langle J'M'|2-1;JM\rangle - \langle J'M'|11;JM\rangle\langle J'M'|21;JM\rangle)]$$
$$- \frac{ik}{2m}\langle \beta' J' \| X_1 \| \beta J \rangle^* \langle \beta' J' \| T_3 \| \beta J \rangle$$
$$\times \Bigg[(3\cos^2\theta-1)\Bigg[\frac{\langle J'M'|11;JM\rangle\langle J'M'|31;JM\rangle}{\sqrt{15}} + \frac{\langle J'M'|1-1;JM\rangle\langle J'M'|3-1;JM\rangle}{\sqrt{15}} + \frac{\langle J'M'|10;JM\rangle\langle J'M'|30;JM\rangle}{\sqrt{10}} \Bigg]$$
$$+ \cos^2\theta(3-5\cos^2\theta)\frac{\langle J'M'|10;JM\rangle\langle J'M'|30;JM\rangle}{\sqrt{10}}$$
$$+ (\tfrac{3}{5})^{1/2}\frac{1}{4}(1-\cos^2\theta)(1-5\cos^2\theta)(\langle J'M'|11;JM\rangle\langle J'M'|31;JM\rangle$$
$$+ \langle J'M'|1-1;JM\rangle\langle J'M'|3-1;JM\rangle) \Bigg] \Bigg\} . \quad (8)$$

Special cases of interest are found in Ref. 21.

III. NUMERICAL EVALUATION

Equation (8) is quite general and should be valid in any potential model. Only the reduced matrix elements will depend on the form of the potential used. From now on we specialize our discussion to the radiative decays of charmonium: $\psi'' \to \chi_J + \gamma$, $\psi' \to \chi_J + \gamma$, and $\chi_J \to \psi + \gamma$ ($J=0,1,2$). Even though in general there are $E1$, $M2$, and $E3$ contributions, to order v^2/c^2, because of the properties of Clebsch-Gordan coefficients, for ψ' or $\psi'' \to \chi_1 + \gamma$ and $\chi_1 \to \psi + \gamma$ the $E3$ contribution vanishes, and for ψ' or $\psi'' \to \chi_0 + \gamma$ and $\chi_0 \to \psi + \gamma$ $E3$ and $M2$ contributions vanish and only $E1$ contribution survives. For the decays ψ' or $\psi'' \to \chi_2 + \gamma$ and $\chi_2 \to \psi + \gamma$ all three multipoles $E1$, $M2$, and $E3$ contribute. These results will also follow from the conservation of angular momentum in the radiative decays. In order to find the relative strengths of the different multipole contributions, we have to calculate the reduced matrix elements. Before we do that we have to specify the initial and the final states. For a given magnetic quantum number M,

$$|\psi'\rangle = a'|2\,^3S_{1M}\rangle + b'|1\,^3D_{1M}\rangle ,$$
$$|\psi''\rangle = -b'|2\,^3S_{1M}\rangle + a'|1\,^3D_{1M}\rangle , \quad (9a)$$

where a' and b' are real coefficients satisfying

$$a'^2 + b'^2 = 1 . \quad (9b)$$

In Eq. (9a),

$$|2\,^3S_{1M}\rangle = R_{2S}(r)Y_{00}(\theta,\phi)\chi_{1M} ,$$
$$|1\,^3D_{1M}\rangle = R_{1D}(r)\sum_{m_s,m_l}\langle 1M|2m_l;1m_s\rangle Y_{2m_l}(\theta,\phi)\chi_{1m_s} . \quad (9c)$$

Also,

$$|\chi_J\rangle = |1\,^3P_{JM}\rangle \quad (J=0,1,2) , \quad (10a)$$

where

$$|1\,^3P_{JM}\rangle = R_{1P}(r)\sum_{m_s,m_l}\langle JM|1m_l;1m_s\rangle Y_{1m_l}(\theta,\phi)\chi_{1m_s} \quad (10b)$$

and

$$|\psi\rangle = a|1\,^3S_{1M}\rangle + b|1\,^3D_{1M}\rangle , \quad (11a)$$

where

$$a^2 + b^2 = 1 . \quad (11b)$$

In the above equations the radial wave functions R_{1S}, R_{2S}, R_{1P}, etc., need not be nonrelativistic eigenfunctions, but could include mixing due to relativistic terms in the Hamiltonian. For example,

$$|n\,^3S_{1M}\rangle = |n\,^3S_{1M}\rangle_0 + \sum_{m\neq n} a_m |m\,^3S_{1M}\rangle_0 , \quad (12a)$$

$$|n\,^3P_{JM}\rangle = |n\,^3P_{JM}\rangle_0 + \sum_{m\neq n} b_m |m\,^3P_{JM}\rangle_0 , \quad (12b)$$

where $|n\,{}^3S_{1M}\rangle_0$ and $|n\,{}^3P_{JM}\rangle_0$ are eigenfunctions of the nonrelativistic Hamiltonian. The coefficients a_m and b_m can be calculated[5] in first-order perturbation theory treating the relativistic terms in the charmonium Hamiltonian as a perturbation to the nonrelativistic Hamiltonian. By calculating the matrix element of a particular component of an irreducible tensor operator between given initial and final states and then comparing the result with Eq. (7), the Wigner-Eckart theorem, we can obtain all the reduced matrix elements occurring in Eq. (8) for the different radiative decays in charmonium. They can be expressed in terms of four types of radial integrals:

$$\begin{aligned} I_1 &= \int_0^\infty r^3 dr\, R_{nS}(r) R_{mP}(r)\ , \\ I_2 &= \int_0^\infty r^3 dr\, R_{nD}(r) R_{mP}(r)\ , \\ I_3 &= \int_0^\infty r^4 dr \frac{dR_{nS}}{dr} R_{mP}(r)\ , \\ I_4 &= \int_0^\infty r^4 dr \frac{dR_{nD}}{dr} R_{mP}(r)\ . \end{aligned} \tag{13}$$

See Ref. 21 for the needed reduced matrix elements, in terms of these four integrals. Since we find that the reduced matrix element of T_3 is given entirely in terms of I_2 and I_4 which involves the D-state radial wave function, the $E3$ contribution of $\psi' \to \chi_2 + \gamma$ and $\chi_2 \to \psi + \gamma$ will vanish if there is no D-state admixture in the ψ' and ψ wave functions. The $E3$ contribution to $\psi'' \to \chi_2 + \gamma$ will be significant since the ψ'' is expected to be predominantly D. But the small admixture of S state in ψ'' given by the coefficient b' is important to explain the observed leptonic annihilation rate of ψ''. We also find that the reduced matrix element of T_2 is real, as is the reduced matrix element of X_1, and so T_2 does not contribute to Eq. (8). The radial integrals of Eq. (13) are calculated using Eqs. (12). For example,

$$\begin{aligned} \int_0^\infty R_{2S}(r) R_{1P}(r) r^3 dr &= \int_0^\infty R_{2S}^{(0)}(r) R_{1P}^{(0)}(r) r^3 dr \\ &\quad + \sum_{m\neq 2} \int_0^\infty a_m R_{mS}^{(0)}(r) R_{1P}^{(0)}(R) r^3 dr \\ &\quad + \sum_{b\neq 1} b_n \int_0^\infty R_{2S}^{(0)}(r) R_{nP}^{(0)}(r) r^3 dr\ . \end{aligned} \tag{14}$$

We use as mixing coefficients $a_1 = -0.187$ ($R_{2S} = R_{2S}^{(0)} + a_1 R_{1S}^{(0)}$) and $a_2 = +0.187$ ($R_{1S} = R_{1S}^{(0)} + a_2 R_{2S}^{(0)}$) for Buchmüller-Tye, and $a_1 = -0.0617$ and $a_2 = +0.0617$ for Gupta-Radford-Repko[22,23] (GRR). The S-D mixing coefficients a, b, and a', b' are treated as phenomenological parameters to be determined from fitting the leptonic annihilation rate of ψ'' and the angular distribution of photons in the radiative decays of ψ' and ψ''.

We note here that we have used the relativistically corrected wave functions in calculating the integrals I_1, I_2, I_3, and I_4 of Eq. (13). The relativistically corrected integrals are significantly different from the nonrelativistic integrals even though the mixing coefficients are small.

We have plotted the differential decay rate [Eq. (8)] versus $\cos\theta$ for the different charmonium decays and for various S-D mixings in Figs. 1–9. The plotted angular distributions are normalized such that

$$\frac{1}{2J+1} \sum_{M=-J}^{J} \frac{d\Gamma}{d\Omega} = 1\ , \tag{15}$$

where J is the total angular momentum of the initial state. For the decays $\psi' \to \chi_J + \gamma$ and $\psi'' \to \chi_J + \gamma$ we choose $-0.1 \le b' \le 0.0$. This range is motivated by the coupled-channel mixing result from Ref. 1 (ψ' contains -3.1% $1\,{}^3D_1$) and our calculation of the mixing necessary to bring the theoretical prediction of the ψ'' leptonic width into agreement with the measured value ($b' = -10\%$). For the decays $\chi_J \to \psi + \gamma$ we use $b = -1.5\%$, as calculated in first-order relativistic perturbation theory, as we expect coupled-channel effects to be small. We note here that we are not specifying the origin of the mixing, but rather we are hoping to use the experimental measurement of the angular distribution to determine the mixing, whatever the source. The angular distributions as calculated in the Buchmüller-Tye model and in the GRR model are very nearly identical, so our plots represent results from both models.

IV. CONCLUDING COMMENTS

States directly produced at e^+e^- colliders are the ψ, ψ', and ψ'' in an $M = \pm 1$ state (angular momentum, parity, and charge conjugation implies $J^{PC} = 1^{--}$, and helicity arguments imply $M = \pm 1$), and so our Figs. 1, 2, 8, and 9 represent expected results from directly produced ψ states at an e^+e^- collider. In $p\bar{p}$ (or pp) collisions, P-wave states (e.g., χ_J) can also be directly produced, in addition to the ψ states. Similar helicity arguments imply $M = \pm 1$ here also (for $p\bar{p} \to$ quarkonia), but there is experimental evidence for the presence of $M = 0$ (Ref. 24). Thus for $\chi_J \to \psi + \gamma$ ($J = 2, 1$) we include both $M = \pm 1$ and $M = 0$ figures, which must be combined with production fractions in order to predict the overall angular distribution. For completeness we include the angular distribution for χ_2 in an $M = 2$ state. As a reminder, use of our Figs. 1, 2, 8, and 9, along with the normalization condition, Eq. (15), can provide the angular distributions for ψ states produced with $M = 0$.

As previously noted, our results do not differ significantly for the two potential models used. This is not entirely unexpected, as in the important range for charmonium (0.1 fm $\le r \le$ 1 fm), phenomenological potentials tend to be nearly indistinguishable. However, relativistic corrections are handled in very different manners—the Buchmüller-Tye result is calculated in standard perturbation theory, while the GRR result is calculated in a semirelativistic approximation with variational techniques. Given the above observation we can conclude that our result is, to a very good approximation, model independent. Thus, experimental tests of our results are, in reality, tests of the potential model approach to quarkonia.

There is a considerable difference between the nonrelativistic (i.e., pure $E1$) and the relativistic predictions.

This demonstrates the importance of dealing with a relativistically corrected transition operator, not only relativistically corrected states. This difference can be observed experimentally. Although there has been some experimental work on angular distributions of charmonium decays (i.e., the Crystal Ball Collaboration,[13] and the R704 Collaboration[15]), we feel that much higher statistics are needed. Such an experiment is being planned (E-760 at the Fermilab $\bar{p}$ accumulator) and we eagerly await results.

Finally, due to the small spread in the angular distribution in $\psi' \to \chi_J + \gamma$ over the range of S-D state mixing we consider, it seems extremely likely that any reasonable amount of S-D state mixing (i.e., that necessary to explain the large leptonic width of the ψ'') will be acceptable, that is consistent with angular distribution data. We had hoped to use experimental observations to extract the D-state mixing, which is expected to be due to a combination of relativistic and coupled-channel effects, but this now appears unlikely.

This research was supported by NSF Grant No. PHY-8819727.

[1]E. Eichten *et al.*, Phys. Rev. D **17**, 3090 (1978); **21**, 203 (1980); **21**, 313(E) (1980).

[2]W. Buchmüller and S.-H. H. Tye, Phys. Rev. D **24**, 132 (1981).

[3]P. Moxhay and J. L. Rosner, Phys. Rev. D **28**, 1132 (1983).

[4]R. McClary and N. Byers, Phys. Rev. D **28**, 1692 (1983).

[5]H. Grotch, D. A. Owen, and K. J. Sebastian, Phys. Rev. D **30**, 1924 (1984).

[6]S. N. Gupta, S. F. Radford, and W. W. Repko, Phys. Rev. D **26**, 3305 (1982); **31**, 160 (1985); **34**, 201 (1986).

[7]J. S. Whitaker *et al.*, Phys. Rev. Lett. **37**, 1596 (1976).

[8]H. Spitzer, in *Summer Institute for Particle Physics,* proceedings, Kyoto, Japan, 1978, edited by M. Ida, T. Kamae, and M. Kobayashi (Research Institute for Fundamental Physics, Kyoto, 1978), p. 47.

[9]W. Bartel *et al.*, Phys. Lett. **79B**, 492 (1978).

[10]W. Tanenbaum *et al.*, Phys. Rev. D **17**, 1731 (1978).

[11]R. Brandelik *et al.*, Nucl. Phys. **B160**, 426 (1979).

[12]T. M. Himel *et al.*, Phys. Rev. Lett. **44**, 920 (1980).

[13]M. Oreglia *et al.*, Phys. Rev. D **25**, 2259 (1982).

[14]J. E. Gaiser *et al.*, Phys. Rev. D **34**, 711 (1986).

[15]C. Baglin *et al.*, Phys. Lett. B **195**, 85 (1987).

[16]A. D. Martin, M. G. Olsson, and W. J. Stirling, Phys. Lett. **147B**, 203 (1984).

[17]M. G. Olsson, C. J. Suchyta III, A. D. Martin, and W. J. Stirling, Phys. Rev. D **31**, 1759 (1985).

[18]M. G. Olsson and C. J. Suchyta III, Phys. Rev. D **34**, 2043 (1986).

[19]K. J. Sebastian, Phys. Rev. A **23**, 2810 (1981); we begin with Eq. (49) of this work and after some rearranging arrive at our Eq. (1).

[20]R. A. Krajcik and L. L. Foldy, Phys. Rev. D **10**, 1777 (1974).

[21]M. A. Doncheski, Ph.D. thesis, Pennsylvania State University, 1990.

[22]X. Zhang (private communication).

[23]X. Zhang, Ph.D. thesis, Pennsylvania State University, 1988; the source for GRR model nonrelativistic wave functions.

[24]Suppression of $M=0$ states in $p\bar{p}$ reactions implies similar suppression in the time-reversed reaction, e.g., η_c $(1\,^1S_0) \to p\bar{p}$. However, $B(\eta_c \to p\bar{p}) = (0.10 \pm 0.02)\%$ is of the same order of magnitude as $B(\psi \to p\bar{p}) = (0.22 \pm 0.01)\%$. See Particle Data Group, G. P. Yost *et al.*, Phys. Lett. B **204**, 31 (1988).