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Anyonic superconductivity

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The renormalized Chern-Simons term at finite density is shown to vanish when the renormalized coefficient at zero density takes values $Ne^2/2\pi$. Thus in the Chern-Simons description a system of anyons at zero temperature is a superfluid. This result is shown to hold to all orders in perturbation theory by generalizing a nonrenormalization theorem of the zero-density case. We also discuss the finite-temperature case, where a perturbative Chern-Simons mass appears.

It was first suggested by Laughlin[1,2] that anyons[3] (particles with fractional statistics in two spatial dimensions) may exhibit superconductivity. In particular, the work of Fetter, Hanna, and Laughlin,[4] which was expanded upon by Chen, Halperin, Wilczek, and Witten,[5] showed that in the random-phase approximation a free gas of anyons with the statistics parameter $\gamma=\pi(1-1/N)$, where N is a large integer, has a massless pole in the current-current correlation at zero temperature and thus exhibits superfluidity. This then implies that a charged gas of anyons would be superconducting at zero temperature. The main limitation of these results is that it was not known to what extent the random-phase approximation is valid and, in particular, whether these results would survive improvements in this approximation. It was also not known how a nonzero temperature affects the results.

Banks and Lykken[6] studied a field-theoretic realization of an anyonic system in which charged fermions in 2+1 dimensions were coupled to ordinary photons plus an additional "statistics" gauge field possessing a Chern-Simons (CS) term. They argued that superconductivity (at zero temperature) occurs if and only if the renormalized CS term vanishes—i.e., if the quantum corrections to the bare CS term precisely cancel it.

In this Rapid Communication we follow the approach of Ref. 6 and calculate the renormalized CS coefficient. The system we analyze is with a nonzero anyon density.

We present the results of a detailed calculation[7] showing that the renormalized Chern-Simons term at finite density vanishes if and only if the zero-density renormalized Chern-Simons coefficient $2\pi\theta_R/e^2$ is a positive integer N. This establishes that this system of anyons is a superfluid at zero temperature.

We then show that, at zero temperature, this result extends to all orders in perturbation theory and thus does not depend on the mean-field approximation, nor on the large-N limit.[8] We do this by showing that the nonrenormalization theorem of Coleman and Hill[9] can be extended to the Chern-Simons theory at finite density.

We also show that at finite temperature the Chern-Simons term does *not* cancel, at least perturbatively. This gives an infrared effective theory consisting of a real massive scalar which, since it is massive, cannot be associated with the phase of a local order parameter. If this perturbative result holds it would imply thermally activated dissipation in the Chern-Simons superfluid, rather than the Kosterlitz-Thouless behavior observed in other descriptions of the anyon gas.[10]

We consider a single two-component massive fermion field coupled to a fictitious ("statistics") U(1) gauge field A_μ, which has a Chern-Simons term but no Maxwell term. (In such a pure Chern-Simons theory, a Maxwell term is, in fact, generated at one loop. This implies that the results of our calculation should hold even if a bare Maxwell term is present. We believe that our calculation can be extended to this case.) The relationship of the Chern-Simons Lagrangian to anyons has been studied by many authors.[11-15] The Euclidean path-integral expression for the $T=0$ partition function of the Chern-Simons theory at finite chemical potential μ is given by

$$Z=\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A_\mu \exp(-S_E) \tag{1}$$

with

$$S_E=\int d^3x\left[\bar{\psi}(\not{D}-m)\psi+i\frac{\theta}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda-\mu\psi^\dagger\psi\right]. \tag{2}$$

We shall work throughout with a non-negative chemical potential μ. We choose to work in the Coulomb gauge ($\partial_i A^i=0$). We proceed by integrating out the A_0 field, which simply gives the Gauss-law constraint $\delta(B-(e/\theta)\psi^\dagger\psi)$. This δ function now allows us to do the integration over A_1 and A_2 by setting

$$A_i=\frac{e}{\theta}\epsilon_{ij}\frac{\partial_j}{\nabla^2}\psi^\dagger\psi. \tag{3}$$

This leads to the following effective four-Fermi theory:

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left[-\int d^3x \left(\bar{\psi}(\tilde{\partial\!\!\!/} - m - \mu\gamma^0)\psi - \frac{ie^2}{\theta}(\bar{\psi}\gamma^i\psi)\frac{\epsilon_{ij}\partial^j}{\nabla^2}(\bar{\psi}\gamma^0\psi)\right)\right]. \tag{4}$$

We use the γ matrices $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$, and $\gamma_0 = \sigma_3$, where σ_i are the Pauli spin matrices. Notice that the effect of the chemical potential is simply to replace ∂_0 by $\partial_0 - \mu$. We thus define $\tilde{\partial}_\nu$ to be equal to ∂_ν unless $\nu = 0$, in which case $\tilde{\partial}_0 = \partial_0 - \mu$.

We begin by studying the fermion propagator $S(x,y)$ for this theory. The bare fermion propagator $S_0(x,y)$ is simply $1/(\tilde{\partial\!\!\!/} - m)$. When including perturbative corrections to this propagator we notice that there are tadpole contributions to S. These are shown diagrammatically in Fig. 1. These tadpoles are nonvanishing since $\langle J_0 \rangle = \rho_0$ is nonzero when μ is nonzero. Our first key observation is that we can compute the entire contribution of these tadpoles as a function of the mean density ρ_0. Each tadpole contributes an amount

$$i\frac{e^2}{\theta}\gamma^i \epsilon_{ij}\frac{\partial_j}{\nabla^2}\rho_0 \tag{5}$$

to the fermion propagator. This can be written as $ie\gamma^i\mathcal{A}_i$ where

$$\mathcal{A}_i = \frac{e}{\theta}\epsilon_{ij}\frac{\partial_j}{\nabla^2}\rho_0 . \tag{6}$$

Notice that $\mathcal{A}_i$ is precisely the gauge potential one would obtain from a constant fictitious magnetic field $\mathcal{B} = (e/\theta)\rho_0$. The full tadpole contribution is found by summing the geometric series of Fig. 1 which results in the expression

$$\begin{aligned} S_T &= (S_0^{-1} - ie\gamma^i\mathcal{A}_i)^{-1} \\ &= [\gamma^\mu(\partial_\mu - ie\mathcal{A}_\mu) - m - \mu\gamma^0]^{-1} , \end{aligned} \tag{7}$$

where $\mathcal{A}_0 = 0$. Thus the tadpole-corrected propagator S_T is the Green's function for a free fermion in a constant magnetic field $\mathcal{B} = (e/\theta)\rho_0$ and with chemical potential μ.

The next step is to find the fermion propagator S_T under this circumstance. We have done this by two methods: using the Euclidean version of Schwinger's proper-time method[16] and by directly solving the Green's-function equation. In this Rapid Communication we concentrate on the latter method. We choose an asymmetric gauge in which $\mathcal{A}_y = \mathcal{B}x$, $\mathcal{A}_x = 0$, $\mathcal{A}_0 = 0$ which is consistent with the Coulomb gauge. To find S_F we want to invert the operator $\tilde{D\!\!\!/} - m$ where $\tilde{D}_\mu = \tilde{\partial}_\mu - ie\mathcal{A}_\mu$. Note that

$$\begin{aligned} (\tilde{D\!\!\!/} - m)^{-1} &= (\tilde{D\!\!\!/} + m)[(\tilde{D\!\!\!/} - m)(\tilde{D\!\!\!/} + m)]^{-1} \\ &= (\tilde{D\!\!\!/} + m)(\tilde{D}^2 - m^2 + e\sigma_3\mathcal{B})^{-1} . \end{aligned} \tag{8}$$

We thus first invert the operator $(\tilde{D}^2 - m^2 + e\sigma_3\mathcal{B})$ by finding its eigenvalues and eigenfunctions. The result is given by

$$\begin{aligned} \frac{1}{\tilde{D}^2 - m^2 + e\sigma_3\mathcal{B}} = -\sum_{n=0}^{\infty}\int\frac{d\omega}{2\pi}\int\frac{dP_y}{2\pi}\Bigg\{&\left[\frac{1}{2}\left(\frac{1}{d_{n+1}} + \frac{1}{d_n}\right) + \frac{1}{2}\left(\frac{1}{d_{n+1}} - \frac{1}{d_n}\right)\sigma_3\right] \\ &\times e^{-i\omega(t-t')}e^{-iP_y(y-y')}\Psi_n\left(x - \frac{p_y}{e\mathcal{B}}\right)\Psi_n^*\left(x' - \frac{p_y}{e\mathcal{B}}\right)\Bigg\} , \end{aligned} \tag{9}$$

where Ψ_n is the nth normalized eigenfunction of a harmonic oscillator with frequency $e\mathcal{B}$ and where $-d_n$ and $-d_{n+1}$ are the eigenvalues given by $d_n = (\omega - i\mu)^2 + 2ne\mathcal{B} + m^2$.

Now that we have the fermion propagator, we calculate ρ_0 as a function of μ and $\mathcal{B}$. We will argue below that the lowest-order calculation presented here is in fact an exact result:

$$\begin{aligned} \rho_0 &= \langle\psi^\dagger(x)\psi(x)\rangle = -\mathrm{Tr}[\gamma_0 S_T(x,x)] \\ &= \frac{-ie\mathcal{B}}{2\pi}\sum_{n=0}^{\infty}\int\frac{d\omega}{2\pi}\left[(\omega - i\mu)\left(\frac{1}{d_{n+1}} + \frac{1}{d_n}\right) + im\left(\frac{1}{d_{n+1}} - \frac{1}{d_n}\right)\right] \\ &= \frac{e\mathcal{B}}{4\pi}\sum_{n=0}^{\infty}\{\Theta(\mu - (2ne\mathcal{B} + m^2)^{1/2}) + \Theta(\mu - [2(n+1)e\mathcal{B} + m^2]^{1/2})\} - \frac{e\mathcal{B}}{4\pi}\frac{m}{|m|}\Theta(|m| - \mu) \\ &= \frac{e\mathcal{B}}{2\pi}\left[\mathrm{Int}\left(\frac{\mu^2 - m^2}{2e\mathcal{B}}\right) + \frac{1}{2}\right]\Theta(\mu - |m|) - \frac{e\mathcal{B}}{4\pi}\frac{m}{|m|}\Theta(|m| - \mu) , \end{aligned} \tag{10}$$

where Int stands for the integer part of its argument and μ and $\mathcal{B}$ are taken to be positive. Notice that when $(\mu^2 - \mu^2)/2e\mathcal{B}$ is itself an integer, the value of the density ρ_0 is ambiguous. Notice also that the density ρ_0 is nonzero at $\mu = 0$. This is due to the spectral asymmetry of our parity-noninvariant Hamiltonian and to our definition of ρ_0 as $\langle\psi^\dagger\psi\rangle$. One thus has to define the physical density

$$\rho_{\mathrm{ph}} = \rho(\mu) - \rho(\mu = 0) = \frac{e\mathcal{B}}{2\pi}\left[\mathrm{Int}\left(\frac{\mu^2 - m^2}{2e\mathcal{B}}\right) + 1\right] \tag{11}$$

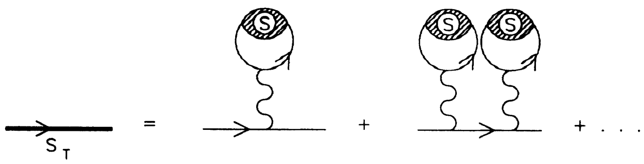

FIG. 1. Tadpole contributions to the fermion propagator. The bold line indicates a tadpole-corrected fermion line. S represents the full fermion propagator.

for $\mu > m > 0$. The final result for $\rho_{\rm ph}$ when $\mu > m > 0$ is drawn in Fig. 2. We now see the correspondence of $\rho_{\rm ph}$ to the density of fermions in Landau levels. Notice that the filled Landau levels are described by the horizontal sections (constant $\rho_{\rm ph}$) in Fig. 2. These occur when $2\pi\rho_{\rm ph}/e\mathcal{B}$ is an integer N. This implies that $2\pi\rho_0/e\mathcal{B} = N - \frac{1}{2}$ (for $m > 0$). Using the definition of $\mathcal{B}$, i.e., $\mathcal{B} = (e/\theta)\rho_0$ we see that a filled Landau level occurs when $\theta/2\alpha = N - \frac{1}{2}$ with $\alpha = e^2/4\pi$. This is precisely the condition that the renormalized Chern-Simons coefficient θ_R at zero density is 2α times an integer. (See Ref. 6.) (We shall see below that this is the case for the regularization scheme which we chose. The point is that the zero-density one-loop corrections to the Chern-Simons coefficient are regularization dependent as is shown in Ref. 11. It is only the renormalized θ which is related to the statistics parameter of the anyons.)

Furthermore an unfilled level corresponds to the vertical parts of Fig. 2 for which $(\mu^2 - m^2)/2e\mathcal{B}$ is an integer.

We proceed now to evaluate the one-loop contribution to the CS term in the effective action, which is given by the parity-odd part $\Pi_{\rm odd}(k^2=0)$ of the vacuum polariza-

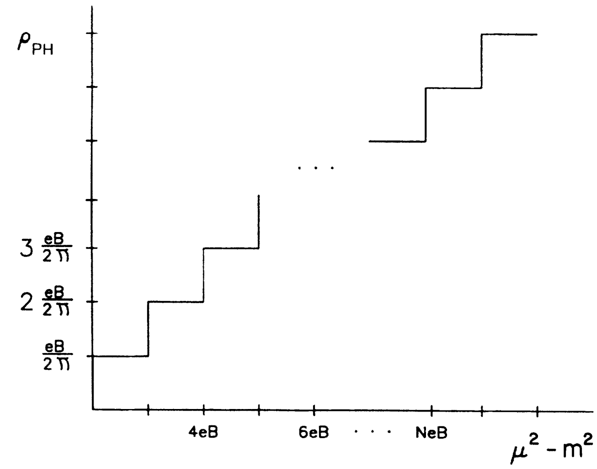

FIG. 2. The fermion density as a function of the chemical potential μ and the magnetic field $\mathcal{B}$.

tion

$$\Pi_{\mu\nu}(k) = \Pi^e_{\mu\nu}(k^2) + \epsilon_{\mu\nu\lambda}k^\lambda \Pi_{\rm odd}(k^2)\,, \qquad (12)$$

where Π^e is symmetric under interchange of μ and ν. Note that gauge invariance requires the odd part to have the above form even at finite density where Lorentz invariance is lost. The one-loop expression for $\Pi_{\mu\nu}$ is given by

$$\Pi_{\mu\nu}(x,y) = -e^2 {\rm Tr}[\gamma_\mu S_T(x,y)\gamma_\nu S_T(y,x)]\,. \qquad (13)$$

To evaluate $\Pi_{\rm odd}$ we extract the term in Eq. (13) which is proportional to $\epsilon_{ij}k_0$. The expression for $\Pi_{\rm odd}(0)$ using both Schwinger's method[7] and the method presented above is

$$\Pi_{\rm odd} = e^2 \frac{e\mathcal{B}}{\pi} \sum_{n=0}^{\infty} \int \frac{d\omega}{2\pi} \left(\frac{-m}{d_n d_{n+1}} + 2ie\mathcal{B} \frac{(\omega - i\mu)[(\omega - i\mu)^2 + m^2]}{d_n^2 d_{n+1}^2} \right). \qquad (14)$$

We now compare this expression for $\Pi_{\rm odd}$ to the previous expression for ρ_0. The term proportional to m is identical to the similar term in ρ_0 (divided by $\mathcal{B}/e$). The second term in the integrand can be rewritten as

$$2ie\mathcal{B} \left[\frac{\omega - i\mu}{(2e\mathcal{B})^2} \left(\frac{1}{d_{n+1}} + \frac{1}{d_n} \right) - \frac{2n}{2e\mathcal{B}} \frac{\omega - i\mu}{[(\omega - i\mu)^2 + M^2(n)]^2} \right], \qquad (15)$$

where $M^2(n) = m^2 + 2en\mathcal{B}$. The first term Eq. (15) is identical to the remaining term in ρ_0. For the second term of Eq. (15) we perform the ω integration which results in an expression which is proportional to $\delta(\mu - M(n))$. This term vanishes whenever $(\mu^2 - m^2)/2e\mathcal{B} \neq$ integer. Therefore, for any μ such that $(\mu^2 - m^2)/2e\mathcal{B} \neq$ integer, we get the following relation between $\Pi_{\rm odd}(0)$ and ρ_0:

$$\Pi_{\rm odd}(k^2=0) = \frac{e^2\rho_0}{e\mathcal{B}} = \theta\,, \qquad (16)$$

where we have used the tadpole relation between ρ_0 and $\mathcal{B}$. We see that the one-loop correction to the Chern-Simons term precisely cancels the bare Chern-Simons term *provided* $(\mu^2 - m^2)/2e\mathcal{B}$ is not an integer. As discussed previously and as is seen from Fig. 1, this occurs precisely when we have any number N of filled Landau levels, in which case $\theta/2\alpha = N - \frac{1}{2}$ (for $m > 0$). If we now apply the above equation and Eq. (10) to the case $\mu = 0$ we see that $\Pi_{\rm odd}(k^2=0, \mu=0) = -\alpha$. This renormalizes the *zero-density* Chern-Simons term from θ to $\theta_R = \theta - \alpha$. Thus at *nonzero* μ the condition for having N filled Landau levels becomes $\theta_R/2\alpha = N$. We thus conclude that to first order in the tadpole-corrected perturbation calculation a CS coefficient whose renormalized value at zero density is $Ne^2/2\pi$ is, at any finite density, *renormalized to zero.* Note that this value of θ_R corresponds precisely to anyons with a statistics parameter $\gamma = \pi(1 - 1/N)$.

The above result for $\Pi_{\rm odd}$ could have been anticipated without an explicit calculation once the result for ρ_0 was calculated. The reason for this is that at any order, there is a general relation between those diagrams which contribute to $\Pi_{\rm odd}(k=0)$ and those which contribute to ρ. Here $\Pi_{\rm odd}(0)$ is defined by summing one-particle-irreducible diagrams to any order in tadpole-corrected perturbation theory. To see this imagine taking $\delta/\delta\mathcal{B}$ (for fixed μ) of any diagram which contributes to ρ. This has the effect of removing a tadpole insertion and replacing it by $(1/e)\epsilon^{ij}\gamma_j\partial_i/\nabla^2$. One easily sees that the resulting diagram is one-particle irreducible (in terms of tadpole-corrected lines) and contributes precisely to the odd part of the $\Pi(0)/e$, i.e., $\Pi_{\rm odd}(0)$. We obtain the general relation

$$\frac{\delta\rho}{\delta e\mathcal{B}}\bigg|_{\mu}=\frac{1}{e^2}\Pi_{\rm odd}(0)\,. \tag{17}$$

Thus, having calculated ρ_0 we can simply differentiate with respect to $\mathcal{B}$ at fixed μ. From Fig. 2 we see that for filled Landau levels ρ_0 is simply proportional to $\mathcal{B}$ and thus $\Pi_{\rm odd}(0)$ is simply equal to $e\rho_0/\mathcal{B}$ which agrees with our previous result.

We will now show that ρ, $\Pi_{\rm odd}(0)$, and thus Eq. (16), are unaffected by higher-order radiative corrections in tadpole-corrected perturbation theory. The nonrenormalization theorem can be proven either by a topological argument,[17] or by a direct extension of the Coleman-Hill theorem[9] to the finite-density case. Here we will present the latter approach (for more details, see Ref. 7).

Consider the Euclidean n-photon effective vertex, at finite density, given by summing all graphs consisting of a single tadpole-corrected fermion loop with n external photons attached. We denote this by

$$\Gamma^{(n)}_{\mu_1\cdots\mu_n}(k_1\cdots k_n)\,. \tag{18}$$

All diagrams in tadpole-corrected perturbation theory which contribute to ρ or $\Pi_{\rm odd}(0)$ can be constructed from the $\Gamma^{(n)}$'s, by sewing together photon lines. To prove our nonrenormalization theorem for ρ and $\Pi_{\rm odd}(0)$, it suffices to show that, for $k_1, k_2\to 0$,

$$\begin{aligned}&\Gamma^{(n)}_{\cdots}(k_1\cdots)=O(k_1),\quad n>1\,,\\&\Gamma^{(n)}_{\cdots}(k_1,k_2,\cdots)=O(k_1k_2),\quad n>2\,.\end{aligned} \tag{19}$$

By gauge invariance and the argument of Ref. 6, these relations are true provided that $k\to 0$ is in the region of analyticity of the $\Gamma^{(n)}$.

We prove the nonrenormalization theorem, therefore, by demonstrating the analyticity of $\Gamma^{(n)}$ as $k^2\to 0$ in the Euclidean region. This is obvious for the zero-density system, since the physical (Minkowski) threshold for fermion-antifermion pairs begins at $k^2=4m^2$. At finite density, however, one must also worry about the production of fermion-hole pairs. In our case, since $\Gamma^{(n)}$ are defined in tadpole-corrected perturbation theory, this corresponds to a (Minkowski) photon being absorbed by a fermion in a Landau level, causing a transition to an unoccupied state. The Landau levels allow continuous values of momentum but are discretely spaced in energy, with spacing $e\mathcal{B}/m$. Therefore, when we have N completely filled Landau levels, physical singularities are absent for (Minkowski) $k_0<e\mathcal{B}/m$. Thus as we approach $k^2\to 0$ from the Euclidean region the $\Gamma^{(n)}$ are analytic, and the nonrenormalization theorem holds precisely for $\theta=Ne^2/2\pi$.

Next we analyze the finite-temperature behavior of the anyonic system, namely, that of the partition function given in Eq. (2). The evaluation of the fermion propagator follows the same lines as for the zero-temperature case apart from replacing the integral over ω with the sum over discrete Matsubara frequencies $\omega_l=(2\pi/\beta)(l+\frac{1}{2})$. Inserting the resulting propagator into (10) and using standard contour integrals[7] to replace the summations we get the following expression for the density at finite temperature:

$$\rho_0=\frac{e\mathcal{B}}{4\pi}\left\{\sum_{n=0}^{\infty}\left[\tanh\left(\frac{\beta}{2}[\mu+M(n)]\right)+\tanh\left(\frac{\beta}{2}[\mu-M(n)]\right)\right]+\frac{m}{|m|}\tanh\left(\frac{\beta}{2}(\mu-|m|)\right)\right\}. \tag{20}$$

In the limit $\beta\to\infty$ this expression reduces to the one given in Eq. (10).

We now compute the renormalized CS term at finite temperature. The simplest way to do this is to use Eq. (17) which is valid at finite temperature. We see immediately that ρ is no longer proportional to $\mathcal{B}$ and, in fact, $\rho/\mathcal{B}$ is a monotonic function of $\mathcal{B}$. Equation (16) is thus never valid, and the renormalized CS term is nonzero for any finite temperature. Instead, for fully filled levels, the renormalized CS term is given by $\theta_R(\mu,T)=-e\mathcal{B}d(\rho/\mathcal{B})/d\mathcal{B}$. Alternately we could repeat the steps that led to Eq. (14). We now get the same expression with the sum over ω_l replacing integration over ω. Recall that the cancellation of the bare and one-loop CS term was a result of the vanishing of the second term in Eq. (15) for the filled Landau levels—i.e., $(\mu^2-m^2)/2e\mathcal{B}\neq$integer. At finite temperature this term does not vanish.

Using either of the above methods we find

$$\theta_R(\mu,T)=-\frac{\alpha e\mathcal{B}}{2}\beta\sum_{n=0}^{\infty}\frac{n}{M(n)}\left[\tanh^2\left(\frac{\beta}{2}[\mu+M(n)]\right)-\tanh^2\left(\frac{\beta}{2}[\mu-M(n)]\right)\right]. \tag{21}$$

For small temperature and large mass $\beta m\gg 1$, $|\mu^2-m^2|\ll m^2$ this reduces to

$$|\theta_R(\mu,T)|=2\alpha e\mathcal{B}\beta\sum_{n=0}^{\infty}\frac{n}{M(n)}e^{-\beta|\mu-M(n)|}\,. \tag{22}$$

In this limit the only significant terms are those with $2ne\mathcal{B}\sim|\mu^2-m^2|\ll m^2$ so that $\theta_R(\mu,T)\sim\alpha\beta[(\mu^2-m^2)/m]\times(e^{-\beta(e\mathcal{B}/m)\delta}+e^{-\beta(e\mathcal{B}/m)(1-\delta)})$, where $\delta=(\mu^2-m^2)/2e\mathcal{B}-{\rm Int}((\mu^2-m^2)/2e\mathcal{B})$. Note that θ_R vanishes exponentially at $T\to 0$.

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