Nonperturbative time-dependent classical string solutions for the closed, bosonic string

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(Received 3 May 1990)

We show that the conformal anomaly vanishes nonperturbatively in the σ model for the bosonic string in a large class of time-dependent backgrounds.

Recently a large class of time-dependent classical string solutions in the form of plane-fronted waves has been found.[1–3] These solutions satisfy the σ-model β-function equations to all orders in α', the inverse string tension. In addition to being time dependent many of these solutions have spacetime singularities. It has been shown that for the special case of gravitational plane waves the conformal anomaly also vanishes nonperturbatively in the σ model.[2] In this paper, we extend this to the other massless fields of the bosonic string by showing that the conformal anomaly vanishes nonperturbatively in the presence of the axion and dilation as well. We also discuss the implications for string propagation in singular backgrounds.[4]

We will first find the quantum-mechanical effective σ model action in the presence of the metric, axion, and dilaton background. We will then calculate the conformal anomaly and find the equation of motion which the background fields must satisfy in order for the anomaly to vanish. This will coincide with the ordinary perturbative β-function result. Finally, we will discuss the implications for string scattering in singular backgrounds.

Let us consider an exact plane wave with metric of the form

$$ds^2=-dU\,dV+dX^i dX_i+W_{ij}(U)X^iX^j dU^2 \ . \tag{1}$$

We also include an axion field of the form

$$B_{\mu\nu}=-\tfrac{1}{3}A_{ij}(U)X^i l_{[\mu}\partial_{\nu]}X^j \ , \tag{2}$$

where

$$l^\mu=\left[\frac{\partial}{\partial V}\right]^\mu \tag{3}$$

is the null Killing field of the plane wave. Finally, we include a scalar dilaton that depends only on the coordinate U:

$$\Phi=\Phi(U) \ . \tag{4}$$

Here, we are restricting ourselves to fields that are bounded and defined for all U. At the end of the paper we will weaken this restriction and consider the case of singular backgrounds of the type discussed in Ref. 4. The two-dimensional action for the string reads

$$\begin{aligned}S=-\tfrac{1}{2}\int d^2\sigma\sqrt{h}\,[&-\nabla_a U\nabla^a V+\nabla_a X^i\nabla^a X_i\\&+W_{ij}X^iX^j(\nabla U)^2\\&-\tfrac{1}{3}A_{ij}(U)X^i\nabla_a U\nabla_b X^j\epsilon^{ab}\\&-\tfrac{1}{4}R\,\Phi(U)] \ ,\end{aligned} \tag{5}$$

where X^μ are now fields on the two-dimensional world sheet, h_{ab} is the world-sheet metric, and R is the world-sheet scalar curvature. Following Ref. 2 we now proceed to calculate the generating functional

$$\begin{aligned}Z[J,h_{ab}]&=e^{iW[J,h_{ab}]}\\&=\int DX^\mu\exp\left[iS(X^\mu,h_{ab})\right.\\&\qquad+i\int(J_V V+J_U U+J_i X^i)\\&\qquad\left.\times\sqrt{h}\,d^2\sigma\right] \ ,\end{aligned} \tag{6}$$

where the J's are current sources. Integration over V yields a δ functional:

$$\delta(-\tfrac{1}{2}\nabla^2 U+J_V) \ . \tag{7}$$

We then integrate over U thereby obtaining the factor

$$\det{}^{-1}(\nabla^2) \tag{8}$$

from the δ-function Jacobian. Finally, one can integrate over X^i since the action is quadratic in those fields. This yields for the partition function

$$\begin{aligned}Z[J]=&(\det{}^{-1}\nabla^2)[\det{}^{-1/2}(M_{ij})]\\&\times\exp\{i[-\tfrac{1}{2}J_iM_{ij}^{-1}J_j+\tfrac{1}{8}R\,\Phi(\bar U)+J_U\bar U]\} \ ,\end{aligned} \tag{9}$$

where

$$M_{ij}=\delta_{ij}\nabla^2-W_{ij}(\bar U)(\nabla\bar U)^2+\tfrac{1}{3}A_{ij}(\bar U)\nabla_a\bar U\epsilon^{ab}\nabla_b \tag{10}$$

and $\bar U=2(\nabla^2)^{-1}J_V$. The next step is to calculate the effective action $\Gamma[X_{\rm cl}]$ by taking the Legendre transform of $W[J]$ where

$$X^\mu_{\rm cl}=\frac{\delta W}{\delta J_\mu} \ . \tag{11}$$

In particular, one has $U_{\rm cl}=\bar U$. Calculating the transform, one finds

$$e^{i\Gamma}=(\det{}^{-1}\nabla^2)[\det{}^{-1/2}(M_{ij})]e^{iS(X_{\rm cl})} \ . \tag{12}$$

We thus see that the quantum-mechanical effective action is the classical action along with corrections coming from the two determinants.

Having obtained the effective action, we now proceed to calculate the conformal anomaly $\delta\Gamma/\delta\phi$ where

$$h_{ab}=e^{\phi}\eta_{ab} \ . \tag{13}$$

There are four contributions to the anomaly. First, as a result of going to the conformal gauge, one obtains the standard contribution from the Faddeev-Popov determinant:[5,6]

$$\frac{\delta\Gamma}{\delta\phi}=-\frac{26}{48\pi}R \ , \tag{14}$$

where R is the world-sheet scalar curvature. The restriction that $D=26$ in flat space-time arises from this factor. There are also the two contributions from the two determinants. Finally, since the dilaton term in the σ model is not conformally invariant, it too will make a contribution to the anomaly.

We now proceed to calculate the contribution from the determinant of M in Eq. (12). To do this we will use the Euclidean heat-kernel method.[7,8] Let λ_n be the eigenvalues of M. Then one has

$$\ln \det M=\sum_n \ln\lambda_n=-\sum_n \int_\epsilon^\infty t^{-1}e^{-t\lambda_n}dt=-\int_\epsilon^\infty t^{-1}\left[\int \langle z|\mathrm{Tr}e^{-tM}|z\rangle d^2z\right]dt \ , \tag{15}$$

where the trace is over the index i and where we have used the representation

$$\ln x=-\int_\epsilon^\infty t^{-1}e^{-tx}dt+(\text{an } x\text{-independent constant})+O(\epsilon x) \ . \tag{16}$$

Here, ϵ is playing the role of a cutoff. From the form of M in Eq. (10) one notes that it depends on the conformal factor in the following way:

$$M=e^{-\phi}\times(\phi\text{-independent piece}) \ . \tag{17}$$

One then obtains

$$\begin{aligned}\frac{\delta \ln \det M}{\delta\phi(z)}&=\int_\epsilon^\infty\int\left\langle z'\left|\mathrm{Tr}\left[\frac{\delta M}{\delta\phi(z)}e^{-tM}\right]\right|z'\right\rangle d^2z'dt=-\int_\epsilon^\infty\langle z|\mathrm{Tr}(Me^{-tM})|z\rangle dt\\ &=\langle z|\mathrm{Tr}e^{-tM}|z\rangle|_\epsilon^\infty=-\mathrm{Tr}\,Y(z,z;\epsilon) \ ,\end{aligned} \tag{18}$$

where

$$Y(z,w;t)=\langle z|e^{-tM}|w\rangle \tag{19}$$

is called the heat kernel of M. It satisfies the heat equation

$$-\frac{\partial Y(z,w;t)}{\partial t}=MY(z,w;t) \tag{20}$$

with initial condition

$$Y(z,w;0)=\delta(z,w) \ . \tag{21}$$

Durhuus, Oleson, and Petersen[8] have obtained asymptotic expressions for Y for an arbitrary elliptic operator in two dimensions. If M has the form

$$-M=\frac{1}{\sqrt{h}}(\nabla_a+B_a)\sqrt{h}\,h^{ab}(\nabla_b+B_b)+B_0 \ , \tag{22}$$

where B_a and B_0 can be any matrix-valued function of z, then they found that

$$Y(z,z;t)=\frac{1}{4\pi t}\sqrt{h}-\frac{1}{24\pi}R\sqrt{h}+\frac{1}{4\pi}B_0\sqrt{h}+O(t) \ . \tag{23}$$

Rewriting M from Eq. (10) in the form of Eq. (22), we find, for B_0,

$$B_0=\tfrac{1}{2}\nabla_a A_{ij}^a+W_{ij}(\nabla U_{\rm cl})^2+\tfrac{1}{4}A_{ik}^aA_{kja} \ , \tag{24}$$

where

$$A_{ij}^a=\tfrac{1}{3}A_{ij}\epsilon^{ba}\nabla_b U_{\rm cl} \tag{25}$$

is the coefficient of the single derivative in the third term of Eq. (10). Since A_{ij}^a is antisymmetric, the first term is traceless, and we find

$$\mathrm{Tr}B_0=(W_i^i+\tfrac{1}{36}A_{ij}A^{ij})(\nabla U_{\rm cl})^2 \ . \tag{26}$$

Thus, from Eqs. (18), (23), and (26) we obtain the contribution of M to the conformal anomaly

$$\begin{aligned}\frac{\delta\Gamma_M}{\delta\phi}&=\frac{1}{2}\frac{\delta}{\delta\phi}\ln\det M\\ &=-\tfrac{1}{2}\mathrm{Tr}\,Y(z,z;\epsilon)\\ &=-\frac{1}{8\pi\epsilon}(D-2)\sqrt{h}+\frac{1}{48\pi}(D-2)R\sqrt{h}\\ &\quad-\frac{1}{8\pi}(W_i^i+\tfrac{1}{36}A_{ij}A^{ij})(\nabla U_{\rm cl})^2\sqrt{h} \ .\end{aligned} \tag{27}$$

The factor of $D-2$ comes from tracing over the $D-2$ transverse directions.

The third contribution to the conformal anomaly comes from the other determinant

$$\det{}^{-1}\nabla^2=[\det{}^{-1/2}(\nabla^2)]^2 \ . \tag{28}$$

This is the same as the anomaly from two free scalars, and hence we have

$$\frac{\delta\Gamma}{\delta\phi}=-\frac{2}{8\pi\epsilon}\sqrt{h}+\frac{2}{48\pi}R\sqrt{h} \ . \tag{29}$$

The $1/\epsilon$ poles in Eqs. (27) and (29) can be canceled by adding a local counterterm to the action in the form of a cosmological constant. Combining the three contributions from Eqs. (14), (27), and (29) and rotating back to Minkowski space, we find that the conformal anomaly vanishes only if $D=26$ and the background fields satisfy

$$W_i^i+\tfrac{1}{36}A_{ij}A^{ij}=0 \ . \tag{30}$$

One may include the dilaton as well. The dilaton only appears in the classical action. Hence, its contribution to the anomaly is

$$\begin{aligned}\frac{\delta S_{\text{dil}}}{\delta\phi}&=-\frac{1}{8\pi}\sqrt{h}\,\nabla^2\Phi\\&=-\frac{1}{8\pi}\sqrt{h}\,[\nabla^2U_{\text{cl}}\Phi'+(\nabla U_{\text{cl}})^2\Phi''] \ ,\end{aligned} \tag{31}$$

where ∇_a is a world-sheet derivative. We now use the equation of motion for U_{cl} obtained by varying the effective action with respect to V_{cl}. However, since the determinants are independent of V_{cl}, the equation of motion is just the classical equation of motion

$$\nabla^2U_{\text{cl}}=0 \ . \tag{32}$$

Using this in Eq. (31), we find that the conformal anomaly will vanish if the background field satisfy the equation of motion

$$W_i^i+\tfrac{1}{36}A_{ij}A^{ij}+\phi''=0 \ . \tag{33}$$

This is the same equation of motion as was obtained from the β functions in Ref. 3. Here, however, we have shown that these backgrounds are also solutions nonperturbatively.

We now proceed to consider briefly the effects of the nonperturbative corrections as they relate to the discussion of string propagation in Ref. 4. In that treatment the light-cone gauge was imposed. After doing this, aside from zero-mode contributions coming from V, one is left with the transverse X^i as the only physical degrees of freedom. The only change in the string action due to nonperturbative effects is the contribution from the determinants. However, because these depend only on U, the equation of motion for X^i is unchanged. Hence, the propagation of a first-quantized string remains unaffected. In the same paper solutions which became singular at finite U were also considered. Does the conformal anomaly vanish nonperturbatively for these singular solutions as well? The only potential problem would arise in the integration over U. Since the spacetime is now only defined up to some finite U, one should restrict the integration range. If U_{cl} takes on a value outside this range at some point, the δ functional will give zero and the Legendre transformation will become singular. However, for U_{cl} everywhere within the integration range, that is, for U_{cl} within the physical spacetime, the calculation of the conformal anomaly goes through unaltered. Hence, the conformal anomaly vanishes for these singular spacetimes as well. We thus conclude that a large class of time-dependent backgrounds for the bosonic string are nonperturbative classical string solutions.

It is a pleasure to thank Gary Horowitz for helpful discussions and for useful comments on the manuscript. This work was supported by NSF Grant No. PHY85-06686.

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