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Eigenstates of the Schwinger-model Hamiltonian

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We solve the Schwinger model on the circle in the canonical Hamiltonian formalism without bosonizing. Working in the $A_0=0$ gauge, we explicitly construct the exact, physical states of the theory in the fermionic Fock space using the Schrödinger-picture representation for the gauge field degrees of freedom. The Θ vacuum is obtained naturally and its detailed structure is exhibited.

INTRODUCTION

Electrodynamics in two spacetime dimensions with massless fermions was first solved by Julian Schwinger in 1962, and hence named the Schwinger model.[1] He obtained the exact Green's functions for the theory, and since then other authors have obtained them in various gauges.[2] Path-integral solutions[3] and operator solutions[4] have also since been given. A solution which gives all the exact eigenstates of the Hamiltonian (on the circle) has been obtained by Hetrick and Hosotani[5] by bosonizing the model. It would certainly be of interest to write this solution down in its fermionic representation, and in fact that is what we do here by solving the model directly.

The various solutions have allowed many properties, often nonperturbative, to be elucidated. In particular, the global chiral anomaly and the Θ angle have received much attention. The Θ vacuum is usually obtained by demonstrating that there is an infinite number of degenerate, lowest-energy states, related by large gauge transformations, and these are summed to a single gauge-invariant vacuum which gives the theory the cluster property. This vacuum is here obtained quite naturally as the lowest-energy state of a set of stationary states which are obtained by a linear vector space transformation (the transformation matrix has entries which are functionals of the gauge field) from a basis of states constructed in the usual manner with elementary fermionic creation operators. The solution has the usual properties of invariance under small and large gauge transformations, and variance of Θ under chiral transformations.

In 1985, Manton constructed explicitly (without bosonizing) a set of approximate eigenstates for the Hamiltonian in the $\partial_x A_x=0$ gauge.[6] (Hence we follow quite closely his lead and regulate the theory by taking space to be a circle and working with the discrete momentum modes.) He solved the eigenstate problem by neglecting the Coulomb interaction to find a set of eigenstates without fermionic excitations. What we will do here is find a canonical transformation which yields a Hamiltonian for which his states are exact. These correspond to the zero-momentum scalars of the bosonized version of the theory, and the rest of the states are then easily found from these by using bosonic creation operators. Transforming back gives a complete solution in terms of the original fermionic variables. The canonical transformation is closely related to bosonization and was first discovered by Mattis and Lieb to solve the Luttinger model.[7]

PURE ELECTRODYNAMICS ON THE CIRCLE

The Lagrangian density for two-dimensional QED (QED_2) without matter is

$$\mathcal{L}=\tfrac{1}{2}(\partial_t A_x-\partial_x A_t)^2 . \tag{1}$$

We choose the gauge $A_t=0$ and are left with the residual freedom of performing time-independent gauge transformations, which are of the form

$$A_x \rightarrow A_x - i(\partial_x g)g^{-1}, \quad \text{where } g(x)=e^{i\Lambda(x)} . \tag{2}$$

Demanding that g be single valued on the circle of length 2π forces,

$$\Lambda(2\pi)=\Lambda(0)+2\pi m , \tag{3}$$

where m is an integer which labels what we call the homotopy class of Λ. When $m=0$ the transformation is called a small gauge transformation, and when $m\neq 0$ it is called a large gauge transformation. Notice that the zeroth Fourier component of A_x may be brought to the interval [0,1], with the end points identified, by a large gauge transformation:

$$A_0 \equiv \frac{1}{2\pi}\int_0^{2\pi} A_x dx \rightarrow A_0+m . \tag{4}$$

The electric field is the canonically conjugate momentum to A_x,

$$E_x=\frac{\partial\mathcal{L}}{\partial \dot{A}_x}=\partial_t A_x , \tag{5}$$

and the Hamiltonian is

$$H=\tfrac{1}{2}\int_0^{2\pi} E_x^2 dx . \tag{6}$$

We quantize this system in the field version of the Schrödinger picture, obtaining a representation of the commutator algebra

$$[A_x(x), E_x(y)] = i\delta(x-y) \quad (7)$$

by taking A_x to be the multiplication operator and E_x to be the derivative operator:

$$E_x = -i\frac{\delta}{\delta A_x} + \frac{i}{2\pi}\Theta \ . \quad (8)$$

Θ is a real consant which will later be seen as the vacuum angle. The solution of this system is easy and can be found, for example, in Ref. 6.

THE SCHWINGER-MODEL HAMILTONIAN

Upon addition of the Dirac Lagrangian for matter, the previous Lagrangian becomes that of the Schwinger model:

$$\mathcal{L} = \tfrac{1}{2}(\partial_t A_x - \partial_x A_t)^2 + i\bar{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi \ , \quad (9)$$

where $\bar{\psi} = \psi^\dagger\gamma^0$ $\gamma^0 = \sigma^1$ $\gamma^1 = -i\sigma^2$ $\gamma^5 = \gamma^0\gamma^1 = \sigma^3$—the σ's being the Pauli matrices. Standard manipulations yield the $A_t = 0$ gauge Hamiltonian

$$H = \int_0^{2\pi}(\tfrac{1}{2}E_x^2 + i\psi^\dagger h\psi)dx \quad (10)$$

with the first-class constraint known as Gauss's law:

$$\partial_x E_x = e\psi^\dagger\psi \ . \quad (11)$$

Upon integration over the circle this constraint implies that the total electric charge must be zero. Here

$$h = \sigma^3\left[i\frac{\partial}{\partial x} - eA_x\right] \quad (12)$$

is the single-particle Dirac Hamiltonian.

To quantize this system we impose the canonical anticommutator

$$\{\psi_\alpha^\dagger(x), \psi_\beta(y)\} = \delta(x-y)\delta_{\alpha\beta} \quad (13)$$

and the canonical commutator (7). The Hamiltonian becomes the operator

$$H = \int_0^{2\pi}\left[-\frac{1}{2}\left[\frac{\delta}{\delta A(x)} + \frac{i\Theta}{2\pi}\right]^2 + i\psi^\dagger h\psi\right]dx$$
$$\equiv H_1 + H_2 \ , \quad (14)$$

and Gauss's law is imposed as a constraint which defines physical states:

$$\left[i\partial_x\frac{\delta}{\delta A(x)} + e\psi^\dagger\psi(x)\right]|\psi_{\text{phys}}\rangle = 0 \ , \quad (15)$$

where $A(x) \equiv A_x(x)$. These operators will be properly defined, i.e., regulated, in momentum space shortly.

We will expand ψ in the eigenmodes of the single-particle Dirac Hamiltonian which solve the eigenvalue equation

$$h\begin{bmatrix}\varphi_n^1 \\ \varphi_n^2\end{bmatrix} = \omega_n\begin{bmatrix}\varphi_n^1 \\ \varphi_n^2\end{bmatrix} \ . \quad (16)$$

These are

$$\varphi_n^1(x) = \exp\left[-i\omega_n x - ie\int_0^x A(x')dx'\right] \ ,$$
$$\varphi_n^2(x) = -\varphi_n^1(x) \ . \quad (17)$$

Periodic boundary conditions imply that

$$\omega_n = n + eA_0 \ , \quad (18)$$

where A_0 is the previously defined zeroth Fourier component of $A(x)$. Notice that under the gauge transformation (2) the energy levels undergo an overall shift, $\omega_n \to \omega_{n+m}$, by the integer m which labels the homotopy class of the gauge transformation.

The Fourier expansion for the field operator

$$\psi_i(x) = \frac{1}{\sqrt{2\pi}}\sum_n a_n^i\varphi_n^i(x) \quad (19)$$

allows the fermion kinetic energy part of the Hamiltonian to be written as

$$H_2 = \sum_n \omega_n(a_n^{1\dagger}a_n^1 - a_n^{2\dagger}a_n^2) \ , \quad (20)$$

with the anticommutators

$$\{a_n^i, a_m^j\} = \{a_n^{i\dagger}, a_m^{j\dagger}\} = 0 \ ,$$
$$\{a_n^{i\dagger}, a_m^j\} = \delta_{ij}\delta_{mn} \ . \quad (21)$$

Thus $a_p^{1\dagger}$ is the creation operator for a positive-chirality particle of energy ω_p, which we call left handed, and $a_p^{2\dagger}$ is the creation operator for a negative-chirality particle of energy $-\omega_p$, which we call right handed. We use these operators to define the fermionic Fock space as follows: The vacuum of H_2 is defined by the conditions

$$a_n^1|0\rangle = 0 \ \text{ for } \omega_n \geq 0 \ ,$$
$$a_n^{1\dagger}|0\rangle = 0 \ \text{ for } \omega_n < 0 \ ,$$
$$a_n^2|0\rangle = 0 \ \text{ for } \omega_n < 0 \ ,$$
$$a_n^{2\dagger}|0\rangle = 0 \ \text{ for } \omega \geq 0 \ . \quad (22)$$

Then applying these operators appropriately to $|0\rangle$ gives, for fixed $A(x)$, a basis of states which we call $\{|F\rangle\}$, in which each energy level is specified as either empty or filled; and in which all but a finite number of negative-energy levels are filled, and only a finite number of positive-energy levels are filled.

A solution for the fully interacting system can be written

$$|\psi\rangle = \sum_F |F\rangle\chi_F[A] \ , \quad (23)$$

and by solving the system we mean finding all $\chi_F[A]$ such that $|\psi\rangle$ is an eigenstate of H and obeys the constraint. Since we wish to work with the discrete momentum modes of the theory, we Fourier analyze Gauss's law. Use of

$$A(x) = \sum_{-\infty}^{\infty} A_n e^{-inx} \quad (24)$$

and the chain rule for differentiation yields the momentum space form for Gauss's law:

$$G(n)\equiv n\frac{\partial}{\partial A_n}-e\sum_{k,i}a_k^{i\dagger}a_{k-n}^i=0\ . \qquad (25)$$

It is shown in the Appendix that the basis states $|F\rangle$ obey Gauss's law. Hence Gauss's law $G(n)|\psi\rangle=0$, where $|\psi\rangle$ is the state (23), implies

$$\sum_F|F\rangle n\frac{\partial}{\partial A_n}\chi_F[A(x)]=0\ , \qquad (26)$$

which by the completeness of $\{|F\rangle\}$ implies $\partial_{A_n}\chi_F[A]=0$ for $n\neq 0$. This is just a statement of invariance under small gauge transformations: $\chi_F[A(x)]=\chi_F(A_0)$. It is also shown in the Appendix that $\partial_{A_0}|F\rangle=0$.

The standard vector $j^\mu=\bar\psi\gamma^\mu\psi$ and axial-vector $j_5^\mu=\bar\psi\gamma^\mu\gamma^5\psi$ currents with our choice of gamma matrices have the components

$$\begin{aligned}j^0&=j_5^1=\psi_1^\dagger\psi_1+\psi_2^\dagger\psi_2\ ,\\ j^1&=j_5^0=\psi_1^\dagger\psi_1-\psi_2^\dagger\psi_2\ .\end{aligned} \qquad (27)$$

Thus we define the momentum-space chiral charge densities as

$$\rho_\alpha(p)=\int_0^{2\pi}\psi_\alpha^\dagger\psi_\alpha(x)e^{ipx}dx\ , \qquad (28)$$

which upon substitution of the expansion (19) for ψ become

$$\rho_\alpha(p)=\sum_k a_k^{\alpha\dagger}a_{k+p}^\alpha\ . \qquad (29)$$

Alas we may write Gauss's law as

$$G(n)=n\frac{\partial}{\partial A_n}-ej^0(-n)=0\ . \qquad (30)$$

We also Fourier transform H_1 and use this equation to substitute for $\partial/\partial A_n$ when $n\neq 0$, obtaining

$$H_1=\frac{-1}{4\pi}\left[\frac{\partial}{\partial A_0}+i\Theta\right]^2+\frac{1}{4\pi}\sum_{n\neq 0}\frac{1}{n^2}j^0(-n)j^0(n)\ . \qquad (31)$$

Since we have used Gauss's law, the Hamiltonian is now restricted to act on physical states only. The first term of H_1 represents the energy in the electric field, and the second term is the Coulomb energy. When $\Theta=0$ and A_0 is identified with Manton's A_x, we obtain the Hamiltonian he wrote down.

We will rewrite the fermion kinetic term H_2 in terms of currents using the Sugawara formula; hence we pause to examine the chiral charge density operators. Acting on $\{|F\rangle\}$ these have the bosonic commutation relations[6]

$$\begin{aligned}&[\rho_1(-p),\rho_1(q)]=p\delta_{p,q}\ ,\\ &[\rho_2(-p),\rho_2(q)]=-p\delta_{p,q}\ ,\\ &[\rho_1(p),\rho_2(q)]=0\ .\end{aligned} \qquad (32)$$

For $n\neq 0$, $\rho_i(n)$ is a finite operator; but for $n=0$ this is not so and it is necessary to regulate the ultraviolet infinity. We do so with the following gauge-invariant (called heat-kernel) regularization, which exponentially damps the high-momentum modes:

$$\begin{aligned}\rho_1^\lambda(0)&=\sum_p a_p^{1\dagger}a_p^1e^{\lambda\omega_p}\ ,\\ \rho_2^\lambda(0)&=\sum_p a_p^{2\dagger}a_p^2e^{-\lambda\omega_p}\ .\end{aligned} \qquad (33)$$

These operators are gauge invariant under both large and small gauge transformations, because when $\omega_p\to\omega_{p+m}$ then also $a_p\to a_{p+m}$. Acting on a basis state $|F\rangle$ the regularized chiral charge is the same as when it acts on a basis state without fermionic excitations but the same numbers of left- and right-handed particles (see Ref. 6). This associated state is denoted $|M,N\rangle$ to indicate that the left-handed particles fill the levels $\leq M$, and the right-handed particles fill the levels $\geq N$. In the limit $\lambda\to 0$, after a subtraction of $1/\lambda$, the chiral charges are

$$\begin{aligned}\rho_1^{\text{reg}}(0)&=M+eA_0+\tfrac{1}{2}\ ,\\ \rho_2^{\text{reg}}(0)&=-N-eA_0+\tfrac{1}{2}\ .\end{aligned} \qquad (34)$$

Demanding that the total electric charge be zero forces $N=M+1$; and hence, the absolute regularized axial charge is

$$Q_5^{\text{reg}}=2M+2eA_0+1\ . \qquad (35)$$

The fermion kinetic term of the Hamiltonian is also infinite; and when regularized in the same manner as above, acting in the space $\{|F\rangle\}$ (in the limit $\lambda\to 0$, after a $1/\lambda^2$ subtraction) it is equal to its Sugawara[8] form[6]

$$H_2=\tfrac{1}{2}\sum_n\rho_1(n)\rho_1(-n)+\rho_2(-n)\rho_2(n)\ . \qquad (36)$$

SOLUTION OF THE EIGENSTATE PROBLEM

We proceed to find the eigenstates for the complete, physical Hamiltonian:

$$H=H_0+H_I\ ,$$

where

$$\begin{aligned}H_0&=\frac{-1}{4\pi}\left[\frac{\partial}{\partial A_0}+i\Theta\right]^2+\tfrac{1}{2}[\rho_1^{\text{reg}}(0)]^2+[\rho_2^{\text{reg}}(0)]^2\ ,\\ H_I&=\tfrac{1}{2}\sum_{p\neq 0}\Bigg\{\rho_1(p)\rho_1(-p)+\rho_2(-p)\rho_2(p)\\ &\quad+\frac{e^2}{2\pi p^2}[\rho_1(p)+\rho_2(p)][\rho_1(-p)+\rho_2(-p)]\Bigg\}\ .\end{aligned} \qquad (37)$$

First we look for a canonical transformation which eliminates the off-diagonal terms $\rho_1\rho_2$:

$$H'=e^{iS}He^{-iS}\ . \qquad (38)$$

Here S is the Hermitian operator

$$S=i\sum_{p\neq 0}\frac{\phi(p)}{p}\rho_1(p)\rho_2(-p)\ , \tag{39}$$

where $\phi(p)$ is a real, even function of p to be determined, with $\phi(0)$ defined to be 0. From the charge-density commutators (32) we have that

$$\begin{aligned}[][iS,\rho_1(p)]&=\phi(p)\rho_2(p)\ ,\\ [iS,\rho_2(p)]&=\phi(p)\rho_1(p)\ ,\end{aligned} \tag{40}$$

which in turn, upon use of the operator expansion

$$e^{A}Be^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\cdots\ , \tag{41}$$

gives us

$$\begin{aligned}e^{iS}\rho_1(p)e^{-iS}&=\rho_1(p)\cosh\phi(p)+\rho_2 p\sinh\phi(p)\ ,\\ e^{iS}\rho_2(p)e^{-iS}&=\rho_2(p)\cosh\phi(p)+\rho_1(p)\sinh\phi(p)\ .\end{aligned} \tag{42}$$

It is now easy to apply the canonical transformation to H. Clearly H_0 transforms to itself; and we find that choosing $\phi(p)$, for $p\neq 0$, to satisfy

$$\coth[2\phi(p)]=-\left[1+\frac{2\pi p^2}{e^2}\right] \tag{43}$$

results in H_I being transformed to

$$H_I'=\sum_{p>0}\frac{\sqrt{\mu^2+p^2}}{p}[\rho_1(p)\rho_1(-p)+\rho_2(-p)\rho_2(p)]\ , \tag{44}$$

where $\mu=e/\sqrt{\pi}$.

We now exhibit the eigenstates of H_0 in the subspace spanned by basis states without fermionic excitations, i.e., basis states of form $|M,M+1\rangle$, denoted by $|M\rangle$. The important observation to make here is that these states are annihilated by $\rho_1(-p)$ and $\rho_2(p)$ for $p>0$; which means $H_I'|M\rangle=0$. Hence these states will be exact eigenstates for H', whereas for Manton's Hamiltonian they only are approximate (he dropped the Coulomb term to find these states[6]).

Writing

$$|\psi\rangle=\sum_M|M\rangle\chi_M(A_0) \tag{45}$$

and using the expression (37) for H_0, along with the result (34) for $\rho_i(0)$, yields

$$H'|\psi\rangle=\sum_M|M\rangle\left[\frac{-e^2}{4\pi}\left[\frac{\partial}{\partial A_0}+i\Theta\right]^2+(M+A_0+\tfrac{1}{2})^2\right]\chi_M(A_0)\ . \tag{46}$$

Thus a stationary state $|\psi\rangle$ is specified by the set of wave functions $\{\chi_M(A_0)|M\in Z,\ 0\leq A_0\leq 1\}$ satisfying

$$\left[\frac{-e^2}{4\pi}\left[\frac{\partial}{\partial A_0}+i\Theta\right]^2+(M+A_0+\tfrac{1}{2})^2\right]\chi_M(A_0)=E\chi_M(A_0)\ , \tag{47}$$

with boundary conditions

$$\begin{aligned}\chi_M(1)&=\chi_{M+1}(0)\ ,\\ \partial_{A_0}\chi_M(1)&=\partial_{A_0}\chi_{M+1}(0)\ .\end{aligned} \tag{48}$$

The first boundary condition states that $|M\rangle$ when $A_0=1$ is the same state as $|M+1\rangle$ when $A_0=0$. The second says the two should join smoothly, due to the angular nature of $2\pi A_0$. Then upon the definition

$$\chi_M(A_0)=\chi(M+A_0)=\chi(\tilde{A})\ , \tag{49}$$

the system of equations reduces to an eigenvalue equation for a wave function over the real line:

$$\left[\frac{-e^2}{4\pi}\left[\frac{\partial}{\partial\tilde{A}}+i\Theta\right]^2+(\tilde{A}+\tfrac{1}{2})^2\right]\chi(\tilde{A})=E\chi(\tilde{A})\ . \tag{50}$$

As observed by Manton, this is essentially the harmonic-oscillator problem, and has the solutions

$$\begin{aligned}\chi^n(\tilde{A})&=\mathcal{H}_n(\tilde{A}+\tfrac{1}{2})e^{(-1/\mu)(\tilde{A}+1/2)^2-i\Theta(\tilde{A}+1/2)}\ ,\\ E_n&=(n+\tfrac{1}{2})\mu\ ,\end{aligned} \tag{51}$$

where $\mathcal{H}_n$ is the nth Hermite polynomial.

We denote by $|\psi_n\rangle$ the nth eigenstate:

$$|\psi_n\rangle=\sum_M|M\rangle\chi_M^n(A_0)\ . \tag{52}$$

Although every term in this sum is by itself an eigenstate, obeying Gauss's law, with energy E_n; it is only the sum which is invariant under large gauge transformations, by which $A_0\to A_0+m$ and $|M\rangle\to|M+m\rangle$, where m is the homotopy class of the gauge transformation.

From the zero-momentum sector $\{|\psi_n\rangle\}$ we can easily construct the rest of the eigenstates as follows. Define, for $p>0$,

$$\begin{aligned}p^{-1/2}\rho_1(p)&=B^\dagger(p)\ ,\\ p^{-1/2}\rho_2(-p)&=B^\dagger(-p)\ ,\\ p^{-1/2}\rho_1(-p)&=B(p)\ ,\\ p^{-1/2}\rho_2(p)&=B(-p)\ .\end{aligned} \tag{53}$$

The B's have the commutators of a scalar, and in terms of them

$$H_I'=\sum_{p\neq 0}\sqrt{\mu^2+p^2}B^\dagger(p)B(p)\ . \tag{54}$$

The operators $B(p)$ annihilate the states $|\psi_n\rangle$, whereas the operators $B^\dagger(p)$ create excitations of energy $\sqrt{\mu^2+p^2}$. For example,

$$H'B^\dagger(p)|\psi_n\rangle=[(n+\tfrac{1}{2})\mu+\sqrt{\mu^2+p^2}]B^\dagger|\psi_n\rangle\ . \tag{55}$$

Application of arbitrary sequences of these bosonic creation operators to $\{|\psi_n\rangle\}$ yield all the stationary states, $\{|\psi\rangle\}$, of the transformed Hamiltonian. $|\psi_n\rangle$ corresponds to the state with n zero-momentum scalars in the bosonized version of the theory. The stationary states of the original Hamiltonian are

$$e^{-iS}|\psi\rangle \,, \tag{56}$$

and by construction obey the constraint equation (31).

$$|\Theta\rangle = \frac{1}{N^{1/2}} \sum_{M=-\infty}^{M=+\infty} e^{(-1/\mu)(eA_0+M+1/2)^2 - i\Theta(A_0+M+1/2)} e^{-iS}|M\rangle \,, \tag{57}$$

where the normalization factor is easily determined to be $N=\sqrt{\mu\pi/2}$ (when calculating scalar products one must not forget to integrate over A_0 from 0 to 1). In the limit that the coupling constant μ goes to zero, the value of A_0 is arbitrary and we choose it to be $\frac{1}{2}$, in which case, since the canonical transformation (38) becomes the identity $|\psi_0\rangle \to |-1\rangle$, which is precisely the ground state of the free, purely fermionic theory. When the coupling constant is nonzero there is a finite probability of finding any particular number of fermions, in a charge neutral configuration, in the vacuum.

Having obtained the states, we may calculate matrix elements for the observables and compare our solution with that obtained in Ref. 5. Because the nonzero-momentum sector is identical in form for both treatments we only give matrix elements for the zero-momentum eigenstates $|\psi_\Theta^n\rangle \equiv \exp(-iS)|\psi_n\rangle$, where $|\psi_n\rangle$ is given by (52). The electric field is given by

$$\langle \psi_\Theta^m | E | \psi_\Theta^n \rangle = -i \int_{-\infty}^{+\infty} dx\, \mathcal{U}^m(x) \frac{\partial}{\partial x} \mathcal{U}^n(x) \,, \tag{58}$$

where $\mathcal{U}^m$ is the mth level harmonic-oscillator wave function with frequency μ. The nonintegrable phase $A = e^{2\pi i e A_0}$ has matrix elements

$$\langle \psi_\Theta^m | A | \psi_\Theta^n \rangle = -\int dx\, \mathcal{U}^m(x) e^{2\pi i x} \mathcal{U}^n(x) \,. \tag{59}$$

The vector charge is zero, whereas the axial charge is

$$\langle \psi_\Theta^m | \psi^\dagger \gamma^5 \psi | \psi_\Theta^n \rangle = \frac{1}{\pi} \int dx\, \mathcal{U}^m(x) x \mathcal{U}^n(x) \,. \tag{60}$$

These are all in precise agreement with Ref. 5 provided that we set their constants to $\alpha = \frac{1}{2}$, $L = 2\pi$. In particular these imply the vacuum expectations

$$\langle E \rangle = \langle \psi^\dagger \gamma^5 \psi \rangle = 0 \quad \text{and} \quad \langle A \rangle = -e^{-\mu\pi^2/2} \,. \tag{61}$$

To calculate the matrix elements of $\frac{1}{2}\bar\psi(1-\gamma_5)\psi$, which with our choice of γ matrices is $\psi_1^\dagger(x)\psi_2(x)$, we first obtain

$$\langle M | e^{iS} \psi_1^\dagger \psi_2 e^{-iS} | N \rangle = -\frac{1}{2\pi} C(\mu) \delta(N, M-1) \,, \tag{62}$$

where $C(\mu) = 1 + \sum_p' 2/|p|\phi(p) + \cdots$ is a positive constant whose closed form we have not yet been able to give. From this we obtain

$$\langle \psi_\Theta^m | \psi_1^\dagger \psi_2 | \psi_\Theta^n \rangle = -\frac{1}{2\pi} C(\mu) e^{i\Theta} \times \int dx\, \mathcal{U}^m(x+\tfrac{1}{2}) \mathcal{U}^n(x-\tfrac{1}{2}) \,, \tag{63}$$

which for the case of the vacuum gives

$$\langle \psi_1^\dagger \psi_2 \rangle = -\frac{\mu}{2\sqrt{\pi}} C(\mu) e^{-1/2\mu} e^{i\Theta} \,. \tag{64}$$

Again this is in complete agreement with Ref. 5 provided that $\mu(C) = 2\sqrt{\pi} B(\mu, 2\pi)$, where B is given in Ref. 5. Matrix elements of products of operators can now be calculated by inserting a resolution of unity between factors.

Because Θ is multiplied by the axial charge Q_5 [recall formula (35)], the sole effect of a chiral transformation

$$|\psi_\Theta^n\rangle \to e^{i\omega Q_5} |\psi_\Theta^n\rangle \tag{65}$$

is to shift the value of the Θ angle by 2ω, as has been noted in the literature. Recently,[9] authors have found two vacuum angles in the Schwinger model, the other of which enters similarily to Θ, but multiplies the vector charge. This cannot appear here because we consider only physical states, and these have zero vector charge.

The sum (52) may be viewed as a basis transformation from the subbasis $\{|M\rangle\}$, to the subbasis $\{|\psi_n\rangle\}$. In older treatments,[4] this basis transformation was introduced to diagonalize certain local observables; however, only one row of the transformation matrix was known—giving us only one state of the new basis, namely, the Θ vacuum. Our result may be summarized as follows. Fully gauge invariant, nondegenerate eigenstates to the canonically transformed problem are obtained by applying the transformation matrix χ_M^n [given by (51)] to the subbasis $\{|M\rangle\}$ to obtain the subbasis $\{|\psi_n\rangle\}$. The rest of the eigenstates are obtained by applying nonzero-momentum creation operators to $\{|\psi_n\rangle\}$. The eigenstates for the Schwinger model on the circle are then given by applying e^{-iS}, and are in fact precisely the particle states of the free massive scalar theory.

DISCUSSION

The normalized ground state is the $n=0$ state which we have just constructed:

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APPENDIX

Here we proceed to demonstrate that $|F\rangle$ obeys Gauss's law (15). To do so we must first obtain its derivative by $A(x)$. Hence we claim that the operator $\mathcal{A}$ defined (up to a to-be-fixed additive constant) by

$$\frac{\delta a}{\delta A(x)}=[\mathcal{A}(x),a] , \tag{A1}$$

where a is any of the elementary creation or annihilation operators, gives the derivative

$$\frac{\delta|F\rangle}{\delta A}=\mathcal{A}|F\rangle . \tag{A2}$$

The proof is as follows.

Since $|F\rangle=a\cdots a^{\dagger}|0\rangle$ for some string of elementary creation and annihilation operators we may write

$$\begin{aligned}\frac{\delta|F\rangle}{\delta A}&=\left[\frac{\delta}{\delta A},a\cdots a^{\dagger}\right]|0\rangle+a\cdots a^{\dagger}\frac{\delta}{\delta A}|0\rangle\\&=[\mathcal{A},a\cdots a^{\dagger}]|0\rangle+a\cdots a^{\dagger}\frac{\delta}{\delta A}|0\rangle\\&=\mathcal{A}|F\rangle-a\cdots a^{\dagger}\left[\mathcal{A}-\frac{\delta}{\delta A}\right]|0\rangle .\end{aligned} \tag{A3}$$

Hence it remains to show that (A2) is true for $|F\rangle=|0\rangle$. We may use the fact that $\mathcal{A}$ is still only defined up to a constant to set $\langle 0|\mathcal{A}|0\rangle=\langle 0|(\delta/\delta A)|0\rangle$. It is easily shown that for some other abritrary basis state $|G\rangle=a\cdots a^{\dagger}|0\rangle$ that $\langle G|\mathcal{A}|0\rangle=\langle G|(\delta/\delta A)|0\rangle$. The arbitrariness of $|G\rangle$ allows us to say our claim is true, and we go on to find an explicit expression for $\mathcal{A}$.

Since the electric field operator and the Fermi field operator commute,

$$\left[\frac{\delta}{\delta A(y)},\sum_n a_n^i\varphi_n^i(x)\right]=0 , \tag{A4}$$

Eq. (A1) and the orthogonality of the φ_n^i's imply

$$[\mathcal{A}(y),a_m^i]=-\sum_n A_{mn}^i(y)a_n^i , \tag{A5}$$

where

$$A_{mn}^i(y)=\frac{1}{2\pi}\int_0^{2\pi}\varphi_m^{i*}\frac{\delta\varphi_n^i(x)}{\delta A(y)}dx . \tag{A6}$$

Equation (A5) is to be understood as acting on $|F\rangle$, and in turn implies

$$\mathcal{A}(x)=\sum_{mni} A_{mn}^i(x)a_m^{i\dagger}a_n^i+f , \tag{A7}$$

where f is some complex-valued function which we set zero in order to obtain $\langle 0|\mathcal{A}|0\rangle=\langle 0|E_x|0\rangle=0$. Substituting the eigenmodes (17) into (A6) gives A_{mn}^i, which in turn gives $\mathcal{A}$ as

$$\mathcal{A}(x)=-\frac{ie}{2\pi}\sum_j\left[(\pi-x)\sum_m a_m^{j\dagger}a_m^j+\sum_{m\neq n}\frac{e^{i(m-n)x}}{m-n}a_m^{j\dagger}a_n^j\right] , \tag{A8}$$

so that finally we get Gauss's law for the state $|F\rangle$:

$$\partial_x\mathcal{A}(x)=ie\psi^{\dagger}(x)\psi(x) . \tag{A9}$$

A heat-kernel regularization, as in the main text, does not change the conclusions.

Using the chain rule for differentiation we may rewrite Eq. (A2) as

$$\frac{\partial|F\rangle}{\partial A_m}=\int_0^{2\pi}e^{-imx}\mathcal{A}(x)dx|F\rangle , \tag{A10}$$

which for the case $m=0$ gives $\partial_{A_0}|F\rangle=0$, and for the case $m\neq 0$ this is Gauss's law in momentum space for the state $|F\rangle$.

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