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Schrödinger equation for the nonrelativistic particle constrained on a hypersurface in a curved space

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Dirac-bracket quantization of the nonrelativistic particle whose motion is constrained on the hypersurface $f(x)=\text{const}$ embedded in a general curved space is discussed. The noncanonical nature of commutation relations makes it difficult to obtain the coordinate representation. Then the system with the derivative-type constraint $df(x)/dt=0$ is alternatively quantized, treating carefully the operator-ordering problem. In this system it is shown that there exist no constraints on coordinates and momenta and that one can thus have a straightforward representation, which leads, in turn, to the representation and the Schrödinger equation in the former system.

I. INTRODUCTION

Quantization of constrained Hamiltonian systems was first established by Dirac[1,2] in the 1950s, and has been applied to various kinds of singular systems. With the Dirac method one introduces Dirac brackets and, upon quantization, replaces them by $-i$ times commutators. However, in many cases the underlying Dirac brackets have rather complicated forms, and owing to this one is led to several difficulties associated with the quantization. We are, therefore, concerned with this problem in the present paper. Namely, we want to have a general method of quantization where the coordinate representation of momentum operators, with the commutation relations being complicated, acquires highly nontrivial expressions and cannot easily be found in contrast with the conventional case in which momentum operators are represented by $-i$ times the derivatives with respect to coordinate variables. Finding this representation is quite important in writing down the Schrödinger equation. Another difficulty, which we take up, is related to the operator-ordering problem.[3] This problem arises not only in defining various observables such as the Hamiltonian, but in defining commutators and constraint operators containing the products of noncommuting operators.

In a previous work[4] we have studied the Hamiltonian formulation of a rigid rotator,[5] i.e., a free particle constrained on a sphere. The constraint $x^2=A$ [$x=x(t)$ stands for the position of the particle in the n-dimensional space and A the squared radius of the sphere] was imposed through the introduction of a Lagrange multiplier. As is well known, the corresponding Dirac brackets differ from the Poisson brackets and involve the products of coordinates and momenta. When one quantizes such a system, in order to avoid the difficulties stated above, one often transforms, *at the classical level,* to a specific coordinate system so that the Dirac brackets for independent variables may reduce to "Kronecker's deltas." Then, apart from the operator-ordering problem, the quantization in the Schrödinger representation will easily be carried out. However, it is completely unclear whether or not the transformation of variables (i.e., the canonical transformation) is uniquely defined *at the quantum level.* This will produce an additional ambiguity. It is hence desirable that one can find the representation of operators without any recourse to specific coordinate systems.

As a first step toward the resolution of these problems, we have considered the system subject to the constraint $x\dot{x}=0$ ($\dot{x}=dx/dt$) instead of $x^2=A$ (Ref. 4). Evidently the former constraint is equivalent to the set of the constraints of the latter type for all values of A. But, somewhat surprisingly, the corresponding Hamiltonian systems are *apparently quite different.* In the former case the Dirac brackets for the phase-space variables, say, x and π, have the same expressions as those of Poisson brackets; i.e., they behave as unconstrained variables, and the constraint $x\dot{x}=0$ is implemented in the equation of motion[6] such that $\{x^2,\text{Hamiltonian}\}_D=0$. Then, setting the value of the constant x^2 to be A, we have shown that the tangential components P of the momenta π ("tangential" in the meaning of "perpendicular to vectors normal to the hypersphere defined by $x^2=A$") satisfy the same Dirac brackets as the ones appearing in $x^2=A$ theory. Thus the canonical momenta in $x^2=A$ theory are identified with P (with a fixed constant A). The advantage of choosing the constraint $x\dot{x}=0$ instead of $x^2=A$ becomes manifest when we proceed to the quantization. Namely, we can straightforwardly obtain the coordinate representation of the momenta π, and then their tangential components P give the representation of the momenta in $x^2=A$ theory.[7] Consequently we have the Schrödinger equation without referring to a "convenient" coordinate system such as a polar coordinate.

In the present paper, we extend the above arguments to the case of a general hypersurface $f(x)=B$ ($B=\text{const}$). Here the space in which the hypersurface is embedded need not be flat, but is assumed to be a curved space endowed with a (Reimannian) metric structure. We impose the constraint with the use of a Lagrange multiplier and construct the Dirac brackets. Next we consider the system under the derivative-type constraint $\dot{f}(x)=0$, which includes the constraint $f(x)=B$ as a "subset." The momenta in this system are unconstrained, but the Hamiltonian involves them in combination with a certain projection matrix picking up the components tangential to the hypersurface $f(x)=\text{const}$. The actual motion is thereby restricted to the hypersurface and we recover the relation $\dot{f}(x)=0$ through the equation of motion. Then, arranging the order of operators in a certain way and setting the value of the constant $f(x)$ to be B, we will find that the quantum mechanics for this system is equivalent to that with the constraint $f(x)=B$. [Of course, there is not a complete equivalence, in the sense that in the system subject to the derivative-type constraint one can arbitrarily choose the value of $f(x)$ as an initial data, not just as B.] Further, the tangential components of π (including quantum corrections) precisely give the coordinate representation of the momenta for the $f(x)=B$ theory, with which we can obtain the Schrödinger equation. We also propose some principles that can be used to determine the order of noncommuting operators.

The present paper is organized as follows. In Sec. II, we study the Dirac-bracket quantization of the system under the constraint $f(x)=B$. We find the four second-class constraints and with these we construct the Dirac brackets. In Sec. III, we consider the case of the derivative-type constraint $\dot{f}(x)=0$. In the Hamiltonian formulation there appear no constraints on coordinates and momenta, but the equation of motion tells us that the particle is still "constrained" on the hypersurface $f(x)=\text{const}$. We study in Sec. IV the relation between these systems, paying attention to the problem of operator ordering, and the Schrödinger equation is derived in both systems. Section V is devoted to the application of our method to the simple example: the rigid rotator in a flat space. We give concluding remarks in Sec. VI.

II. QUANTIZATION OF THE SYSTEM UNDER THE CONSTRAINT $f(x)=B$

Let x^i $(i=1,\ldots,n)$ be the coordinates of the n-dimensional curved space equipped with the Riemannian metric $g_{ij}(x)$. We consider a nonrelativistic particle of mass m whose motion is constrained on the hypersurface defined by

$$f(x)=B \quad (B=\text{const}) . \tag{2.1}$$

In the presence of the vector and the scalar potentials, $A_i(x)$ and $V(x)$, the Lagrangian is given by

$$L=\tfrac{1}{2}mg_{ij}\dot{x}^i\dot{x}^j+A_i(x)\dot{x}^i-V(x)+\lambda[f(x)-B] ,$$
$$\dot{x}^i\equiv dx^i/dt , \tag{2.2}$$

where $x^i(t)$ denotes the position of the particle and λ is a Lagrange multiplier. The Lagrangian is singular because it does not contain the "velocity" $\dot{\lambda}$. Hence we apply Dirac's method.[1,2] The canonical momenta conjugate to x^i and λ are

$$p_i=mg_{ij}\dot{x}^j+A_i , \tag{2.3a}$$

$$p_\lambda=0 , \tag{2.3b}$$

respectively, and Eq. (2.3b) represents the primary constraint:

$$\phi_0\equiv p_\lambda\approx 0 . \tag{2.4}$$

The primary Hamiltonian (the total Hamiltonian) H_p is then written as

$$H_p=\frac{1}{2m}g^{ij}(p_i-A_i)(p_j-A_j)+V(x)$$
$$-\lambda[f(x)-B]+vp_\lambda , \tag{2.5}$$

where g^{ij} is the inverse of g_{ij}, and $v(t)$ is the multiplier associated with the primary constraint (2.4) ($v=\dot{\lambda}$ in this case). Requiring the time derivatives of the (primary and/or secondary) constraints to vanish, we have, successively,

$$\phi_1\equiv f(x)-B\approx 0 , \tag{2.6}$$

$$\phi_2\equiv\frac{1}{m}g^{ij}\frac{\partial f}{\partial x^i}(p_j-A_j)\approx 0 , \tag{2.7}$$

$$\begin{aligned}\phi_3\equiv\frac{1}{m^2}\Bigg\{&g^{kl}\left[\frac{\partial}{\partial x^k}\left[g^{ij}\frac{\partial f}{\partial x^i}\right]p_j-\frac{\partial}{\partial x^k}\left[g^{ij}\frac{\partial f}{\partial x^i}A_j\right]\right](p_l-A_l)\\&-\frac{1}{2}\frac{\partial g^{kl}}{\partial x^i}g^{ij}\frac{\partial f}{\partial x^j}p_kp_l+\frac{\partial}{\partial x^j}(g^{kl}A_l)g^{ij}\frac{\partial f}{\partial x^i}p_k\Bigg\}-\frac{1}{m}g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial V}{\partial x^j}+\frac{1}{m}D\lambda\approx 0 .\end{aligned} \tag{2.8}$$

Here the function D appearing in the last line of Eq. (2.8) is defined by

$$D=g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} . \tag{2.9}$$

Namely, D is the squared length of the vector $\partial f(x)/\partial x^i$ normal to the hypersurface (2.1). We can assume without loss of generality $D>0$ with g_{ij} being the Riemannian metric. As a consequence the requirement $\dot{\phi}_3\approx 0$ solves $v(t)$ as a function of x^i and p_i. We now have the four constraints, and the Poisson brackets satisfied by them are

$$\{\phi_0,\phi_3\}=-\frac{1}{m}D , \tag{2.10a}$$

$$\{\phi_1,\phi_2\}=\frac{1}{m}D , \tag{2.10b}$$

and those which are not needed in the following calculations. All the constraints are thus second class. In the next step we construct the Dirac brackets by making use of their iterative property.[2] The results are[8]

$$\{x^i,x^j\}_D=0\ , \tag{2.11a}$$

$$\{x^i,p_j\}_D=F^i{}_j(B)\ , \tag{2.11b}$$

$$\{p_i,p_j\}_D=-\frac{1}{D}\left[\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}\left[g^{kl}\frac{\partial f}{\partial x^k}\right]p_l-\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}\left[g^{kl}\frac{\partial f}{\partial x^k}A_l\right]\right]\Bigg|_B-(i\leftrightarrow j)\ , \tag{2.11c}$$

where $F^i{}_j(B)$ is given by $F^i{}_j(B)=F^i{}_j|_B$, with

$$F^i{}_j=\delta^i{}_j-\frac{1}{D}g^{ik}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^j}\ . \tag{2.12}$$

Because of the constraints which are now imposed *strongly,* the Hamiltonian reduces to

$$H_p=\frac{1}{2m}g^{ij}(p_i-A_i)(p_j-A_j)+V(x)\ . \tag{2.13}$$

The classical dynamics of the system can be described by this Hamiltonian and the Dirac brackets (2.11).

In order to quantize the system, we must replace Eqs. (2.11) by commutators. However not only the Hamiltonian (2.13) but also the Dirac brackets (2.11c) contain the products of the coordinates and momenta, which do not commute in the quantum theory. Thus the commutators are not uniquely determined. Among the various possibilities, the simplest choices for the fundamental commutation relations are

$$[x^i,x^j]=0\ , \tag{2.14a}$$

$$[x^i,p_j]=iF^i{}_j(B)\ , \tag{2.14b}$$

$$[p_i,p_j]=-i\left\{\frac{1}{2}\left[\frac{1}{D}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}\left[g^{kl}\frac{\partial f}{\partial x^k}\right](p_l-A_l)+(p_l-A_l)\frac{\partial}{\partial x^j}\left[g^{kl}\frac{\partial}{\partial x^k}\right]\frac{\partial f}{\partial x^i}\frac{1}{D}\right]-\frac{1}{D}g^{kl}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^k}\frac{\partial A_l}{\partial x^j}\right\}\Bigg|_B-(i\leftrightarrow j)\ . \tag{2.14c}$$

The quantum Hamiltonian also is not unique and we assume

$$H_p=\frac{1}{2m}(p_i-A_i)g^{ij}(p_j-A_j)+V(x)\ . \tag{2.15}$$

In principle, the quantum-mechanical property of this system is entirely characterized by the commutation relations (2.14) and the quantum Hamiltonian (2.15). But it is rather unclear how the constraints are realized in the quantum theory. Indeed it comes to a nontrivial problem because, for example, the constraint (2.7) suffers from the ambiguity in the order of operators. In the classical description, the constraints are implemented in the introduction of Dirac brackets such that they have vanishing Dirac brackets with an arbitrary function of phase-space variables, which fact leads us, at the quantum level, to the natural requirement that the constraint operator ϕ should obey

$$[\phi,G(x,p)]=0\ , \tag{2.16}$$

where $G(x,p)$ is an arbitrary operator. This condition imposes a strict restriction on the operator form of the constraint. For example, the expression of Eq. (2.7) with x^i and p_i replaced by operators is forbidden when we adopt the commutation relations (2.14) [especially, Eq. (2.14c)]. In fact the "symmetrization" of the right-hand side of Eq. (2.7) gives us the simplest solution satisfying the condition (2.16) for $\phi=\phi_2$:

$$\phi_2=\frac{1}{2m}\left[g^{ij}\frac{\partial f}{\partial x^i}(p_j-A_j)+(p_j-A_j)\frac{\partial f}{\partial x^i}g^{ij}\right]\approx 0\ , \tag{2.17}$$

which we take up as *an operator form of the constraint* (2.7). If one chose the commutator different from Eq. (2.14c), the allowable expressions for the constraint operator ϕ_2 would also be different. Hence choosing specific forms of commutators amounts, in general, to restricting the operator forms of constraints to some extent.[9]

Now we are left with an important task to write down the coordinate representation of operators and the Schrödinger equation of motion. The commutation relations (2.14) are quite complicated and moreover they are singular (i.e., all the p_i's are no longer independent operators): it is thus too hard to construct a representation from Eqs. (2.14) by a trial-and-error method. As was briefly commented in Introduction, the Dirac-bracket quantization of constrained Hamiltonian systems often encounters such a difficulty and, in many cases, one carries out the calculation in the reduced phase space. In fact, the (reduced) phase space is a symplectic manifold in the mathematical language,[10] and then Darboux's theorem ensures that one can find at least locally the coordinates in terms of which the Poisson brackets (defined on the reduced phase space in the presence of constraints) have the canonical forms. Therefore the *classical* description of constrained systems can always be performed in the reduced phase space equipped with a canonical Poisson-bracket structure. Then one can quantize the system as in the conventional quantum mechanics. However the procedure stated above cannot in general be completely justified, because it is not evident whether or not the transformation to the reduced phase space has a uniquely defined *quantum* analogue. It has been an open problem.

In relation to this point, the case of special interest worth being mentioned here is that in which the representation of operators is fixed through certain physical or mathematical principles, without referring to the canonical coordinates in the reduced phase space. We will find a situation such as this in Sec. IV, where we consult a somewhat indirect procedure to obtain the representation of operators.

III. QUANTIZATION OF THE SYSTEM UNDER THE CONSTRAINT $\dot{f}(x)=0$

In this section we consider the quantization of the system subject to the constraint defined by the time derivative of the function $f(x)$ $[x=x(t)]$:

$$\dot{f}(x)=0 \; . \tag{3.1}$$

Incorporation of the constraint into the Lagrangian formulation is again performed by using the Lagrange multiplier λ:

$$L=\tfrac{1}{2}mg_{ij}\dot{x}^i\dot{x}^j+A_i(x)\dot{x}^i-V(x)+\lambda\dot{f}(x) \; . \tag{3.2}$$

A glance at the constraints (2.1) and (3.1) would imply that both systems lead to, up to the constant B, the essentially equivalent (classical or quantum) dynamics. But as will be seen, *it is far from a trivial problem.*

We start with the definition of the canonical momenta:

$$\pi_i=mg_{ij}\dot{x}^j+A_i+\lambda\frac{\partial f}{\partial x^i} \; , \tag{3.3a}$$

$$\pi_\lambda=0 \; . \tag{3.3b}$$

The latter becomes the primary constraint:

$$\chi_0\equiv\pi_\lambda\approx 0 \; . \tag{3.4}$$

The primary Hamiltonian is given by

$$H_p=\frac{1}{2m}g^{ij}\left[\pi_i-A_i-\lambda\frac{\partial f}{\partial x^i}\right]\left[\pi_j-A_j-\lambda\frac{\partial f}{\partial x^j}\right] +V(x)+u\pi_\lambda \; , \tag{3.5}$$

where $u(=\dot{\lambda})$ is a multiplier for the primary constraint. The consistency condition provides the following secondary constraint:

$$\chi_1\equiv\frac{1}{m}g^{ij}\frac{\partial f}{\partial x^i}\left[\pi_j-A_j-\lambda\frac{\partial f}{\partial x^j}\right]\approx 0 \; , \tag{3.6}$$

and the requirement $\dot{\chi}_1\approx 0$ enables us to solve $u(t)$. The constraints (3.4) and (3.6) are second class because $\{\chi_0,\chi_1\}=D/m$ [D is defined by Eq. (2.9)], which does not vanish on the constraint surface. Owing to these constraints λ and π_λ are no longer independent variables: not so, however, for the x^i's and the π_i's in contrast with the system based on the Lagrangian (2.2). The Dirac brackets for these variables consequently reduce to the same forms as those of Poisson brackets,

$$\{x^i,x^j\}_D=\{\pi_i,\pi_j\}_D=0 \; , \tag{3.7a}$$

$$\{x^i,\pi_j\}_D=\delta^i{}_j \; , \tag{3.7b}$$

while the Dirac brackets for λ and π_λ are used to impose

$$\lambda=\frac{1}{D}g^{ij}\frac{\partial f}{\partial x^i}(\pi_j-A_j) \tag{3.8a}$$

and

$$\pi_\lambda=0 \tag{3.8b}$$

as strong equations. These relations are inserted into the Hamiltonian (3.5) to give

$$H_p=\frac{1}{2m}g^{kl}F^i{}_k(\pi_i-A_i)F^j{}_l(\pi_j-A_j)+V(x) \; . \tag{3.9}$$

The matrix $F^i{}_j$ [defined by Eq. (2.12)] satisfies $F^i{}_jF^j{}_k=F^i{}_k$, and

$$F^j{}_i\frac{\partial f}{\partial x^j}=g^{jk}\frac{\partial f}{\partial x^k}F^i{}_j=0 \; . \tag{3.10}$$

These identities show that $F^i{}_j$ is the matrix which projects all vectors onto the hypersurface defined by $f(x)=$const. The form of the Hamiltonian (3.9) therefore tells that it contains only the "tangential" components of the vector π_i-A_i. Here it should be noted that the third term on the right-hand side of Eq. (3.3a) is proportional to the vector normal to the hypersurface, and thus the π_i's are certainly *unconstrained* (with nonzero λ). But the Hamiltonian, as it stands, prohibits the motion away from the hypersurface:

$$\frac{df(x)}{dt}=\{f(x),H_p\}_D=0 \; , \tag{3.11}$$

where use has been made of Eq. (3.10). Namely, the relation (3.1) is recovered through an equation of motion and then *$f(x)$ is a constant of motion.*

The quantization is carried out with the Dirac brackets (3.7) replaced by the commutation relations

$$[x^i,x^j]=[\pi_i,\pi_j]=0 \; , \tag{3.12a}$$

$$[x^i,\pi_j]=i\delta^i{}_j \; . \tag{3.12b}$$

The coordinate representation of the momentum operator involves the corrections originating from the curved nature of the underlying space,[11] and is given by

$$\pi_i=-i\frac{\partial}{\partial x^i}-\tfrac{1}{2}i\begin{Bmatrix} j \\ ji\end{Bmatrix}(x) \; , \tag{3.13}$$

where

$$\begin{Bmatrix} i \\ jk\end{Bmatrix}=\tfrac{1}{2}g^{il}(\partial_j g_{kl}+\partial_k g_{jl}-\partial_l g_{jk}) \tag{3.14}$$

is a Christoffel symbol constructed with the Riemannian metric $g_{ij}(x)$. One can arrive at the expression (3.13) without suffering from the problem of operator ordering as in the conventional quantum mechanics, and such a remarkable feature (compared to the case treated in Sec. II) comes from the "constraint-free" property of the system. The Hamiltonian, however, has a more complicated form than that of the system under the constraint (2.1) and is not free from the operator-ordering problem. At this stage, aside from the condition of Hermiticity, we have no appropriate guiding principles to determine the ordering of the Hamiltonian (3.9). We return to this point in the next section.

IV. EQUIVALENCE OF THE TWO ALTERNATIVE FORMULATIONS

Now we have the two theories describing a particle moving on a hypersurface of the type $f(x)=\text{const}$—the theory with the constraint $f(x)-B=0$ (system I) and the theory with $\dot{f}(x)=0$ (system II). As mentioned before, these systems are expected to give essentially equivalent dynamics once one assigns, in system II, the initial value B to the constant of motion $f(x)$. (Henceforth we refer to the equivalence of both systems in this narrow sense unless otherwise stated.)

In the first place, we consider at the classical level and the vector potential A_i is set to be zero for simplicity. The notable aspect of system II is that the Hamiltonian (3.9) contains the matrix of rank $n-1$, and one degree of freedom for the unconstrained momentum π_i happens to be dropped out of the Hamiltonian. It implies that the only $n-1$ coordinates (i.e., the conjugate partners for $F^j{}_i\pi_j$) acquire "dynamics" through the Hamiltonian, and hence we have the identity (3.11). On the basis of the observation stated above, we introduce the new momentum variables which are defined by the "tangential" components of π_i:

$$P_i=F^j{}_i\pi_j \ . \tag{4.1}$$

Then the classical Hamiltonian simply becomes

$$H_p=\frac{1}{2m}g^{ij}P_iP_j+V(x) \ , \tag{4.2}$$

resulting in the form quite similar to that of the Hamiltonian (2.13) of system I (in the absence of the vector potential). Remembering the fact that $f(x)$ has been found to be a constant of motion, we specifically set $f(x)=B$ on the right-hand sides of Eqs. (4.1) and (4.2) to find $H_p=g^{ij}P_i(B)P_j(B)/2m+V$, where

$$P_i(B)=F^j{}_i(B)\pi_j \ . \tag{4.3}$$

The question is now that whether or not one can *identify* the $P_i(B)$'s with the momenta p_i in system I. As regards to this, one is to notice that the following relation holds:

$$\{P_i(B),P_j(B)\}_D=\{P_i,P_j\}_D|_B \ , \tag{4.4}$$

which can be verified with the help of the identity

$$F^j{}_i(B)\frac{\partial f}{\partial x^j}\bigg|_B=0 \tag{4.5}$$

or equivalently $\{P_i(B),f(x)\}_D|_B=0$. We can then prove by a straightforward calculation that the $P_i(B)$'s obey, with the x^i's, the same Dirac brackets as those in Eqs. (2.11). In addition, it follows from Eq. (4.5) that the $P_i(B)$'s are subject to

$$g^{ij}\frac{\partial f}{\partial x^i}\bigg|_B P_j(B)=0 \ , \tag{4.6}$$

which is equivalent to Eq. (2.7). Now that we have the $P_i(B)$'s satisfying the Dirac brackets (2.11) and the constraint relation (2.7), we are led to identify $P_i(B)$ with the momentum p_i in system I:

$$P_i(B)\sim p_i \ . \tag{4.7}$$

Consequently Hamilton's equation for system II is found to be equivalent to the one for system I. This equivalence is essentially represented by Eq. (4.7).

We now make a comparison, at the quantum level, between system I and system II. In fact we have not yet had a quantum theory for system II, mainly owing to the operator-ordering problem in the Hamiltonian (3.9). It is here to be stressed that one can have the quantum theory equivalent to system I only when picking up a certain specific ordering of operators out of various possibilities, and thus the equivalence at the quantum level is not a direct consequence of the classical results. We therefore define the quantum theory for system II *by demanding the equivalence of the two systems as a first principle, now at the quantum level.* With the requirement one is able to quantize system II almost uniquely, or turning to the different standpoints, we will see that the requirement enables us, through the quantum version of Eq. (4.1), to have the representation of the commutation relations (2.14) and hence to have the Schrödinger equation in system I.

We first "symmetrize" the P_i's defined by Eq. (4.1) in order that they become Hermitian operators. Since the π_i's defined by Eq. (3.13) are Hermitian, we replace the P_i's by

$$\begin{aligned}P_i&=\tfrac{1}{2}(F^j{}_i\pi_j+\pi_jF^j{}_i)\\&=F^j{}_i\pi_j-\frac{1}{2}i\frac{\partial F^j{}_i}{\partial x^j} \ .\end{aligned} \tag{4.8}$$

This is not, of course, the unique choice. There is also the ambiguity in writing down the quantum analogue of the classical Hamiltonian (3.9). Then we determine the order of operators so that the quantum Hamiltonian should be equivalent to the Hamiltonian (2.15) when we identify the P_i's defined above [with a suitable value for the constant $f(x)$] with the momenta in system I. The quantum Hamiltonian is thus assumed to have the form

$$H_p=\frac{1}{2m}P_ig^{ij}P_j+V(x) \ . \tag{4.9}$$

Here we have used the notation of Eq. (4.8). Fixing the value of the constant $f(x)$ as B, the momentum P_i in the Hamiltonian is to be realized by $P_i(B)$, where

$$P_i(B)=F^j{}_i(B)\pi_j-\frac{1}{2}i\frac{\partial F^j{}_i}{\partial x^j}\bigg|_B \ , \tag{4.10}$$

which is just the quantum analogue of Eq. (4.3). We now understand that *the commutation relations among the $P_i(B)$'s and the x^i's reduce to the same forms as those of Eqs. (2.14).* Moreover, with Eq. (4.5) [or Eq. (3.10)], we have the operator identity for the $P_i(B)$'s:

$$g^{ij}\frac{\partial f}{\partial x^i}\bigg|_B P_j(B)+P_j(B)\frac{\partial f}{\partial x^i}g^{ij}\bigg|_B=0 \ , \tag{4.11}$$

which has the same content as the constraint (2.17). We can thus identify the $P_i(B)$'s with the momentum operators in system I to obtain the same commutation relations and the Hamiltonian that we derive in Sec. II. In other words, employing expression (3.13), one has *in system I* the coordinate representation of the operators satisfying the commutation relations (2.14) and the constraint (2.17), which fact manifests the great merit of considering the quantization of system II rather than system I. That is to say, one can write down the Schrödinger equation *in system I* via the formulation for system II.

Let us now comment on the operator-ordering problem. In Sec. II, we have chosen the order of operators in the commutation relations and the Hamiltonian as in Eqs. (2.14) and (2.15). However it is, of course, possible to make a different choice, and in such a case, we need to symmetrize the $P_i(B)$'s in the way different from Eq. (4.8) *so as to have the same commutation relations and the Hamiltonian in system II.* Namely, for the various choices of the order of operators in system I, one can accordingly construct many quantum theories (having the same classical limit) for system II *under the requirement of the equivalence of both systems.*

When it comes to the operator-ordering problem in the constraint, we point out that the form of constraint (4.11) has been obtained unambiguously once the expression (4.8) is adopted. The ambiguity is again translated into the nonuniqueness in defining the quantum analogue of Eq. (4.1).

Here we consider the case in which the vector potential is present. We can define at the classical level the P_i's in the same way as in Eq. (4.1), now replacing the π_i's and the P_i's by $\pi_i - A_i$ and $P_i - A_i$, respectively, so that $P_i - A_i = F^j{}_i(\pi_j - A_j)$. As a result, we obtain

$$P_i - A_i = \tfrac{1}{2}[F^j{}_i(\pi_j - A_j) + (\pi_j - A_j)F^j{}_i] \tag{4.12}$$

in the quantum theory. Setting the value of the constant $f(x)$ to be B, the P_i's turn out to be

$$P_i(B) - A_i(B) = F^j{}_i(B)[\pi_j - A_j(B)] - \frac{1}{2} i \left.\frac{\partial F^j{}_i}{\partial x^j}\right|_B , \tag{4.13}$$

where $A_i(B) = A_i|_B$. We thus have the commutation relations and the Hamiltonian in both systems being equivalent. By means of Eq. (3.13), the $P_i(B)$'s are found to acquire the coordinate representation

$$P_i(B) = F^j{}_i(B)\left[-i\frac{\partial}{\partial x^j} - \tfrac{1}{2} i \left.\left\{ {k \atop kj} \right\}\right|_B \right] - \tfrac{1}{2} i \left.\frac{\partial F^j{}_i}{\partial x^j}\right|_B + [\delta^j{}_i - F^j{}_i(B)] A_j(B) , \tag{4.14}$$

which straightforwardly gives the representation of the momenta in system I. After all, one obtains the Schrödinger equation *for system I* by inserting Eq. (4.14) into $i\partial\psi/\partial t = H_p \psi$, with ψ, a wave function.

V. EXAMPLE: THE RIGID ROTATOR IN A FLAT SPACE (REF. 4)

Let us give a simple application of our formulation developed in the preceding sections. As an example, we consider the motion of a particle constrained on the n-dimensional sphere of the constant (squared) radius A. The Lagrangian for the system (system I) is given by

$$L = \tfrac{1}{2} m \dot{x}_i \dot{x}_i + \lambda(x^2 - A), \quad x^2 \equiv x_i x_i , \tag{5.1}$$

where the underlying space is assumed to be flat for simplicity's sake. Then we have the following Dirac brackets[5] and the Hamiltonian H_p:

$$\{x_i, x_j\}_D = 0 , \tag{5.2a}$$

$$\{x_i, p_j\}_D = \delta_{ij} - \frac{x_i x_j}{A} , \tag{5.2b}$$

$$\{p_i, p_j\}_D = \frac{p_i x_j - p_j x_i}{A} , \tag{5.2c}$$

$$H_p = \frac{1}{2m} p_i p_i . \tag{5.3}$$

Here the constraints with which the fundamental Dirac brackets are constructed are

$$\phi_1 \equiv x^2 - A \approx 0 , \tag{5.4a}$$

$$\phi_2 \equiv x_i p_i \approx 0 . \tag{5.4b}$$

In the quantum theory, Eqs. (5.2) are replaced by the commutation relations. The situation characteristic to this simple example is that the "symmetrization" of the right-hand side of Eq. (5.2c) does not cause any change in the expression. Namely, from Eqs. (2.14) we immediately have

$$[x_i, x_j] = 0 , \tag{5.5a}$$

$$[x_i, p_j] = i\left[\delta_{ij} - \frac{x_i x_j}{A}\right] , \tag{5.5b}$$

$$[p_i, p_j] = i\frac{p_i x_j - p_j x_i}{A} . \tag{5.5c}$$

Once we fix the commutation relations as above, there remains no ambiguity in the order of operators [except for the operator form of the constraint (5.4b)]. The Schrödinger equation can then be obtained by representing the momenta so as to satisfy the commutation relations (5.5). Finding the representation is carried out along the same line as in the previous sections, we consider the system constrained by the condition $x_i \dot{x}_i = 0$ (system II). The Lagrangian is given by

$$L = \tfrac{1}{2} m \dot{x}_i \dot{x}_i + \lambda(x_i \dot{x}_i) . \tag{5.6}$$

Eliminating the Lagrange multiplier and its conjugate partner, one finds the Hamiltonian H_p:

$$H_p = \frac{1}{2m}(M_{ij}\pi_j)^2 , \tag{5.7}$$

where π_i is the momentum conjugate to x_i and

$M_{ij}=\delta_{ij}-(x_i x_j)/x^2$ is a projection matrix which satisfies $M_{ij}x_j=0$ (x_i is the vector normal to the hypersphere $x^2=$const). Now the new variables P_i are defined as

$$P_i=M_{ij}\pi_j \ . \tag{5.8}$$

By this the Hamiltonian (5.7) is rewritten as $H_p=P_iP_i/2m$. Since $dx^2/dt=\{x^2,H_p\}_D=0$, and hence imposing the initial condition $x^2=A$ on the P_i's, we obtain the same Dirac brackets as Eqs. (5.2) in terms of the new variables.

Upon quantization, we replace the P_i's defined above with

$$P_i=\tfrac{1}{2}(M_{ij}\pi_j+\pi_j M_{ij}) \ , \tag{5.9}$$

according to Eq. (4.8). The quantum Hamiltonian consequently becomes $H_p=(M_{ij}\pi_j+\pi_j M_{ij})^2/8m$. The coordinate representation of the momentum is given by $\pi_i=-i\partial/\partial x_i$, and for a fixed value of x^2, say A, the operator expression for the P_i's yields

$$P_i(A)=-iM_{ij}(A)\frac{\partial}{\partial x^j}+\tfrac{1}{2}i\frac{n-1}{A}x_i \ , \tag{5.10}$$

where $M_{ij}(A)=M_{ij}|_A=\delta_{ij}-(x_ix_j)/A$. It gives the representation of the commutation relations (5.5) with the identification $P_i(A)\sim p_i$. Now having obtained the representation, one has the Schrödinger equation and furthermore, one can write unambiguously the quantum analogue of the constraint (5.4b) as $x_ip_i+p_ix_i=0$. [The difference between x_ip_i and p_ix_i is a c number and we cannot specify the form of the constraint operator as above with the condition (2.16) only, which situation is slightly different from the general case treated in the previous sections. One thus needs an additional principle, that is, in our case, the equivalence of both systems.]

VI. CONCLUSION

We have discussed the quantization of the system describing the motion of a nonrelativistic particle with the constraint $f(x)-B=0$ (system I). In the Hamiltonian formulation, all the constraints are second class and thereby the Dirac-bracket quantization has been applied. The resulting Dirac brackets, however, contain the products of phase-space variables which do not commute in the quantum theory. Having defined the commutation relations from the Dirac brackets through the simplest "symmetrization" method as in Eqs. (2.14), we obtain the equations of motion with the appropriate quantum Hamiltonian (2.15).

For writing down the Schrödinger equation we need, further, the coordinate representation of the momenta obeying the commutation relations (2.14). The representation can be constructed by considering the system constrained by the relation $\dot{f}(x)=0$ (system II) instead of system I. That is, the commutation relations reduce to the canonical forms in this case: we can get the coordinate representation quite easily in system II. On the other hand, the difficulty in quantizing system II mainly originates from the operator-ordering problem in the Hamiltonian (3.9). We have resolved the ambiguity *by demanding* that the Hamiltonian and the commutation relations have the same expressions as those in system I. As a result, we can construct the quantum theory for system II with the fixed value B of $f(x)$, and at the same time, we have the representation of the commutation relations (2.14), by which one can write down the Schrödinger equation for system I.

The essential point in the above prescription to obtain the Schrödinger equation is that we have determined the order of operators [especially in Eq. (4.8) or (4.10)] so as to get the equivalent quantum systems in both cases. Namely, we have first determined the quantum theory for system I, and accordingly we have chosen the expression of Eq. (4.8). In fact, the most general form of the momentum operator $P_i(B)$ is given by

$$P_i(B)=F^j{}_i(B)\pi_j+\Delta_i(B{:}x) \ , \tag{6.1}$$

where $\Delta_i(B{:}x)$ is a certain function of the x^i's representing the quantum corrections due to the noncommutativity of operators. Evidently the function $\Delta_i(B{:}x)$ admits various expressions, all of which vanish in the classical limit. It is thus expected that for a given $\Delta_i(B{:}x)$ we can correspondingly construct the quantum theory for system I under the requirement of the equivalence. The quantum theory with Eqs. (2.14) and (2.15) is, for example, realized by

$$\Delta_i(B{:}x)=-\tfrac{1}{2}i\left.\frac{\partial F^j{}_i}{\partial x^j}\right|_B \ , \tag{6.2}$$

as was shown in Sec. IV. When we adopt the different commutation relations in system I, the coordinate representation of the momenta ought to be modified. To obtain the correct representation, we must seek for an appropriate quantum correction $\Delta_i(B{:}x)$ in the quantization of system II.

Furthermore, determining the quantum corrections amounts to determining the operator form of constraints. Namely, Eq. (4.10) leads us uniquely to Eq. (2.17), while condition (2.16) does not give the unique form of constraint operators.

Since our formulation includes an arbitrarily chosen metric, it permits various applications not just as the motion of a particle in a curved space. Indeed the motion in a flat space is often preferably described in terms of curvilinear coordinates, in which one must formulate with a "metric" in the Lagrangian.

In addition, physically, we can conclude our results in this paper, in such a way that we have obtained the Schrödinger equation that describes a particle under some potential constraining its motion on the hypersurface $f(x)=B$.

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[6]Therefore the relation $x\dot{x}=0$ appears as an equation of motion rather than as a constraint in the sense of Dirac.

[7]Strictly speaking, extra terms are needed under the requirement that the tangential components P be Hermitian. These terms also play an important role in determining the order of operators. See the main text.

[8]In the following equations, the symbol $\cdots|_B$ means that "$\cdots$" is to be evaluated modulo the constraints.

[9]Evidently there still remains an ambiguity, for example, in the point that one can add to ϕ_2 a c number proportional to Planck's constant (which is set to be unity throughout this paper).

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