Inertial effects of a Dirac particle

Friedrich W. Hehl* and Wei-Tou Ni
Department of Physics, National Tsing Hua University, Hsinchu, Taiwan, Republic of China
(Received 19 March 1990)

Stationary laboratories on Earth accelerate and rotate relative to the local inertial frames. Any experiment precise enough would detect and/or need to take into account the effects due to acceleration and rotation. We derive these inertial effects for a Dirac particle in a straightforward and unified way within the framework of special relativity. The effects found include the Bonse-Wroblewski phase shift due to acceleration, the Sagnac-type effect, the rotation-spin effect, and the redshift of the kinetic energy.

Because of Earth's gravity and rotation, a local inertial frame accelerates and rotates relative to Earth. According to Einstein's equivalence principle, local physics in this frame is that of special relativity, provided we can neglect curvature (tidal) effects. However, an observer in a *stationary laboratory* on Earth finds himself in a noninertial frame, and inertial effects arise due to acceleration and rotation.[1-4]

This is, of course, well known from classical mechanics, and the Foucault pendulum is the most spectacular evidence for the noninertial nature of our local laboratory frames. The optical "Foucault pendulum" is the Michelson-Gale light interferometer in which the rotation of Earth yielded a Sagnac shift of the light waves.[5]

In the last 15 years, neutron interferometry has been developed with ever increasing accuracy.[6] Interferometers with a typical length of about 10 cm built from silicon monocrystals are in use, and the neutron analog of the Foucault-Michelson-Gale, effect has been found by Werner *et al.*,[7] whereas Atwood *et al.*[8] found the neutron Sagnac effect using an angular velocity of about 30 times that of Earth.

Moreover, Bonse and Wroblewski,[9] with a neutron interferometer positioned horizontally in their laboratory, accelerated it, and found the predicted phase shift. If we apply the equivalence principle, the same effect has to show up in the gravitational field. This has been verified by the celebrated Colella-Overhauser-Werner (COW) experiment,[10] which preceded the neutron experiments mentioned above. The COW and the Bonse-Wroblewski experiments, taken together, attest to the validity of the equivalence principle for neutron waves. Curvature effects in Earth laboratories are small compared to these effects and can be neglected to first order.

It has become feasible to use polarized neutrons in the interferometer experiments. Hence the possible effects of the spin of the neutron come into focus. Mashhoon[11,12] has recently proposed a coupling of the neutron spin to the rotation of a noninertial reference frame. He derived this new rotation-spin coupling from a "simple, yet tentative, extension of the hypothesis of locality."

In this paper[13] we will put the special-relativistic Dirac equation into a noninertial reference frame by standard methods, confining ourselves strictly to flat Minkowski spacetime.[14] We assume that the noninertial observers are equipped with ideal measuring instruments which are insensitive to acceleration ("standard observers").[1] We compute the inertial effects of a Dirac particle exactly. Applying three successive Foldy-Wouthuysen transformations, we find in a nonrelativistic approximation in lowest order, the Bonse-Wroblewski, the Page-Werner, and the Mashhoon coupling terms all at once.

Coordinates and tetrad (frame) field.[15] Consider an inertial frame of reference with Minkowskian coordinates $x^{\mu'}$. In it an observer moves with proper three-acceleration $\mathbf{a}(\tau)$ and proper three-rotation $\boldsymbol{\omega}(\tau)$, τ being the proper time. The orthonormal tetrad (frame) $e_{\hat{\alpha}}$, the observer carries, can be defined as follows. $e_{\hat{0}}(\tau)$ is identical with the four-velocity $u^{\mu'}$ of the observer; the spatial triad $e_{\hat{i}}$ is defined to be orthogonal to $e_{\hat{0}}(\tau)$ and rotates with proper rotation $\boldsymbol{\omega}(\tau)$.[1,2] Put in mathematical terms, the orthonormal tetrad $e_{\hat{\alpha}}(\tau)$ transports according to

$$\frac{de_{\hat{\alpha}}}{d\tau}=\Omega\cdot e_{\hat{\alpha}} \ , \tag{1}$$

where the antisymmetric rotation tensor Ω splits into a Fermi-Walker part Ω_F and a spatial rotation part Ω_R:

$$\Omega^{\mu\nu}\equiv\Omega_F^{\mu\nu}+\Omega_R^{\mu\nu}\equiv(a^{\mu}u^{\nu}-a^{\nu}u^{\mu})/c^2+u_{\alpha}\omega_{\beta}\epsilon^{\alpha\beta\mu\nu}/c \ . \tag{2}$$

a^{μ} is the four-acceleration of the observer, ω^{μ} its four-rotation; $\epsilon^{\alpha\beta\mu\nu}$ is the Levi-Civita tensor with $\epsilon^{0'1'2'3'}=-1$. To each point $\mathcal{P}$ on the world line of the observer, associate the *spacelike hyperplane* $S(\mathcal{P})$ orthogonal to it. Define $x^0\equiv ct\equiv c\tau$ and, moreover, x^1,x^2,x^3 as Cartesian coordinates using the triad $e_{\hat{i}}(\tau)$ with the observer at the origin: $x^{\mu}=(x^0,x^1,x^2,x^3)$ are the local coordinates for the observer. From $\mathcal{P}$, parallel transport the tetrad $e_{\hat{\mu}}(\tau)$ to all neighboring points on $S(\mathcal{P})$. This defines the orthonormal tetrad field $e_{\hat{\mu}}(x^{\nu})$. Such a local coordinate system is what we actually use in our laboratory. The world line is that of our reference clock.

Metric and connection. The tetrad field $e_{\hat{\mu}}$ is anholonomic. Define the coordinate tetrad e_{μ} as $\partial/\partial x^{\mu}$. Denote the nonrotating local coordinate system with $\omega(\tau)=0$ by double-primed indices. According to Misner, Thorne, and Wheeler (MTW) (Sec. 6.6),[1]

$$e_{\hat{0}}=e_{\hat{0}''}=\frac{1}{\left[1+\frac{\mathbf{a}\cdot\mathbf{x}}{c^2}\right]}e_{0''} . \tag{3}$$

From rotational coordinate transformation, as in classical mechanics,

$$e_{0''}=e_0-[(\boldsymbol{\omega}/c)\times\mathbf{x}]^k e_k . \tag{4}$$

Combining (3) and (4), and noting that $e_{\hat{i}}=e_i$, as is evident from our construction, we have

$$e_{\hat{0}}=\frac{1}{\left[1+\frac{\mathbf{a}\cdot\mathbf{x}}{c^2}\right]}\{e_0-[(\boldsymbol{\omega}/c)\times\mathbf{x}]^k e_k\} ,$$
$$e_{\hat{i}}=e_i . \tag{5}$$

The tetrad field (5), which is orthonormal, $e_{\hat{\mu}}\cdot e_{\hat{\nu}}=\delta_{\hat{\mu}\hat{\nu}}=\text{diag}\,(+---)$, and which behaves as a rotating Fermi-Walker-transported reference frame, is all that we need as input for the description of the noninertial reference frame. The results of this paper are a consequence of (5), the Dirac equation, and the assumption of the existence of standard observers.

From (5), the dual basis (of one-forms) is

$$\theta^{\hat{0}}=(1+\mathbf{a}\cdot\mathbf{x}/c^2)dx^0 ,$$
$$\theta^{\hat{i}}=dx^i+[(\boldsymbol{\omega}/c)\times\mathbf{x}]^i dx^0 . \tag{6}$$

The metric, as obtained from

$$ds^2=\theta^{\hat{0}}\otimes\theta^{\hat{0}}-\theta^{\hat{1}}\otimes\theta^{\hat{1}}-\theta^{\hat{2}}\otimes\theta^{\hat{2}}-\theta^{\hat{3}}\otimes\theta^{\hat{3}}=g_{\mu\nu}dx^\mu dx^\nu$$
$$=(dx^0)^2[1+2\mathbf{a}\cdot\mathbf{x}/c^2+(\mathbf{a}\cdot\mathbf{x}/c^2)^2+(\boldsymbol{\omega}\cdot\mathbf{x}/c)^2-(\boldsymbol{\omega}\cdot\boldsymbol{\omega}/c^2)(\mathbf{x}\cdot\mathbf{x})]-2\,dx^0 d\mathbf{x}\cdot(\boldsymbol{\omega}/c)\times\mathbf{x}-d\mathbf{x}\cdot d\mathbf{x} , \tag{7}$$

is in agreement with Ref. 2.

The connection expressed with respect to the orthonormal tetrad (5), reads

$$\Gamma_{\hat{\lambda}\hat{\nu}\hat{\mu}}=\tfrac{1}{2}(C_{\hat{\lambda}\hat{\nu}\hat{\mu}}-C_{\hat{\nu}\hat{\mu}\hat{\lambda}}-C_{\hat{\mu}\hat{\lambda}\hat{\nu}}) , \tag{8}$$

where the object of anholonomicity is defined by $C_{\hat{\lambda}\hat{\nu}}{}^{\hat{\mu}}=e^{\lambda}{}_{\hat{\lambda}}e^{\nu}{}_{\hat{\nu}}(\partial_\lambda e_\nu{}^{\hat{\mu}}-\partial_\nu e_\lambda{}^{\hat{\mu}})$.

We find

$$\Gamma_{\hat{i}\hat{j}\hat{0}}=-\frac{\epsilon_{ijk}\omega^k/c}{1+\frac{\mathbf{a}\cdot\mathbf{x}}{c^2}} ,$$
$$\Gamma_{\hat{0}\hat{i}\hat{0}}=-\Gamma_{\hat{i}\hat{0}\hat{0}}=-\frac{a^i/c^2}{1+\frac{\mathbf{a}\cdot\mathbf{x}}{c^2}} , \tag{9}$$
$$\Gamma_{\hat{\mu}\hat{\nu}\hat{i}}=\Gamma_{\hat{0}\hat{0}\hat{0}}=0 ,$$

where ϵ_{ijk} is the three-dimensional Levi-Civita symbol with $\epsilon_{123}=1$.

Dirac Equation in the Observer's Noninertial Frame. The Dirac equation in inertial coordinates reads[16]

$$i\hbar\gamma^{\mu'}\partial_{\mu'}\Psi'=mc\Psi' . \tag{10}$$

In the observer's local frame, (10) becomes[17]

$$i\hbar\gamma^{\hat{\mu}}D_{\hat{\mu}}\Psi=mc\Psi , \tag{11}$$

with

$$D_{\hat{\alpha}}\equiv\partial_{\hat{\alpha}}-\frac{i}{4}\sigma^{\hat{\beta}\hat{\gamma}}\Gamma_{\hat{\beta}\hat{\gamma}\hat{\alpha}} ,$$
$$\sigma^{\hat{\beta}\hat{\gamma}}\equiv\frac{i}{2}(\gamma^{\hat{\beta}}\gamma^{\hat{\gamma}}-\gamma^{\hat{\gamma}}\gamma^{\hat{\beta}}) . \tag{12}$$

Using (5), (6), and (9),

$$D_{\hat{0}}=\frac{1}{1+\frac{\mathbf{a}\cdot\mathbf{x}}{c^2}}\left[\frac{\partial}{\partial x^0}+\frac{1}{2c^2}\mathbf{a}\cdot\boldsymbol{\alpha}-\frac{i}{c\hbar}\boldsymbol{\omega}\cdot\mathbf{J}\right] ,$$
$$D_{\hat{i}}=\frac{\partial}{\partial x^i} , \tag{13}$$

where

$$\mathbf{J}=\mathbf{L}+\mathbf{S}=\mathbf{x}\times\frac{\hbar}{i}\frac{\partial}{\partial\mathbf{x}}+\tfrac{1}{2}\hbar\boldsymbol{\sigma} , \tag{14}$$

is the total angular momentum.

Substituting (13) into (11), the Dirac equation acquires the form

$$i\hbar\frac{\partial\Psi}{\partial t}=H\Psi , \tag{15}$$

with the Hamiltonian

$$H=\beta mc^2+\mathcal{O}+\mathcal{E} , \tag{16}$$

$$\mathcal{O}=c\boldsymbol{\alpha}\cdot\mathbf{p}+\frac{1}{2c}[(\mathbf{a}\cdot\mathbf{x})(\mathbf{p}\cdot\boldsymbol{\alpha})+(\mathbf{p}\cdot\boldsymbol{\alpha})(\mathbf{a}\cdot\mathbf{x})] , \tag{17}$$

$$\mathcal{E}=\beta m(\mathbf{a}\cdot\mathbf{x})-\boldsymbol{\omega}\cdot(\mathbf{L}+\mathbf{S}) . \tag{18}$$

$\mathcal{O}$ is an odd and $\mathcal{E}$ an even operator. Note that this equation is exact. Table I lists the relativistic inertial effects.

Dirac Equation in Nonrelativistic Approximation. After three successive Foldy-Wouthysen transformations, the Hamiltonian can be put into the form[18] (dropping triple primes)

$$H=\beta\left[mc^2+\frac{\mathcal{O}^2}{2mc^2}-\frac{\mathcal{O}^4}{8m^3c^6}\right]+\mathcal{E}-\frac{1}{8m^2c^4}[\mathcal{O},[\mathcal{O},\mathcal{E}]]-\frac{i\hbar}{8m^2c^4}[\mathcal{O},\dot{\mathcal{O}}] . \tag{19}$$

TABLE I. Relativistic inertial effects in the Hamiltonian of a Dirac particle. The Dirac matrices are β and $\boldsymbol{\alpha}$. The mass of the particle is m, $\mathbf{x}$, its local spatial coordinates, $\mathbf{p}$ its momentum, $\mathbf{L}$ its orbital, and $\mathbf{S}$ its spin angular momentum. The noninertial frame is characterized by its proper acceleration $\mathbf{a}$ and its proper rotation $\boldsymbol{\omega}$.

1.	$\beta m(\mathbf{a}\cdot\mathbf{x})$	Energy-momentum redshift effects
2.	$\frac{1}{2c}[(\mathbf{a}\cdot\mathbf{x})(\boldsymbol{\alpha}\cdot\mathbf{p})+(\boldsymbol{\alpha}\cdot\mathbf{p})(\mathbf{a}\cdot\mathbf{x})]$	
3.	$-\boldsymbol{\omega}\cdot\mathbf{L}$	Sagnac-type effect
4.	$-\boldsymbol{\omega}\cdot\mathbf{S}$	Rotation-spin coupling

TABLE II. Inertial effects of a Dirac particle in nonrelativistic approximation. The spin matrix is denoted by $\boldsymbol{\sigma}$, otherwise see Table I, The gravitational effect corresponding to an inertial effect is indicated by an arrow.

1.	$\beta m(\mathbf{a}\cdot\mathbf{x})$	Bose-Wroblewski (Ref. 9) [$\rightarrow$COW (Ref. 10)]
2.	$-\boldsymbol{\omega}\cdot\mathbf{L}$	Page-Werner *et al.* (Ref. 7)
3.	$-\boldsymbol{\omega}\cdot\mathbf{S}$	Mashhoon (Refs. 11 and 19)
4.	$\beta\frac{\mathbf{p}(\mathbf{a}\cdot\mathbf{x})\mathbf{p}}{2mc^2}$	Redshift effect of the kinetic energy
5.	$\frac{\hbar}{4mc^2}\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{p})$	New inertial spin-orbit coupling

Evaluating the operator products to the desired accuracy we find

$$H=\beta mc^2+\frac{\beta}{2m}\mathbf{p}^2+\beta m(\mathbf{a}\cdot\mathbf{x})+\frac{\beta}{2m}\mathbf{p}\left[\frac{\mathbf{a}\cdot\mathbf{x}}{c^2}\right]\cdot\mathbf{p}$$
$$-\boldsymbol{\omega}\cdot(\mathbf{L}+\mathbf{S})+\frac{\hbar}{4mc^2}\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{p})$$
$$+\text{higher-order terms}\;. \tag{20}$$

Inertial effects in H are listed in Table II.

Discussion. As announced we have calculated the inertial effects of a Dirac particle. In particular, these results are valid for a neutron. Therefore the rotation-spin coupling, predicted by Mashhoon for a neutron wave, has been derived in an alternative way. There can be hardly a doubt left that this effect will be found as soon as the necessary experimental accuracy is achieved.

The phase shifts caused by acceleration and rotation, listed under no. 1 and no. 2 in Table II, have been experimentally verified by Bonse and Wroblewski,[9] and Werner *et al.*,[7] respectively. We recovered these effects in our derivation straightforwardly. Their verification and the consistency of our derivation leads us to believe that also the higher-order terms in the Foldy-Wouthuysen procedure presented mirror the actual behavior of Dirac particles in noninertial reference frames. Higher-order inertial effects, the application of the equivalence principle, the discussion of the corresponding gravitational effects, and the tidal gravitational (curvature) effects will be presented in forthcoming publications.

This work was supported in part by the National Science Council of the Republic of China under Contracts Nos. NSC-79-0208-M007-110 and NSC-79-0208-M007-07.

---

*Permanent address: Institute of Theoretical Physics, University of Cologne, D-5000 Cologne 41, Federal Republic of Germany.

[1]For a general discussion, see C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Chap. 6, Sec. 13.6.

[2]W.-T. Ni, Chinese J. Phys. **15**, 51 (1977); W.-Q. Li and W.-T. Ni, *ibid.* **16**, 214 (1978).

[3]W.-T. Ni and M. Zimmermann, Phys. Rev. D **17**, 1473 (1978).

[4]B. DeFacio, P. W. Dennis, and D. G. Retzloff, Phys. Rev. D **18**, 2813 (1978).

[5]For a review of these effects, see E. J. Post, Rev. Mod. Phys. **39**, 475 (1967).

[6]See, *Matter Wave Interferometry,* International Workshop, Vienna, Austria, edited by G. Badurek *et al.* [Physica **B151**, 1–400 (1988)].

[7]S. A. Werner, J.-L. Staudenmann, and R. Colella, Phys. Rev. Lett. **42**, 1103 (1979); this effect has been predicted by L. A. Page, *ibid.* **35**, 543 (1975).

[8]D. K. Atwood, M. A. Horne, C. G. Shull, and J. Arthur, Phys. Rev. Lett. **52**, 1673 (1984).

[9]U. Bonse and T. Wroblewski, Phys. Rev. Lett. **51**, 1401 (1983).

[10]R. Colella, A. W. Overhauser, and S. A. Werner, Phys. Rev. Lett. **34**, 1472 (1975).

[11]B. Mashhoon, Phys. Rev. Lett. **61**, 2639 (1988).

[12]The rotation-spin coupling has also been found by V. Schroth, Diploma thesis, University of Cologne, 1984 (unpublished).

[13]This paper is based on a talk given at the Hsinchu School on Gravitation, Relativity and Cosmology, Hsinchu, Taiwan, Republic of China, 1989.

[14]The Dirac equation in the *gravitational* field of Earth has been discussed by E. Fischbach [in *Cosmology and Gravitation: Spin, Torsion, Rotation, and Supergravity,* proceedings of the International School, Erice, Italy, 1979, edited by P. G. Bergmann and V. de Sabbata (NATO Advanced Studies Institutes—Series B: Physics, Vol. 58) (Plenum, New York, 1980), p. 359] and by D. M. Greenberger and A. W. Overhauser [Rev. Mod. Phys. **51**, 43 (1979)], among others. See also more recent work by C. Q. Xia and Y. L. Wu, Phys. Lett. A **141**, 251 (1989). The Dirac equation in a rotating frame has been discussed by T. C. Chapman and D. J. Leiter [Am. J. Phys. **44**, 858 (1976)], E. Schmutzer and J. Plebanski [Fortschr. Phys. **25**, 37 (1977)], and B. R. Iyer [Phys. Rev. D **26**, 1900 (1982)], among others.

[15]We use latin letters to denote three-indices and greek letters to denote four-indices. Different coordinates systems are denoted by primes or double primes on the indices. A caret on an index denotes an orthonormal-tetrad component. We use the

$(+,-,-,-)$ signature for the metric throughout the paper. The components of a three-vector $\mathbf{v}$ are denoted by (v^1, v^2, v^3).

[16]For the γ^μ, β, and $\boldsymbol{\alpha}$ matrices, we use the Dirac repreesntation of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, San Francisco, 1964).

[17]For the Dirac equation in an arbitrary frame, compare D. W. Sciama, *Recent Developments in General Relativity* (Pergamon, Oxford, 1962), p. 415; T. W. B. Kibble, J. Math. Phys. **2**, 212 (1961), or the discussions in F. W. Hehl, Found. Phys. **15**, 451 (1985).

[18]See Sec. 4.3 of Ref. 16.

[19]S. A. Werner (see Ref. 11).